A VARIATIONAL APPROACH TO NON-RIGID MORPHOLOGICAL IMAGE REGISTRATION*

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Abstract. A variational method to non rigid registration of multi-modal image data is presented. A suitable deformation will be determined via the minimization of a morphological, i.e., contrast invariant, matching functional along with an appropriate regularization energy. The aim is to correlate the morphologies of a template and a reference image under the deformation. Mathematically, the morphology of images can be described by the entity of level sets of the image and hence by its Gauss map. A class of morphological matching functionals is presented which measure the defect of the template Gauss map in the deformed state with respect to the deformed Gauss map of the reference image. The problem is regularized by considering a nonlinear elastic regularization energy. Existence of homeomorphic, minimizing deformation is proved under assumptions on the class of admissible deformations. With respect to actual medical applications suitable generalizations of the matching energies and the boundary conditions are presented. Concerning the robust implementation of the approach the problem is embedded in a multi-scale context. A discretization based on multi-linear finite elements is discussed and first numerical results are presented.

Key words. Image Processing, Image Registration, Mathematical Morphology, Nonlinear Elasticity

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1. Introduction. Nowadays already classical image acquisition machinery such as computer tomography and magnetic resonance tomography and a variety of novel sources for images, such as functional MRI, 3D ultrasound or densiometric computer tomography (DXA) deliver various 3D images of the same human body. Due to different body positioning, temporal difference of the image generation and differences in the measurement process the images frequently can not simply be overlayed. Indeed corresponding structures are situated at usually nonlinearly transformed positions. In case of intra-individual registration, the variability of the anatomy can not be described by a rigid transformation, since many structures like, e. g., the brain cortex may evolve very differently in the growing process. Frequently, if the image modality differs there is also no correlation of image intensities at corresponding positions. What still remains, at least partially, is the local image structure or "morphology" of corresponding objects. Thus, the matching of 2D and especially 3D images – also known as image registration – with respect to their morphology is one of the fundamental tasks in image processing.

One aims to correlate two images – a reference image R and a template image T – via an energy relaxation over a set of in general non-rigid spatial deformations. Let us denote the reference image by $R : \Omega \to \mathbb{R}$ and the template image by $T : \Omega \to \mathbb{R}$. Here, both images are supposed to be defined on a bounded domain $\Omega \in \mathbb{R}^d$ for $d = 1, 2$ or 3 with Lipschitz boundary and satisfying the *cone condition* (cf. e. g. [4]). We ask for a deformation $\phi : \Omega \to \Omega$ such that $T \circ \phi$ is optimally correlated to R. There is a large and diverse body of literature on registration. In particular Grenander, Miller and co-workers contributed different physically motivated and mathematical profound approaches [12, 11, 24, 21, 32]. For an overview in particular on the mathematical

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modeling see references therein. For unimodal images one defines similarity measures for instance by the simple choice $\|T \circ \phi - R\|^2_{L^2}$ [15, 28, 33, 41]. In case T and R are images of different modality, we are left to define what is meant by the correlation of local structures in the image.

Viola, Wells et al. [45, 47] and Collignon [16] presented a information theoretic approach for registration of multi-modal images. It is based on the idea of maximizing the so called mutual information of the deformed template image and the reference image. The mutual information consists of the entropies of both images and the negative joint entropy. It can be interpreted as a measure of variability and uncertainty. Thus, the joint entropy of the images is low, where one image can stochastically be well described by the other and vice versa. Since the entropies of random variables are integrals containing the corresponding density functions, here the intensities, the corresponding local structure analysis is rather implicitly encoded in the global functionals. Viola and Wells performed the maximization process by using a stochastic descent method, in which the gradients are computed via a Parzen windowing function, while Collignon used Powell's method for the optimization. The method is currently restricted to an expression in global parametric form such as rigid transformations or a lower dimensional space of smooth deformations. A different approach of image registration via the matching of objects in images is due to Monasse [35]. He classifies objects by moments and a registration is achieved by aligning these moments under scaling and rigid body motion.

Here, we introduce a different approach based on the definition of a matching energy, which effectively measures the local morphological "defect" of the deformed template and the reference image. The congruence of the shapes instead of the equality of the intensities is the main object of the registration approach presented here. At first, let us define the morphology $M[I]$ of an image I as the set of level sets of I:

$$M[I] := \{\mathcal{M}^I_c \,|\, c \in \mathbb{R}\}, \tag{1.1}$$

where $\mathcal{M}^I_c := \{x \in \Omega \,|\, I(x) = c\}$ is a single level set for the grey value c (For a general overview on image morphology we refer to [40]). I.e. $M[\gamma \circ I] = M[I]$ for any reparametrization $\gamma : \mathbb{R} \to \mathbb{R}$ of the grey values. Obviously, $M[I]$ is uniquely identified by the set of tangent spaces $\mathcal{T}_x\mathcal{M}^I_{I(x)}$ of all level sets $\mathcal{M}^I_c$ or up to the orientation by the normal field N_I on $\mathcal{M}^I_c$. Hence, again up to the orientation the morphology $M[I]$ can be identified with the normal map (Gauss map)

$$N_I : \Omega \to \mathbb{R}^d\,;\; x \mapsto \frac{\nabla I}{\|\nabla I\|}\,. \tag{1.2}$$

Two images I_1 and I_2 are called morphologically equivalent if $M[I_1] = M[I_2]$. Let us emphasize that we deal here with classical level sets, which might not be everywhere defined. The problem related to vanishing image gradients and thus undefined normals will be addressed in Section 4, where we allow for such singularities as long as the measure of the corresponding set in appropriate terms is not too large. A weaker definition of level sets has been introduced by Caselles, Coll and Morel [10]. They consider the so called upper topographic map $\{\{x \,|\, \phi(x) \geq \lambda\} \,|\, \lambda \in \mathbb{R}\}$ to characterize the morphology of an image ϕ. This map uniquely describes the morphology and they prove stability with respect to discretization and quantization.

