On Geometric Evolution and Cascadic Multigrid in Subdivision

U. Diewald *, S. Morigi †, M. Rumpf ‡

*Bonn University, †Bologna University, ‡Duisburg University

Abstract

A new approach to subdivision based on the evolution of surfaces under curvature motion is presented. Such an evolution can be understood as a natural geometric filter process where time corresponds to the filter width. Thus, subdivision can be interpreted as the application of a geometric filter on an initial surface. The approach closely connects subdivision to surface fairing concerning the geometric smoothing and to cascadic multigrid methods with respect to the actual numerical procedure. The derived method does not distinguish between different valences of nodes nor between different mesh refinement types.

Keywords: Variational Subdivision, Surface Fairing, Curves & Surfaces, Geometric Modeling, Image Processing

1 Introduction

Multiresolution mesh representations are a key tool used in computer graphics to achieve real-time interaction with large and complex object models. In a multiresolution modeling environment we are able to deal with global shape and structural details of the same object managing meshes of it at different level of refinement. Starting from coarse representations of surfaces we can generate appealing smooth representation by iterative applications of refinement steps while keeping the connectivity of the original mesh. We propose a refinement approach that combines the advantages of subdivision (arbitrary topology, local control and efficiency) with those of variational design (high quality surfaces) [25].

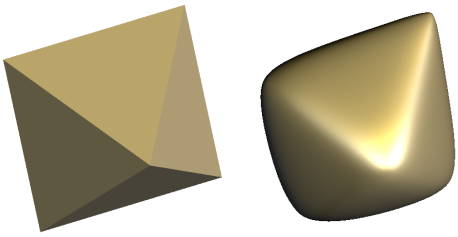

Figure 1: *Starting from a coarse mesh (left) and considering a spatially varying filter width we obtain a limit surface with locally different smoothness modulus.*

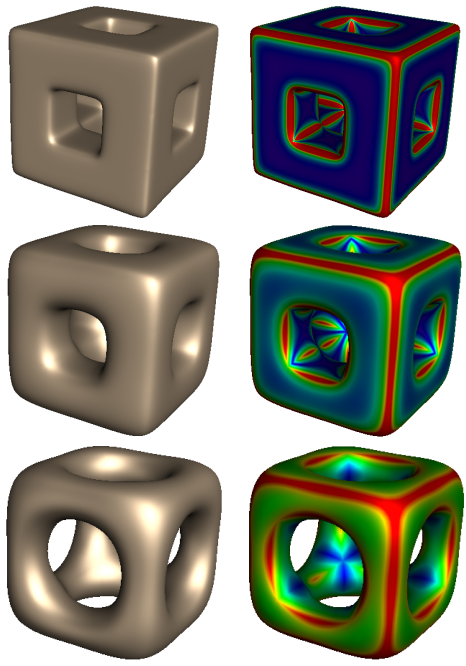

Figure 2: *Increasing a varying filter width one obtains a scale of subdivision surfaces ranging from smoothed sharp edges up to a smoothing of the complete geometry. On the right the corresponding mean curvatures are color coded on the surfaces, especially showing the boundedness of the curvature and giving an indication of bounded second derivatives.*

Subdivision surface modeling is a lively area of research and a promising approach to the efficient design of surfaces with complex geometry. The basic idea behind subdivision is to refine and smooth a given coarse mesh until a smooth surface is obtained. Suppose our surface is represented as a triangular mesh $\mathcal{M}$ embedded in $\mathbb{R}^3$. Starting with an initial coarse mesh $\mathcal{M}^0$, successive meshes are determined iteratively by the equation

$$\mathcal{M}^k = \mathcal{S}^k(\mathcal{M}^{k-1})$$

where $\mathcal{S}^k$ is the subdivision operator at the k^{th} level which takes the points from level k to points on the finer level $k+1$. Assuming that the subdivision converges, the actual subdivision surface is defined as the limit of this sequence of successive refinements:

$$\mathcal{M} := \lim_{k\to\infty} S^k \circ S^{k-1} \circ \cdots \circ S^1 \mathcal{M}^0.$$

The subdivision schemes for arbitrary topology control meshes come in two classes: approximating and interpolating. Many variants of the approximating schemes have been considered; the classical ones are due to Catmull-Clark [2], and Doo-Sabin [9], which considered an extension of quadratic and cubic B-splines on rectangular meshes respectively, while a scheme based on quartic box splines on triangular meshes was presented by Loop [20]. An extension of Loop's scheme, introduced by Hoppe et al. [14] incorporates sharp edges on the final limit surface.

Dyn, Gregory and Levin [10] introduced the butterfly scheme, a simple interpolating subdivision algorithm applicable to arbitrary triangular meshes. Since it only leads to C^1 surfaces in the regular setting (all vertices of the mesh have valence 6), an improved butterfly scheme, the so-called modified butterfly, resulting in smoother surfaces, has been introduced in [28].

Most known stationary subdivision schemes generate at least C^1-continuity surfaces on arbitrary meshes in the regular setting [22, 26]. Recently, the smoothness of the subdivision surfaces in irregular setting (that is near extraordinary vertices) has been rigorously proved in [22, 27].

In [16, 19] an approach to mesh refinement based on variational methods has been proposed in order to define univariate variational subdivision schemes. Kobbelt [18] considered energies which involve curvature quantities. The corresponding evolution problems would lead to Willmore flow and surface diffusion respectively, which are fourth order parabolic problems. Whereas we here restrict to second order mean curvature flow.

In this paper the smoothing step is based on a geometric diffusion and filtering approach related to mean curvature motion, which has already been proved to be very promising for surface fairing purposes [6, 7]. Actually, we consider a single fully implicit timestep of mean curvature motion as our smoothing method. In an iteration we successively refine the surface mesh - which turns this approach into a subdivision scheme - and solve a semiimplicit problem to approximate the fully implicit step. This semiimplicit scheme is explicit with respect to the given metric from the last step and implicit concerning the new positions of the nodes. Thus, our method can be regarded as a fixed point iteration, where we simultaneously expect to improve the metric and the resolution. This leads to a suitable geometric smoothing filter on the initial mesh. Our approach is a usual subdivision scheme, but now founded on tools from the theory of geometric evolution problems (mean curvature motion) and numerical analysis (cascadic multigrid). We try to outrole this new perspective which we believe to offer strong provisions concerning the theoretical analysis of subdivision schemes as well as the range of applications. Furthermore, as already mentioned this approach bridges the gap between subdivision and surface fairing on a rigorous basis. Not very surprisingly, several important question within this new perspective remain open and require further investigation. Concerning the regularity and convergence we only state conjectures here.

The resulting method is closely related to standard subdivision schemes concerning the computational complexity. Our approach however has the following advantages:

- Many qualitative properties of the approach can be studied already on the continuous level and do not require a detailed analysis of the discretization.
- The model is independent of the type of meshes, especially of the valences of the mesh nodes, and the considered refinement rules. These characteristics are naturally incorporated in the finite element matrices and do not influence the method's performance significantly (cf. Fig. 5).

- The resulting scheme can be easily adapted to different applications, e. g. to spatially variation of the corresponding filter width (cf. Fig. 1) or to solely smoothing the edges of coarse polygonal models (cf. Fig. 2).

Furthermore, we expect the resulting limit surfaces to be $C^{2,\alpha}$-smooth for every $\alpha \in [0,1)$.

2 Curvature motion as a natural smoothing process

In this section we will outline evolutionary smoothing methods and their application in surface fairing to motivate our subdivision scheme (cf. Section 6) which generates new surfaces instead of improving the quality of given surfaces reviewed in this section. Simultaneously we will introduce the basic notation of geometry and geometric differential operators. For details we refer to [8] and [3, Chapter 1]. Let us consider a smooth compact embedded manifold $\mathcal{M} \subset \mathbb{R}^3$ without boundary. Let $x : \Omega \to \mathcal{M}\,;\, \xi \mapsto x(\xi)$ be some coordinate map from an atlas. For each point x on $\mathcal{M}$ the tangent space $\mathcal{T}_x\mathcal{M}$ is spanned by the basis $\{\frac{\partial x}{\partial \xi_1}, \frac{\partial x}{\partial \xi_2}\}$. By $\mathcal{T}\mathcal{M}$ we denote the tangent bundle. Measuring length on $\mathcal{M}$ requires the definition of a metric $g(\cdot,\cdot) : \mathcal{T}_x\mathcal{M} \times \mathcal{T}_x\mathcal{M} \to \mathbb{R}$. As the corresponding matrix notation we obtain $g = (g_{ij})_{ij}$ with $g_{ij} = \frac{\partial x}{\partial \xi_i} \cdot \frac{\partial x}{\partial \xi_j}$, where $\cdot$ indicates the scalar product in $\mathbb{R}^3$. The inverse of g is denoted by $g^{-1} = (g^{ij})_{ij}$. The gradient $\nabla_{\mathcal{M}} f$ of a function f is defined as the representation of df with respect to the metric g. In coordinates we obtain

$$\nabla_{\mathcal{M}} f := \sum_{i,j} g^{ij} \frac{\partial (f \circ x)}{\partial \xi_j} \frac{\partial}{\partial \xi_i}\,.$$