Morphological methods in image processing are characterized by an invariance with respect to the morphology. Explicitly speaking, a method is called morphological, if applied to morphologically identical images the resulting images are still morphologically identical [1, 39, 43]. Hence, such methods only effect the morphology of the image, which coincides with the geometry of the level sets. Now, aiming for a morphological registration method, we will ask for a deformation $\phi : \Omega \to \Omega$ such that

$$M[T \circ \phi] = M[R]\,.$$

Thus, we try to align the normal fields (cf. Desolneux et al. [19] where tangent spaces are identified in rigorous statistical terms). We set up a matching functional which locally measures the twist of the tangent spaces of the template image at the deformed position and the deformed reference image or the defect of the corresponding normal fields.

As known from other approaches the corresponding minimization, if settled over an infinite dimensional space of deformations and not ab initio restricted to a small finite dimensional function space, turns out to be ill posed [8, 44]. Hence, we have to ask for a suitable regularization. Various regularization approaches have been considered in the literature [11, 12, 18, 26]. On one hand, a regularization of the energy is taken into account, typically adding a convex energy functional based on gradients to the actual matching energy. The regularization energy is regarded as a penalty for "elastic stresses" resulting from the deformation of the images. This competitive approach is related to the well known classical Tichonov regularization of the originally ill-posed problem. On the other hand, viscous flow techniques are taken into account. They compute smooth paths from some initial deformation towards the set of minimizers of the matching energy [15, 27].

The paper is organized as follows. In Section 2 the morphological matching energies are discussed and in Section 3 the regularization via nonlinear elasticity functionals will be introduced. Then, in Section 4 we prove existence of homeomorphic, minimizing deformations. With respect to the actual application to medical data the model is further generalized in Section 5 and 6, where an additional feature based matching functional is introduced and generalized boundary conditions are discussed. Finally in Section 7 we describe the finite element discretization and the minimization algorithm.

In the present paper, we will prove the existence of a minimizing deformation for a variational approach, which is formulated for 3D images. It is left to the reader to transfer the assumptions and the existence results to the simpler two dimensional case. Here and in what follows we make use of the summation convention. That is, we implicitly sum over every index which appears twice in an expression.

Let us emphasize that the focus of the paper is on the presentation of a new concept in morphological image registration. Details on the implementation will be discussed in a forthcoming publication. Hence, the computational results are currently still restricted to 2D.

2. A morphological registration energy. In this section we will construct a suitable matching energy, which measures the defect of the morphology of the reference image R and the deformed template image T. Thus, with respect to the above identification of morphologies and normal fields we ask for a deformation ϕ such that

$$N_T \circ \phi \,||\, N_R^\phi\,, \tag{2.1}$$

where N_R^ϕ is the transformed normal of the reference image R on $\mathcal{T}_{\phi(x)}\phi(\mathcal{M}^R_{R(x)})$ at position $\phi(x)$. From the transformation rule for the exterior vector product $D\phi\, u \wedge$

$D\phi\, v = \mathrm{Cof}\, D\phi(u \wedge v)$ for all $v, w \in \mathcal{T}_x\mathcal{M}^R_{R(x)}$ one derives

$$N_R^\phi = \frac{\mathrm{Cof}\, D\phi\, N_R}{\|\mathrm{Cof}\, D\phi\, N_R\|}$$

where $\mathrm{Cof}\, A = \det A \cdot A^{-T}$ for invertible $A \in \mathbb{R}^{d,d}$. In a variational setting, optimality can be expressed in terms of energy minimization. Hence, we consider a matching energy

$$E_m[\phi] := \int_\Omega g(N_T \circ \phi, N_R, \mathrm{Cof}\, D\phi)\, \mathrm{d}\mu$$

for some function $g : S^{d-1} \times S^{d-1} \times \mathbb{R}^{d,d} \to \mathbb{R}^+;\ (u, v, A) \mapsto g(u, v, A)$. Here S^{d-1} denotes the unit sphere in $\mathbb{R}^d$ and μ the Lebesgue measure. This matching energy depends on the deformation of normal fields and we are going to relax the energy via a minimizing deformation for fixed image morphologies and hence fixed normal fields. Recently, in image restoration or inpainting energies have been introduced which depend on the normal fields of images represented by BV functions [5, 6, 7]. There, the energy is minimized over an appropriate set of BV functions on a destroyed image region.