We define the divergence $\mathrm{div}_{\mathcal{M}} v$ of a vector field $v \in \mathcal{T}\mathcal{M}$ as the dual operator of the gradient with respect to the L^2-product on $\mathcal{M}$ and obtain in coordinates

$$\mathrm{div}_{\mathcal{M}} v := \sum_i \frac{\partial}{\partial \xi_i}((v_i \circ x)\sqrt{\det g}) \frac{1}{\sqrt{\det g}}\,.$$

Finally, the Laplace Beltrami operator $\Delta_{\mathcal{M}}$ is given by $\Delta_{\mathcal{M}} u := \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} u$. With this operator at hand we can define geometric diffusion in analogy to the linear diffusion problem in the Euclidean space. Furthermore, we can consider a diffusion of the manifold geometry itself. I. e., we seek a one parameter family of embedded manifolds $\{\mathcal{M}(t)\}_{t\geq 0}$ and corresponding parametrizations $x(t)$, such that

$$\begin{aligned} \partial_t x(t) - \Delta_{\mathcal{M}(t)} x(t) &= 0\,, \\ \mathcal{M}(0) &= \mathcal{M}_0\,. \end{aligned}$$

Already in '91 Dziuk [11] presented a semi implicit finite element scheme for geometric diffusion based on this formulation. The fundamental observation is that this geometric diffusion of the coordinate mapping itself coincides with the motion by mean curvature (MCM) [15]; in fact for any manifold $\mathcal{M}$ we have $\Delta_{\mathcal{M}} x = -H(x)\,N(x)$, and thus we obtain $\partial_t x = -H(x)N(x)$, where $H(x)$ is the corresponding mean curvature (here defined as the sum of the two principal curvatures), and $N(x)$ is the normal on the surface at each point x. For the sake of simplicity we define $MCM(t)\mathcal{M}^0 := \mathcal{M}(t)$, where $\mathcal{M}(t)$ is the solution surface for time t. Thus $MCM(\sigma^2/2)\mathcal{M}$ can be regarded as the application of a "geometric " Gaussian filter of width σ to $\mathcal{M}$. The mean curvature motion model is known as the gradient flow with respect to surface area. This is one indication for the strong regularizing effect of MCM.

Previous work on surface fairing has already involved the concept of curvature motion. Taubin [23] and Kobbelt [17] considered an umbrella operator, which is a "spring force type" implementation of the Laplace Beltrami operator. Desbrun et al. [6] discussed an implicit discretization of geometric diffusion closely related to Dziuk's approach to obtain strongly stable numerical smoothing schemes. Here we will apply Dziuks method [11] as the basic discrete smoothing scheme. Hence, in the MCM case in each step of a mean curvature evolution one asks for a family of triangular surfaces $\{\mathcal{M}^k\}_{k\geq 0}$ and corresponding parametrizations $X^k \in (V_k)^3$, where V_k denotes the affine finite element space on the grid $\mathcal{M}_k$, such that

$$\frac{X^k - X^{k-1}}{\tau} - \Delta_{\mathcal{M}^{k-1}} X^k = 0\,,$$

where the discrete Laplace Beltrami operator is defined by $(\Delta_{\mathcal{M}^k} U, \Theta)^h_{\mathcal{M}^k} := -(\nabla_{\mathcal{M}^k} U, \nabla_{\mathcal{M}^k} \Theta)_{\mathcal{T}\mathcal{M}^k}$ for all $\Theta \in V_k$ using the scalar product $(v,w)_{\mathcal{T}\mathcal{M}^k} := \int_{\mathcal{M}^k} g(v,w)\,\mathrm{d}x$ on $\mathcal{T}\mathcal{M}^k$ and the lumped mass L^2 product $(\cdot,\cdot)^h_{\mathcal{M}^k}$, which is defined by $(U,V)^h_{\mathcal{M}^k} := \int_{\mathcal{M}^k} \mathcal{I}_{h_k}(UV)\,\mathrm{d}x$ for two discrete functions

$U, W \in V_k$. Here $\mathcal{I}_{h_k} : C^0(\mathcal{M}^k) \to V_k$ denotes the nodal projection operator (cf. [24]).