As boundary condition we require $\phi = \mathbb{1}$ on $\partial\Omega$, where $\mathbb{1}$ indicates the identity mapping on Ω and simultaneously the identity matrix. So far, we have assumed that the normal fields N_T and N_R are well defined on the whole domain Ω. To be not too restrictive with respect to the space of images we have to take into account the problem of degenerate Gauss maps. Hence, let us define the set $\mathcal{D}_I := \{x \in \Omega \,|\, \nabla I = 0\}$ for $I = T$ or R, where no image normal can be defined. At first, we resolve this problem of undefined normals at least formally by introducing a 0-homogeneous extension $g_0 : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d,d} \to \mathbb{R}^+$ of g in the first and second argument:

$$g_0(v, w, A) = \begin{cases} 0 & ; \quad v = 0 \text{ or } w = 0 \\ g(\frac{v}{\|v\|}, \frac{w}{\|w\|}, A) & ; \quad \text{else} \end{cases} . \tag{2.2}$$

Based on g_0 we can redefine the matching energy E_m and obtain

$$E_m[\phi] := \int_\Omega g_0(\nabla T \circ \phi, \nabla R, \mathrm{Cof}\, D\phi)\, \mathrm{d}\mu. \tag{2.3}$$

In the later analysis we have to take special care of the singularity of g_0 for vanishing first or second argument. Indeed, we will assume that the measure of D_T and D_R is in a suitable sense sufficiently small. Furthermore, in the existence theory we will explicitly control the impact of these sets on the energy. As a first choice for the energy density g let us consider

$$g(v, w, A) := \left(v - \frac{Aw}{\|Aw\|}\right)^2 \tag{2.4}$$

for $v, w \in S^{d-1}$, which corresponds to the energy

$$\int_\Omega \|N_T \circ \phi - N_R^\phi\|^2 .$$

We observe that the energy E_m vanishes if $T \circ \phi = \gamma \circ R$ for a monotone grey value transformation $\gamma : \mathbb{R} \to \mathbb{R}$. If we want E_m to vanish also for non-monotone transformations γ we are lead to the symmetry assumption:

$$g(v, w, A) = g(-v, w, A) = g(v, -w, A)\,. \tag{2.5}$$

EXAMPLE 2.1. *A useful class of matching functionals E_m is obtained choosing functions g which depend on the scalar product $v \cdot u$ or alternatively on $(\mathbb{1} - v \otimes v)u$ (where $\mathbb{1} - v \otimes v = (\delta_{ij} - v_i v_j)_{ij}$ denotes the projection of u onto the plane normal to v) for $u = \frac{Aw}{\|Aw\|}$ and $v, w \in S^{d-1}$, i. e.,*

$$g(v, w, A) = \hat{g}\left((\mathbb{1} - v \otimes v)\frac{Aw}{\|Aw\|}\right)\,. \tag{2.6}$$

Let us remark that $\hat{g}((\mathbb{1} - v \otimes v)u)$ is convex in u, if $\hat{g}$ is convex. With respect to arbitrary grey value transformations mapping morphologically identical images onto each other, we might consider $\hat{g}(s) = \|s\|^\gamma$ for some $\gamma \geq 1$.

3. Hyperelastic, polyconvex regularization. Suppose a minimizing deformation ϕ of E_m is given. Then, obviously for any deformation ψ which exchanges the level sets $\mathcal{M}_c^R$ of the image R, the concatenation $\psi \circ \phi$ still is a minimizer. But ψ can be arbitrarily irregular. Hence, minimizing solely the matching energy is an ill-posed problem. Thus, we consider a regularized energy

$$E[\phi] = E_m[\phi] + E_{reg}[\phi]\,. \tag{3.1}$$

Due to the fact that the matching energy already includes first order derivatives of the deformation ϕ, one might consider a regularization energy which involves higher order derivatives of ϕ [34]. In particular, the existence of minimizers would basically rely on usual compactness arguments. But on the background of elasticity theory, we aim to model the image domain as an elastic body responding to forces induced by the matching energy. Hence, we have to confine with energies as they appear in the usual mechanical approach to elastic bodies. It will turn out in Section 4 that we have nice consistency of the type of nonlinearity in the matching energy with respect to the Jacobian of the deformation and the well-known structure of nonlinear elastic functionals. We have to emphasize, that we do not attempt to model the actual material of the objects represented by the image.

At first, let us briefly recall some background from elasticity. For details we refer to the comprehensive introductions in the books by Ciarlet [13] and Marsden & Hughes [31]. We interprete Ω as an isotropic elastic body and suppose that the regularization energy plays the role of an elastic energy while the matching energy can be regarded as an external potential contributing to the energy. Furthermore we suppose $\phi = \mathbb{1}$ to represent the stress free deformation. Let us consider the deformation of length, area and volume under a deformation ϕ. It is well-known that the norm of the Jacobian of the deformation $\|D\phi\|_2$ controls the isotropically averaged change of length under the deformation, where $\|A\|_2 := \mathrm{tr}\,(A^T A) = \sum_{i,j} A_{ij} A_{ij}$ for $A \in \mathbb{R}^{d,d}$. Secondly, the local volume transformation under a deformation ϕ is represented by $\det D\phi$. If $\det D\phi$ changes sign self-penetration may be observed. Furthermore for $d = 3$, the norm of the matrix of the cofactors of the Jacobian of the deformation $||\mathrm{Cof}\, D\phi||_2 = \mathrm{tr}\,(\mathrm{Cof}\, D\phi^T \mathrm{Cof}\, D\phi)$ is the proper measure for the averaged change of area.