3 The concept of cascadic iterations

Multigrid methods [13] are known to be efficient solvers for systems of linear equations $B\,\bar{X} = R$ characterized by an intrinsic hierarchical structure for example resulting from a finite element discretization. Throughout this paper we alway indicate vectors, especially the vector collecting all nodal coordinates by a bar on top. Typically an underlying grid hierarchy induces a hierarchical structure on the corresponding discrete function spaces. Let us consider a sequence of finite element spaces $\{V_j\}$, with $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots \subset V_{j_{\max}}$ corresponding to a hierarchy of nested grids $\mathcal{M}_0, \cdots, \mathcal{M}_{j_{\max}}$. In the solution process one typically iterates over j, solving an appropriately restricted system $B_j \bar{X}_j = R_j$ on level j and then refines the grid. Here B_j and R_j are restrictions of $B = B_{j_{\max}}$ and $R = R_{j_{\max}}$ respectively.

Thus, the previously calculated solution $\bar{X}_{j-1}$ can be considered as good initial data for an iterative solver on level j. Somewhat surprising this naive strategy turns out to be theoretically well founded and robust [1] as long as an error control in the energy norm is considered. Indeed, the number of required iterations or smoothing steps n_j on level j can be fixed a priori. On coarse levels more iterations are required than on finer levels, where very few iterations are sufficient to ensure a required approximation quality on the finest grid level. Even better, Bornemann and Deuflhard [1] proved optimality for this *cascadic* scheme in the sense that the overall cost of the solution process is $O(m)$ where m is the number of unknowns on the finest grid level. Hence under these circumstances one can save more complex nested iterations in a general multigrid solver.

4 The function graph case

At first, let us consider the case of surfaces $\mathcal{M}$, which are graphs in the x_2 direction over a polygonal domain Ω in the x_0, x_1 plane. Here we will outline a very simple but effective subdivision scheme as a first model case. We denote by u the corresponding graph function. Given an initial graph u^0 - which will later be our coarse polygonal mesh - we can ask for a solution u^* of the elliptic partial differential equation

$$(1 - a\,\Delta)u^* = u^0$$

with natural boundary conditions on $\partial\Omega$. This problem corresponds to a single approximate timestep for the heat equation with timestep size $\tau = a$ or the approximation of a Gaussian filter of width $\sqrt{2\tau}$, respectively. For Lipschitz continuous initial data u_0 it is known [12] that the solution is unique and $C^{2,\alpha}$ regular for any $\alpha \in [0, 1)$. Now we approximate u^* by a sequence of linear finite element solutions $\{U^k\}_{k=0,\ldots}$ on successively refined regular grids Ω_k for the parameter domain Ω. Due to the convergence properties of linear finite elements [4] we know that for $k \to \infty$ and corresponding vanishing grid size h_k on Ω_k the sequence U^k converges to u^* in the energy as well as in the L^∞ norm. In each step of this scheme we have to solve a linear system of equations of the type

$$(M + a\,L)\bar{U}^k = M\overline{I_{h_k} u^0}$$

where M, L are the mass and stiffness matrix respectively, and $I_{h_k} u^0$ is the linear interpolation of u^0 on Ω_k. The solution U^k minimizes the energy

$$E(U) := \int (U - I_{h_k} u^0)^2 + a\,|\nabla U|^2$$

over all admissible functions U in the kth finite element space. Hence, if u^0 is the function corresponding to a polygonal surface graph over Ω, we can interprete this approach as a variational subdivision approach. As usual the linear systems of equations are solved applying iterative solvers. If we consider a cascadic multigrid method, we can reduce the required number of iterations enormously without effecting the convergence properties (cf. Section 3). Given a final level of resolution $k_{\max}$ we a priori fix the required number of smoothing steps within the cascadic algorithm on all levels following the recipe given by Bornemann and Deuflhard [1]. Finally we end up with an subdivision scheme of optimal complexity, i. e. a cost proportional to the number of vertices on the finest grid level. Still we now that the resulting sequence of solutions $\{U_{k_{\max}}\}$ convergences to u^*. Thus, our simple iteration leads to $C^{2,\alpha}$ regularity in the limit, independent of the grid type and the applied refinement scheme.

Unfortunatly, most surfaces in computational geometry are not graphs, except form a local perspective (cf. Mandal et al. [21] for a finite element approach to subdivision which deals with patches of the subdivision surfaces as graphs over the initial, coarse surface polygons). Furthermore, the selection of a parameter domain introduces the metric on that domain as the valid metric. Thus subdivision results significantly depend on this metric and thereby on the proper choice of the parameter domain. It would be much more natural to apply the same concept based on a single diffusion time step but taking into account the metric on the limit surface or suitable approximations, respectively. This will be what we are going to investigate in the following section.

5 Subdivision via cascadic filtering

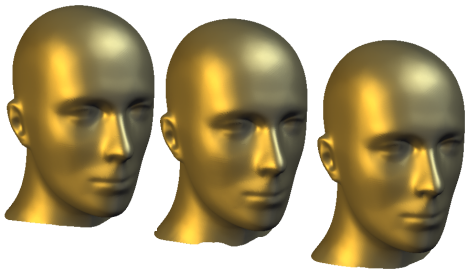

Figure 3: *Comparison of different subdivision results. From left to right the images correspond to exact solution of the linear systems in each subdivision step, cascadic cg-iterations, and cascadic Jacobi-iterations. Visually there is more or less no difference except closed to the boundary in the neck region. This clearly reflects the convergence properties of the cascadic scheme.*

As initial surface we consider any discrete, typically triangulated surface $\mathcal{M}_0$ and denote its parameterization by x_0. To underline the geometric origin and to straighten the presentation we derive our final method in several steps:

Step 1. As it has already been mentioned mean curvature motion is the geometric counterpart of Gaussian filtering and solving the heat equation respectively. Then under reasonable assumptions on the initial surface and for short times a unique solution exists and it is C^∞-regular. Thus we apply the mean curvature motion semigroup (MCM) as a geometric filter of width σ to $\mathcal{M}_0$ and obtain for a time step $\tau = \frac{\sigma^2}{2}$

$$\mathcal{M} := MCM(\tau)\mathcal{M}_0$$

where the corresponding parametrization is defined by $x := x(\tau)$. A suitable choice for the filter width on a polygonal surface with grid size h_0 appears to be $\sigma \approx h_0$. At first, we assume the triangulation of our initial surfaces to be uniform. In Section 7 we will generalize our method to nonuniform filter width. Alternatively, we can incorporate the filter width in the diffusion coefficient and confine to a time step $\tau = 1$. That is we consider for a spatially fixed filter width σ

$$\partial_t x(t) - a\, \Delta_{\mathcal{M}(t)} x(t) = 0$$

with $a := \frac{\sigma^2}{2}$ and evaluate the evolution at time $t = 1$. As one advantage of this rewriting we now can consider a spatially varying filter width σ (cf. Fig. 1).

Step 2. We can replace the continuous nonlinear semigroup by a time discrete evolution and focus on the first step. In the resulting implicit scheme we have to select a metric (cf. Section 2). We approximate a fully implicit scheme, where the metric is evaluated on the unknown surface, by a sequence of semi implicit schemes. Hence, in each iteration we consider the metric from the previous step and calculate parameterizations x^k of surfaces $\mathcal{M}^k$ for $k > 0$ solving the linear problem

$$(x^k - x_0) - a\, \Delta_{\mathcal{M}^{k-1}} x^k = 0\,.$$

Let us emphasize that the index k does not indicate a curvature motion timestep but only successively improved approximations of the fully implicit scheme $(x^* - x_0) - a\Delta_{\mathcal{M}^*} x^* = 0$. For $k \to \infty$ we expect this iteration to be a fix point iteration with a convergence of the parametrizations x^k at least pointwise to a parametrization x^* of a unique fix point surface $\mathcal{M}^*$. Our numerical results give a strong indication for this convergence. In Section 4 we have alread mentioned that such a regularity result holds in the simplified case. Given Lipschitz continuous initial data u_0 the solution u^* of $(\mathrm{Id} - a\Delta)u^* = u_0$ is $C^{2,\alpha}$-continuous for any $\alpha \in [0, 1)$. Our conjecture is that an analogous regularity result holds for the implicit MCM timestep problem and the Laplace Beltrami operator. Thus,

we expect our limit surface to be $C^{2,\alpha}$-continuous for a triangular initial surface (cf. Fig. 5). Experimentally we have verified at least bounded discrete second derivatives (cf. Fig. 2). Rigorous proofs for both the convergence and the smoothness of the limit surface are still missing. We abbreviate the notation and write

$$\mathcal{M}^k = \mathcal{S}(\mathcal{M}^{k-1})\mathcal{M}_0$$

where the argument of the time step operator $\mathcal{S}(\cdot)$ indicates the corresponding metric. The limit surfaces $\mathcal{M}^*$ turns out to be a fix point of the mapping $\mathcal{S}(\cdot)\mathcal{M}_0$, i. e. $\mathcal{M}^* = \mathcal{S}(\mathcal{M}^*)\mathcal{M}_0$. So far we have derived a geometrically natural smoothing method, which results in solving a sequence of spatially continuous but linear problems. Still the surfaces are continuous in space.