EXAMPLE 3.1. *Based on these considerations we can define a simple physically reasonable isotropic elastic energy for d = 3, which separately cares about length, area and volume deformation and especially penalizes volume shrinkage:*

$$E_{reg}[\phi] := \int_\Omega a||D\phi||_2^p + b||\mathrm{Cof}\, D\phi||_2^q + \Gamma(\det D\phi)\, \mathrm{d}\mu \tag{3.2}$$

with $\Gamma(D) \to \infty$ *for* $D \to 0, \infty$*, e. g.,* $\Gamma(D) = \gamma D^2 - \delta \ln D$*. In nonlinear elasticity such material laws have been proposed by Ogden [38] and for p = q = 2 we obtain the Mooney-Rivlin model [13].* More general than in the above example, we will consider a so called polyconvex energy functional [17]

$$E_{reg}[\phi] := \int_\Omega W(D\phi, \mathrm{Cof}\, D\phi, \det D\phi)\, \mathrm{d}\mu \tag{3.3}$$

where $W : \mathbb{R}^{d,d} \times \mathbb{R}^{d,d} \times \mathbb{R} \to \mathbb{R}$ is supposed to be convex. Besides suitable growth conditions to be stated later, we furthermore assume that W and thereby $E_{reg}[\phi]$ again penalizes volume shrinkage, i.e., $W(A, C, D) \stackrel{D\to 0}{\longrightarrow} \infty$. This will enable us to successfully control singularity sets. Such energies have already been introduced to the related optical flow problem by Hinterberger et al. [29]. But their focus was neither on morphological registration nor on the control of singularity sets.

4. An existence result. In this section we will discuss under which conditions there exists a minimizing deformation of the total energy $E[\cdot]$. Let us emphasize that the problem stated here significantly differs from most other regularized image registration problems, e. g., all intensity based approaches, where the matching energy is defined solely in terms of the deformation ϕ and the regularization energy is of higher order and considers the Jacobian $D\phi$ of the deformation. In our case already the matching energy incorporates the cofactor of the Jacobian. Thus, with respect to the direct method in the calculus of variations, we can not use standard compactness arguments due to Rellich's Embedding Theorem to deal with the matching energy on a minimizing sequence [17]. Instead, we will need suitable convexity assumptions on the function g.

REMARK 4.1 (Lack of lower semicontinuity for certain functionals E_m). *Recalling Example 2.1 we might wish to choose a matching energy with an integrand* $g_0(v, w, A) := \hat{g}((1\!\!1 - \frac{v}{\|v\|} \otimes \frac{v}{\|v\|})\frac{Aw}{\|Aw\|})$ *for* $\hat{g} \in C^0(\mathbb{R}^d, \mathbb{R}_0^+)$*. It is well known that the essential condition to ensure weak sequential lower semicontinuity of functionals depending on the Jacobian of a deformation is quasiconvexity [36, 37]. With our special choice of the class of energies (2.3) this requires the convexity of g in the argument A (cf. [17] Section 5.1).*

But, we easily verify that a function

$$f : \mathbb{R}^{d,d} \to \mathbb{R}; A \mapsto f(A)$$

which is 0-homogeneous on $\mathbb{R}^{d,d}$ *and convex has to be constant and thus an existence result for our image matching problem via the direct calculus of variations can only be expected for trivial matching energies, i. e., for* $\hat{g} \equiv const$*. Indeed, suppose* $A, B \in \mathbb{R}^{d,d}$ *with* $f(A) - f(B) = \delta > 0$ *and define* $A_{\alpha,r} := r\,A + \alpha\,(A - B)$ *for* $r > 0$ *and* $\alpha > 0$*, then for* $s = \frac{\alpha}{r}$ *we obtain*