Step 3. Now we discretize in space, considering triangular surfaces $\mathcal{M}_0$ of different grid size h and corresponding linear finite element spaces V^h and ask for parametrizations $X \in (V^h)^3$ of surfaces $\mathcal{M}_h$ (cf. Section 2). Thus, we consider

$$\mathcal{M}_h^k = \mathcal{S}_h(\mathcal{M}_h^{k-1})\mathcal{M}_0$$

where $\mathcal{S}_h(\cdot)$ denotes the corresponding spatially discrete time step operator. I. e. for the parameterization X^k of $\mathcal{M}_h^k$ we get

$$(X^k - \mathcal{I}_h x_0) - a\,\Delta_{\mathcal{M}_h^{k-1}} X^k = 0\,.$$

Here $\mathcal{I}_h$ again indicates the nodal projection onto V^h. We expect $\mathcal{M}_h^k$ to converge to $\mathcal{M}^*$ for $h \to 0$ (cf. the convergence result by Deckelnick and Dziuk [5]).

Step 4. Furthermore, we consider sequences of nested grids generated by any recursive and regular refinement rule and apply a diagonalization argument with respect to the grid level and the above iteration. After each iteration we refine the grid once and obtain the following subdivision scheme:

$$\mathcal{M}_{h_k}^k = \mathcal{S}_{h_k}(\mathcal{M}_{h_{k-1}}^{k-1})\mathcal{M}_0\,.$$

This corresponds to the operator equation

$$(X_{h_k}^k - \mathcal{I}_{h_k} x_0) - a\,\Delta_{\mathcal{M}_{h_{k-1}}^{k-1}} X_{h_k}^k = 0$$

for the parameterizations $X_{h_k}^k$ of $\mathcal{M}_{h_k}^k$. For the sake of simplicity we write X_k^k, S_k, and $\mathcal{M}_k^k$ instead of $x_{h_k}^k$, S_{h_k}, and $\mathcal{M}_{h_k}^k$ respectively. We suppose geometric decay of the sequence of grid sizes h_k, i. e. $\beta^- h_k \le h_{k+1} \le \beta^+ h_k$. In case of a standard butterfly like refinement rule without smoothing, which results in the splitting of each triangle into four up to scaling identical children, we obtain $\beta^+ = \beta^- = \frac{1}{2}$.

Step 5. In each step of the above subdivision scheme the solution of a system of linear equations (cf. Section 2) is required. As usual in finite element calculus this system is sparse and iterative solvers can be applied. In each subdivision step we modify the metric and refine the underlying grid. This obviously is a cascadic strategy (cf. Section 3) and we know that for increasing iteration indices a decreasing number of smoothing steps n_k in the linear solver has to be performed if we consider appropriately prolongated solutions from the previous level as initial data. Let us indicate the number of smoothing steps n_k on grid level k by an upper index. Bornemann and Deuflhard proved that the required number of iteration decays geometrically, i. e. in case of conjugate gradient iterations (CG) $n_k = n_{k_{\max}} 2^{\frac{3}{2}(k_{\max}-k)}$ and for the damped Jacobi iteration $n_k = n_{k_{\max}} 2^{2(k_{\max}-k)}$. Given an error tolerance for the algebraic error we can preset the required number of smoothing steps $n_{k_{\max}}$ on the finest grid level $k_{\max}$. In our application we always set $n_{k_{\max}} = 1$. Thus given a final level of refinement $k_{\max}$ up to which we want to iterate the overall cost of the resulting algorithm has optimal complexity for CG in case of a quatering type refinement and nearly optimal complexity for the damped Jacobi iterations. Optimal here means the cost is proportional to the number of fine grid nodes. If the goal is only to ensure appropriate smoothing results, based on our experience one can further reduce the number of iterations n_k especially on fine grid levels k and confine to a fixed number of iterations n. Thus we obtain a suitable approximation of our original model by the iteration

$$\mathcal{M}_k^k = \mathcal{S}_k^{n_k}(\mathcal{M}_{k-1}^{k-1})\mathcal{M}_0$$

(cf. 3 and Fig. 4). As initial data and for the evaluation of the metric we consider the coordinate vector of $\mathcal{M}_{k-1}^{k-1}$ prolongated to level k by trivial interpolation. The number of considered smoothing steps will correspond to the stencil width of our scheme as it operates on nodal coordinates of a triangular grid. Different from the case of linear problem, where convergence of the cascadic multigrid has

been proven, here the dependence on the metric introduces a nonlinearity. Nevertheless on sufficiently fine grids we expect a neglectable impact of this effect on the performance of a cascadic iteration.