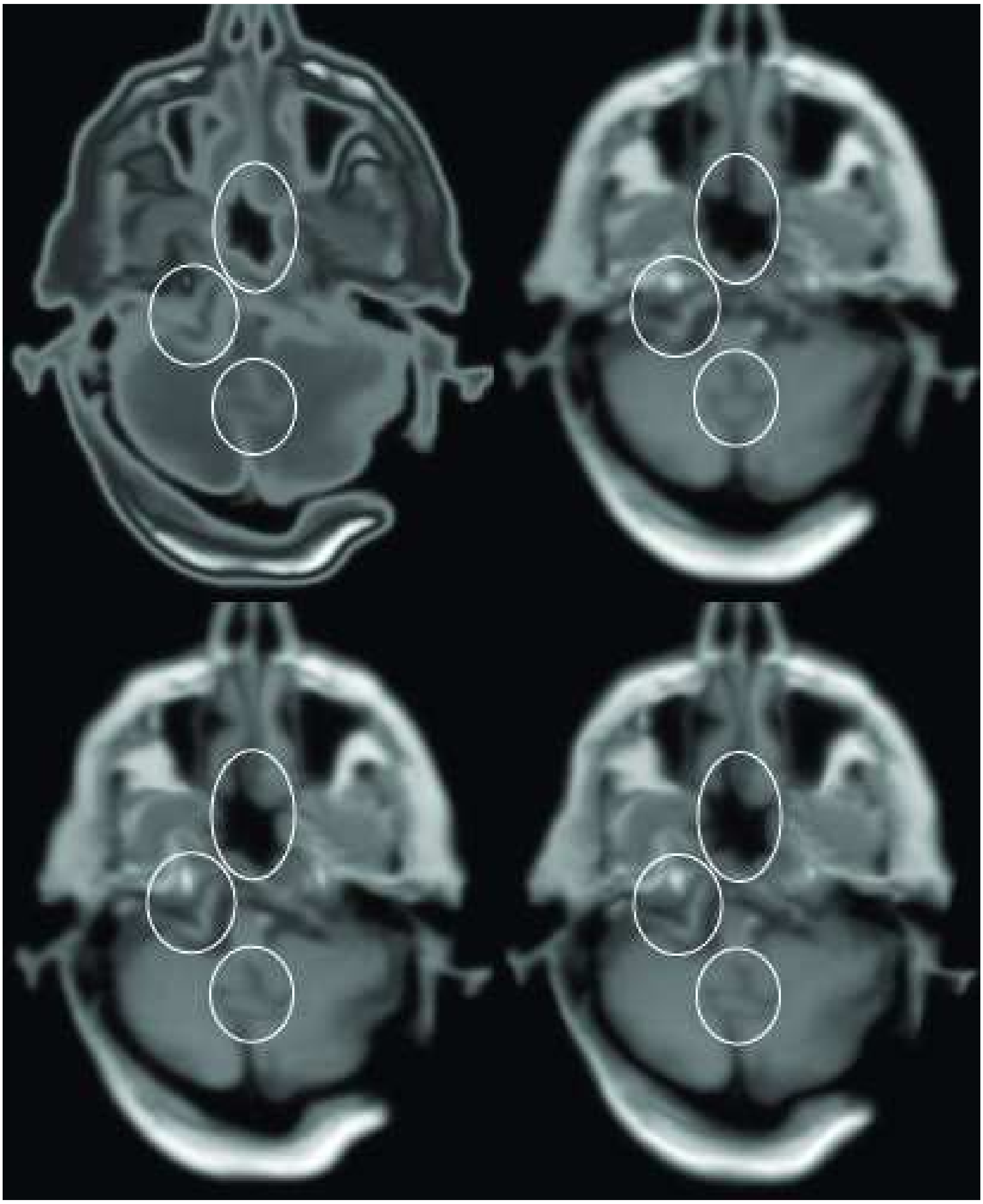


FIG. 3.1. *Test example. Top left: reference image $R = \beta \circ T \circ \psi$, generated from the template image by applying an artificial volume preserving distortion ψ and a non-monotone contrast transformation β. Top right: template image T. Bottom left: reference image $T \circ \psi$ before contrast transformation. Bottom right: registration result $T \circ \phi$, template image applied to the computed deformation ϕ. All images have a resolution of 257^2. Areas of special interest are marked by white circles. See Figure 3 for the corresponding deformation.*

$$\begin{aligned} f(A_{\alpha,r}) &= f(r\,A + s\,r\,(A - B)) \\ &\geq f(r\,A) + s\,(f(r\,A) - f(r\,B)) \\ &= f(A) + \frac{\alpha}{r}(f(A) - f(B)) = f(A) + \frac{\alpha\,\delta}{r} \to \infty \end{aligned}$$

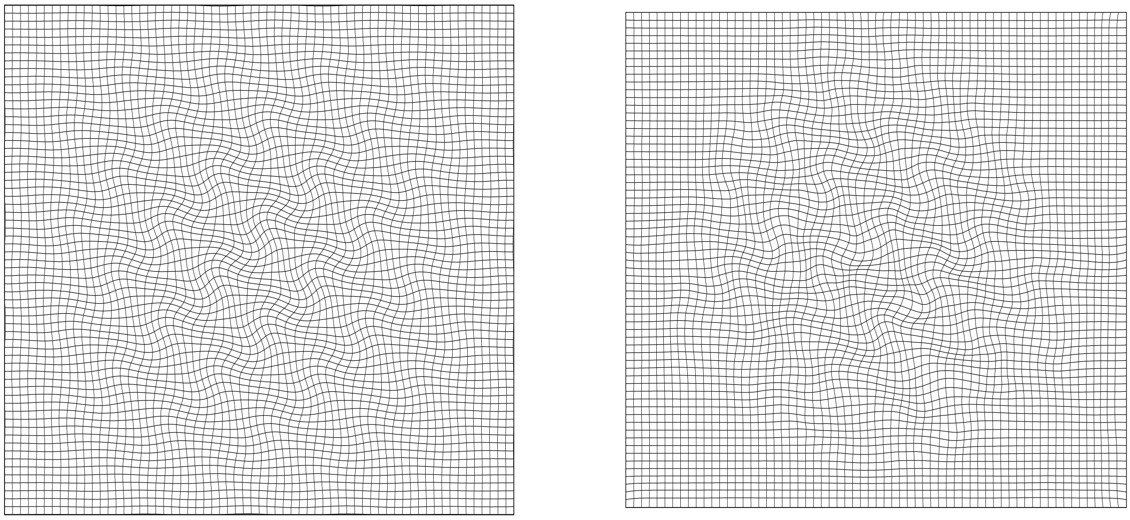
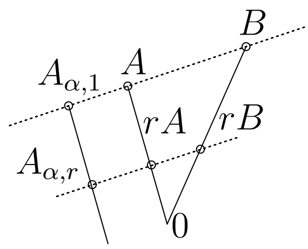


FIG. 3.2. *Exact deformation ψ (left) and computed deformation ϕ for the example in Figure 3.1.*

for $r \to 0$. Finally, we deduce $f(A - B) = \infty$ which contradicts our assumptions on f. Thus, the definition of the matching energy via the above integrand $\hat{g}((\mathbb{1} - v \otimes w)\frac{Aw}{\|Aw\|})$ and especially our first choice of a matching energy in (2.4) is not appropriate with respect to a positive answer to the question of existence of minimizers via direct methods.

So far, we have not discussed the singularities of the normal fields. Hence, let us introduce assumptions, which allow the normals to be undefined on a small set. Indeed, we take into account the space of bounded functions I, which are differentiable and whose gradients are unequal to zero on $\Omega \setminus \mathcal{D}_I$. In particular, the set of degenerate points is defined as $\mathcal{D}_I := \{x \in \Omega \,|\, \nabla I = 0\}$. We suppose that for the Lebesgue measure μ



$$\mu(B_\epsilon(\mathcal{D}_I)) \overset{\epsilon \to 0}{\longrightarrow} 0\,,$$

where $B_\epsilon(\mathcal{D}_I) := \bigcup \{B_\epsilon(x) \,|\, x \in \mathcal{D}_I\}$. Let us introduce a corresponding set of functions

$$\mathcal{I}(\Omega) := \Big\{ I : \Omega \to \mathbb{R} \,\Big|\, I \in C^1(\bar{\Omega}), \exists\, \mathcal{D}_I \subset \Omega \text{ s. t. } \nabla I \neq 0 \text{ on } \Omega \setminus \mathcal{D}_I, \\ \mu(B_\epsilon(\mathcal{D}_I)) \overset{\epsilon \to 0}{\longrightarrow} 0 \Big\}\,.$$

The existence proof for minimizers of nonlinear elastic energies via the calculus of variations and direct methods dates back to the work of J. Ball [3]. Especially the incorporated control of the volume transformation in this theory turns out to be the key to prove existence of minimizing, continuous and injective deformations for the image matching problem discussed here. We consider the following energy (cf. equations (2.3) and (3.3)):

$$E[\phi] := E_m[\phi] + E_{reg}[\phi]\,, \tag{4.1}$$

with

$$E_{reg}[\phi] := \int_\Omega W(D\phi, \mathrm{Cof}\, D\phi, \det D\phi)\, \mathrm{d}\mu$$
$$E_m[\phi] := \int_\Omega g_0(\nabla T \circ \phi, \nabla R, \mathrm{Cof}\, D\phi)\, \mathrm{d}\mu$$

for g_0 defined in (2.2).

Let us denote by L^p for $p \in [1, \infty]$ the usual Lebesgue spaces of functions on Ω into $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{R}^{d,d}$ respectively, by $\|\cdot\|_p$ the corresponding norm, and by $H^{1,p}$ the Banach space of functions in L^p with weak first derivatives also in L^p. For the ease of presentation, we do not exploit the full generality of the corresponding existence theory. Here the reader is for instance referred to [3, 4, 14, 22, 23, 46]. We confine to a basic model and state the following theorem:

Theorem 4.2 (Existence of minimizing deformations). *Suppose $d = 3$, $T, R \in \mathcal{I}(\Omega)$, and consider the total energy defined in (4.1) for deformations ϕ in the set of admissible deformations*

$$\mathcal{A} := \{\phi : \Omega \to \Omega \mid \phi \in H^{1,p}(\Omega), \mathrm{Cof}\, D\phi \in L^q(\Omega), \\ \det D\phi \in L^r(\Omega), \det D\phi > 0 \text{ a.e. in } \Omega, \phi = \mathbb{1} \text{ on } \partial\Omega\}$$

where p, $q > 3$ and $r > 1$. Suppose $W : \mathbb{R}^{3,3} \times \mathbb{R}^{3,3} \times \mathbb{R}^+ \to \mathbb{R}$ is convex and there exist constants $\beta, s \in \mathbb{R}$, $\beta > 0$, and $s > \frac{2q}{q-3}$ such that

$$W(A, C, D) \geq \beta\left(\|A\|_2^p + \|C\|_2^q + D^r + D^{-s}\right) \quad \forall A, C \in \mathbb{R}^{3,3},\ D \in \mathbb{R}^+ \tag{4.2}$$

Furthermore, assume that $g_0(v, w, A) = g(\frac{v}{\|v\|}, \frac{w}{\|w\|}, A)$, for some function $g : S^2 \times S^2 \times \mathbb{R}^{3,3} \to \mathbb{R}_0^+$, which is continuous in $\frac{v}{\|v\|}$, $\frac{w}{\|w\|}$, convex in A and for a constant $m < q$ the estimate

$$g(v, w, A) - g(u, w, A) \leq C_g\, \|v - u\|\, (1 + \|A\|_2^m)$$

holds for all $u, v, w \in S^2$ and $A \in \mathbb{R}^{3,3}$. Then $E[\cdot]$ attains its minimum over all deformations $\phi \in \mathcal{A}$ and the minimizing deformation ϕ is a homeomorphism and in particular $\det D\phi > 0$ *a.e. in Ω.*

Proof. The proof of this result is based on the well known weak continuity results for the principle invariants of the Jacobian of the deformation. We observe that the total energy is polyconvex. Furthermore the volume of the neighborhood sets $B_\epsilon(\mathcal{D}_T)$ and $B_\epsilon(\mathcal{D}_R)$ of the singularity sets $\mathcal{D}_T$ and $\mathcal{D}_R$ respectively can be controlled. Hence, we can basically confine to a set, where the integrand fulfills Carathéodory's conditions. At first, let us recall some well known, fundamental weak convergence results: Given a sequence of deformations $(\phi^k)_k$ in $H^{1,p}$, with $\mathrm{Cof}\, D\phi^k \in L^q$ and $\det D\phi^k \in L^r$, such that the sequence converges weakly in the sense $\phi^k \rightharpoonup \phi$ in $H^{1,p}$, $\mathrm{Cof}\, D\phi^k \rightharpoonup C$ in L^q, and $\det D\phi^k \rightharpoonup D$ in L^r, then $C = \mathrm{Cof}\, D\phi$ and $D = \det D\phi$ (weak continuity). For the proof we refer to Ball [3] or the book of Ciarlet [13] (Section 7.5, 7.6).