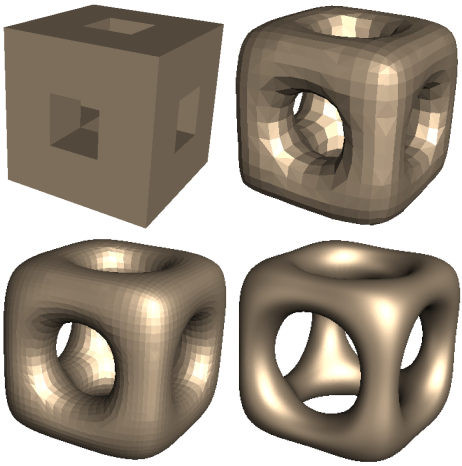

Figure 4: *A sequence of flat shaded subdivision surfaces subdividing each triangle into four children in each refinement step.*

6 Algorithmical aspects

Let us give some details on the non exact solution of the linear system in each step of our subdivision scheme at least in case of damped Jacobi iterations. In fact, we confine to a few smoothing steps. We define $\bar{X}^s := S^{n_k}(\bar{X})$ where S is a suitable smoothing operator and the exponent n_k indicates the number of considered smoothing steps, i. e. $S^{n+1}(\bar{X}) := S \circ S^n(\bar{X})$. The damped Jacobi iteration S_J is defined by

$$S_J \bar{X} := \bar{X} - \theta\, D^{-1}(M^{k-1}(\bar{X} - \bar{X}_0^k) + a\, L^{k-1}\bar{X})$$

where D is the matrix representing the diagonal part of $(M^{k-1} + a\, L^{k-1})$ and θ is the damping factor. Here M^{k-1} and L^{k-1} denote the mass and stiffness matrix on the grid $\mathcal{M}^{k-1}$. We always have set $\theta = \frac{3}{4}$.

So far we have not specified the refinement method to be applied in every iteration of the presented subdivision schemes. As long as a regular refinement rule is considered which guarantees suitable upper and lower bounds for the angles of the

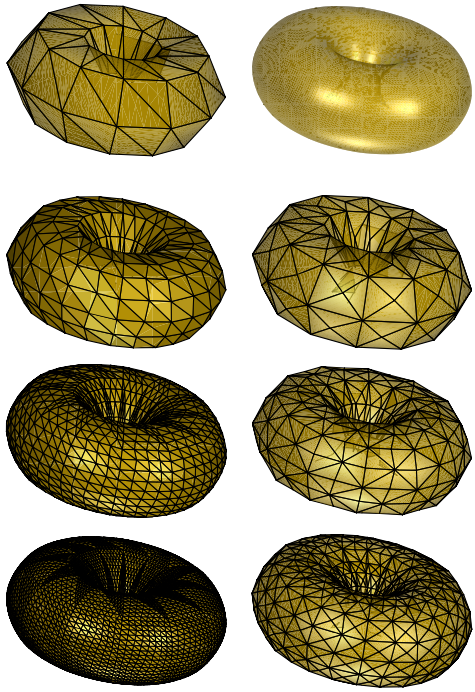

Figure 5: *A coarse torus (top left) is processed by our subdivision method leading to a smooth limit surface (top right). We have used a quatering scheme (left column) and a bisection refinement strategy (right column).*

generated triangles our approach in principle is independent of the concrete scheme. Figure 5 depicts two frequently used refinement schemes, the quatering scheme and the recursive bisection scheme.

Finally, we can write our subdivision scheme in pseudo code notation as follows:

Define $\mathcal{M} = \mathcal{M}^0$ as the initial mesh; $k = 0$;
do {
  $k = k + 1$;
  $\mathcal{M} =$ Refinement($\mathcal{M}$);
  Compute $M =$MassMatrix($\mathcal{M}$)
  and $L =$StiffnessMatrix($\mathcal{M}$);
  $\bar{X} =$ nodal vector of $\mathcal{M}$;
  $\bar{X}^0 = \mathcal{I}_{\mathcal{M}}\mathcal{M}^0$;
  for($n = 0; n < n_k; n = n + 1$)
    $\bar{X} = S(\bar{X}^0, M, L)\,\bar{X}$;
  $\mathcal{M} =$ surface $\mathcal{M}$ with updated nodes $\bar{X}$;
} while ($k \leq k_{\max}$);