The proof of the theorem proceeds in 4 steps:

Step 1. Due to the assumption on the image set $\mathcal{I}(\Omega)$ $E_m[\phi]$ is well defined for $\phi \in \mathcal{A}$. In particular $g_0(\nabla T \circ \phi, \nabla R, \mathrm{Cof}\, D\phi)$ is measurable. Obviously $\mathbb{1} \in \mathcal{A}$ and $E[\mathbb{1}] < \infty$, thus $\underline{E} := \inf_{\phi \in \mathcal{A}} E[\phi] < \infty$ and due to the growth conditions and the assumption of g we furthermore get $\underline{E} \geq 0$. Let us consider a minimizing sequence $(\phi^k)_{k=0,1,\cdots} \subset \mathcal{A}$ with $E[\phi^k] \to \inf_{\phi \in \mathcal{A}} E[\phi]$. We denote by $\overline{E}$ an upper bound of the energy E on this sequence. Due to the growth condition on W we get that $\{(D\phi^k, \operatorname{Cof} D\phi^k, \det D\phi^k)\}_{k=0,1,\cdots}$ is uniformly bounded in $L^p(\Omega) \times L^q(\Omega) \times L^r(\Omega)$. By Poincaré's inequality applied to $(\phi^k - \mathbb{1})$ we obtain that $\{\phi^k\}_{k=0,1,\cdots}$ is uniformly bounded in $H^{1,p}(\Omega)$. Because of the reflexivity of $L^p \times L^q \times L^r$ for p, q, $r > 1$ we can extract a weakly convergent subsequence, again denoted by an index k, such that

$$(D\phi^k, \operatorname{Cof} D\phi^k, \det D\phi^k) \rightharpoonup (D\phi, C, D)$$

in $L^p \times L^q \times L^r$ with $C : \Omega \to \mathbb{R}^{3\times 3}$, $D : \Omega \to \mathbb{R}$. Applying the above results on weak convergence we achieve $C = \operatorname{Cof} D\phi$ and $D = \det D\phi$. In addition, by Rellich's embedding theorem we know that $\phi^k \to \phi$ strongly in $L^p(\Omega)$ and by Sobolev's embedding theorem we obtain $\phi \in C^0(\bar{\Omega})$.

Step 2. Next, we control the set where the volume shrinks by a factor of more than ϵ for the limit deformation. Let us define

$$S_\epsilon = \{x \in \Omega \,|\, \det D\phi \leq \epsilon\}$$

for $\epsilon \geq 0$. Let as assume without loss of generality that the sequence of energy values $E[\phi^k]$ is monotone decreasing and that for given $\epsilon > 0$ we denote by $k(\epsilon)$ the smallest index such that

$$E[\phi^k] \leq E[\phi^{k(\epsilon)}] \leq \underline{E} + \epsilon \qquad \forall k \geq k(\epsilon)\,.$$

From Step 1 we know that $\Psi^k := (D\phi^k, \operatorname{Cof} D\phi^k, \det D\phi^k)$ converges weakly to $\Psi := (D\phi, \operatorname{Cof} D\phi, \det D\phi)$ in $L^p \times L^q \times L^r$. Hence, applying Mazur's Lemma we obtain a sequence of convex combinations of Ψ^k and ϕ^k which converges strongly to Ψ and ϕ in $L^p \times L^q \times L^r \times L^p$. Thus, there exists a family of weights $((\lambda_i^k)_{k(\epsilon)\leq i \leq k})_{k \geq k(\epsilon)}$ with $\lambda_i^k \geq 0$, $\sum_{k(\epsilon)}^k \lambda_i^k = 1$, such that

$$\lambda_i^k \Psi^i \to \Psi \ \text{ and } \ \lambda_i^k \phi^i \to \phi\,.$$

Now, taking into account the growth conditions, the convexity of W and Fatou's lemma we estimate

$$\begin{aligned}
\beta \epsilon^{-s} \mu(S_\epsilon) &\leq \beta \int_{S_\epsilon} (\det D\phi)^{-s} \,\mathrm{d}\mu \leq \int_{S_\epsilon} W(\Psi) \,\mathrm{d}\mu \\
&= \int_{S_\epsilon} \liminf_{k\to\infty} W(\lambda_i^k \Psi^i) \,\mathrm{d}\mu \leq \int_{S_\epsilon} \liminf_{k\to\infty} \lambda_i^k W(\Psi^i) \,\mathrm{d}\mu \\
&\leq \liminf_{k\to\infty} \lambda_i^k \int_{S_\epsilon} W(\Psi^i) \,\mathrm{d}\mu \\
&\leq \liminf_{k\to\infty} \lambda_i^k \int_{\Omega} W(\Psi^i) + g_0(\nabla T \circ \phi^i, \nabla R, \operatorname{Cof} D\phi^i) \,\mathrm{d}\mu \\
&\leq \overline{E}
\end{aligned}$$

and claim $\mu(S_\epsilon) \leq \frac{\bar{E}\epsilon^s}{\beta}$. As one consequence S_0 is a null set and we know that $\det D\phi > 0$ a. e. on Ω. Thus, together with the results form Step 1 we deduce that the limit deformation ϕ is in the set of admissible deformation $\mathcal{A}$. Following Ball [4] we furthermore obtain that ϕ is injective and ϕ is a homeomorphism.