Here the operator *Refinement(·)* denotes any

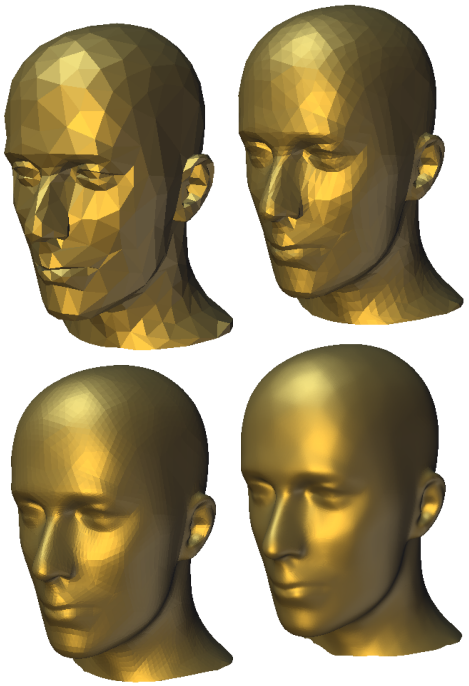

Figure 6: *A sequence of flatshaded subdivision surfaces is generated using the local filter width algorithm. The starting surface comes along with a very irregular triangular grid (different valences of the nodes, thin triangles, nonhomogeneous grid size), which we keep without changes. The proposed method is able to effectively deal with such surfaces.*

regular refinement scheme, *MassMatrix(·)* and *StiffnessMatrix(·)* evaluate the compute the corresponding matrices on a given surface and $\mathcal{I}_{\mathcal{M}}$ represents trivial interpolation of the nodal positions of $\mathcal{M}^0$ on $\mathcal{M}$. By trivial interpolation we mean the build in recursive interpolation due to the applied refinement rules without any smoothing operations.

7 A local filter width expansion

In many applications the initially coarse mesh will be characterized by considerable variations in the local grid size. We can take care of this by adaptation of the filter width in our implicit time step scheme to the local grid size (cf. Fig. 6). Here the idea is to consider a smoothed local grid size of the initial grid as filter width and modify the diffusion coefficient with respect to this filter width. Figure 7 shows a comparison between the expanded model incorporating a smooth local filter width function and the fixed filter width problem studied so far.

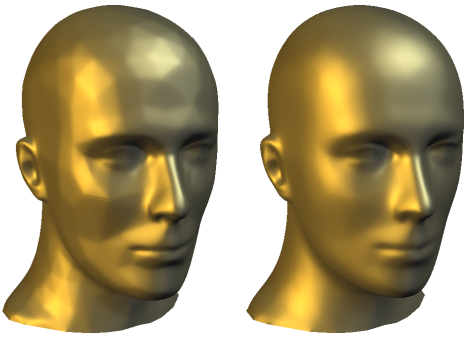

Figure 7: *A comparison of limit surfaces based on a small and spatially fixed filter width (left) and the local filter width expansion (right) for a coarse initial grid with considerable variation in the grid size.*

In our iterative scheme we apply the same smoothing operators in every step to this grid size function to ensure still $C^{2,\alpha}$-smoothness of the limit surface. Hence, we consider the following continuous problem:

$$\begin{aligned}(x^k - x_0) - \mathrm{div}_{\mathcal{M}^{k-1}}\left(a^{k-1}\nabla_{\mathcal{M}^{k-1}} x^k\right) &= 0\\ (h^k - h_0) - \mathrm{div}_{\mathcal{M}^{k-1}}\left(a^{k-1}\nabla_{\mathcal{M}^{k-1}} h^k\right) &= 0\,.\end{aligned}$$

where $a^k := C^2\frac{(h^k)^2}{2}$ and h_0 is the initial grid size function. We expect the sequence $\{x^k, h^k\}_k$ to converge to a solution x^*, h^* of the fully implicit problem:

$$\begin{aligned}(x^* - x_0) - \mathrm{div}_{\mathcal{M}^*}\left(a^*\nabla_{\mathcal{M}^*} x^*\right) &= 0\\ (h^* - h_0) - \mathrm{div}_{\mathcal{M}^*}\left(a^*\nabla_{\mathcal{M}^*} h^*\right) &= 0\,.\end{aligned}$$

where $a^* := C^2\frac{(h^*)^2}{2}$ and $\mathcal{M}^*$ is the resulting mesh with parameterization x^*. Furthermore, we expect the same regularity result as in the case with fixed filter width, i. e. $C^{2,\alpha}$-smoothness of the limit surface $\mathcal{M}^*$ and of the smoothed filter width function h^*.

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