Step 3. Now, we deal with the singularity set of the images T. By our assumption on the image set $\mathcal{I}(\Omega)$ we know that for given $\delta > 0$ there exist $\epsilon_T > 0$ such that $\mu(B_{\epsilon_T}(\mathcal{D}_T)) \leq \delta$. From this and the injectivity (cf. Theorem 1 (ii) in [4]) we especially deduce the estimate

$$\mu\left(\phi^{-1}(B_{\epsilon_T}(\mathcal{D}_T)) \setminus S_\epsilon\right) \leq \frac{1}{\epsilon} \int\limits_{\phi^{-1}(B_{\epsilon_T}(\mathcal{D}_T))} \det D\phi \,\mathrm{d}\mu = \frac{1}{\epsilon} \int\limits_{B_{\epsilon_T}(\mathcal{D}_T)} \mathrm{d}\mu \leq \frac{\delta}{\epsilon}\,.$$

Hence, we can control the preimage of $B_\epsilon(\mathcal{D}_T)$ with respect to ϕ but restricted to $\Omega \setminus S_\epsilon$. Due to the continuous differentiability of both images T and R we can assume that

$$\|\nabla T(x)\| \geq \gamma(\epsilon) \text{ on } \Omega \setminus B_\epsilon(\mathcal{D}_T) \tag{4.3}$$

where $\gamma : \mathbb{R}_0^+ \to \mathbb{R}$ is a strictly monotone function with $\gamma(0) = 0$.

Step 4. Due to Egorov's theorem and the strong convergence of ϕ^k in $L^p(\Omega)$ there is a set K_ϵ with $\mu(K_\epsilon) < \epsilon$ such that a subsequence, again denoted by ϕ^k, converges uniformly on $\Omega \setminus K_\epsilon$. Let us now define the set

$$R_{\epsilon,\delta} := \phi^{-1}(B_{\epsilon_T}(\mathcal{D}_T)) \cup S_\epsilon \cup K_\epsilon\,,$$

whose measure can be estimated in terms of ϵ and δ, i.e.

$$\mu(R_{\epsilon,\delta}) \leq \frac{\delta}{\epsilon} + \frac{\bar{E}\epsilon^s}{\beta} + \epsilon\,.$$

On $\Omega \setminus R_{\epsilon,\delta}$ the sequence $(\nabla T \circ \phi^k)_{k=0,1,\cdots}$ converges uniformly to $\nabla T \circ \phi$. Next, from the assumption on g and the 0-homogeneous extension property of g_0 we deduce that

$$|g_0(u, w, A) - g_0(v, w, A)| \leq C_\gamma \|u - v\| \left(1 + \|A\|_2^m\right) \tag{4.4}$$

for $u, v, w \in \mathbb{R}^3$, $A \in \mathbb{R}^{3,3}$ and $\|u\|, \|v\|, \|w\| \geq \gamma$. To use this estimate for $u = \phi^k$ and $v = \phi$ below, we assume that $k(\epsilon)$ is large enough, such that $\phi^k(x) \in \Omega \setminus B_{\frac{\epsilon_T}{2}}(\mathcal{D}_T)$ for $x \in \Omega \setminus R_{\epsilon,\delta}$ and

$$C_{\gamma(\frac{\epsilon_T}{2})} \left\|\nabla T \circ \phi^k - \nabla T \circ \phi\right\|_{\infty,\Omega\setminus K_\epsilon} \leq \epsilon$$

for all $k \geq k(\epsilon)$. Now we are able to estimate $E[\phi]$ using especially the convexity of W and $g(v, w, \cdot)$, the estimate (4.4), and Fatou's lemma:

$$\begin{aligned}
E[\phi] &= \int\limits_\Omega W(\Psi) + g_0(\nabla T \circ \phi, \nabla R, \operatorname{Cof} D\phi)\,\mathrm{d}\mu \\
&\leq \int\limits_\Omega \liminf_{k\to\infty} \lambda_i^k W(\Psi^i)\,\mathrm{d}\mu + 2\,C_g \int_{R_{\epsilon,\delta}} 1 + \|\operatorname{Cof} D\phi\|^m \,\mathrm{d}\mu \\
&\quad + \int\limits_{\Omega\setminus R_{\epsilon,\delta}} \liminf_{k\to\infty} \lambda_i^k g_0(\nabla T \circ \phi, \nabla R, \operatorname{Cof} D\phi^i)\,\mathrm{d}\mu
\end{aligned}$$

$$
\begin{aligned}
&\leq \liminf_{k\to\infty} \lambda_i^k \int_\Omega W(\Psi^i)\,\mathrm{d}\mu + b(\mu(R_{\epsilon,\delta}))\\
&\quad + \liminf_{k\to\infty} \lambda_i^k \int_{\Omega\setminus R_{\epsilon,\delta}} g_0(\nabla T\circ\phi, \nabla R, \mathrm{Cof}\, D\phi^i) - g_0(\nabla T\circ\phi^i, \nabla R, \mathrm{Cof}\, D\phi^i)\\
&\qquad\qquad + g_0(\nabla T\circ\phi^i, \nabla R, \mathrm{Cof}\, D\phi^i)\,\mathrm{d}\mu
\end{aligned}
$$

where $b(s) := 2\,C_g(s + (\frac{\bar{E}}{\beta})^{\frac{m}{q}} s^{1-\frac{m}{q}})$. Here we have in particular used the a priori estimate $\|\mathrm{Cof}\, D\phi\|_{q,\Omega} \leq (\frac{\bar{E}}{\beta})^{\frac{1}{q}}$. We estimate further and obtain

$$
\begin{aligned}
E[\phi] &\leq \liminf_{k\to\infty} \lambda_i^k \int_\Omega W(\Psi^i) + g_0(\nabla T\circ\phi^i, \nabla R, \mathrm{Cof}\, D\phi^i)\,\mathrm{d}\mu + 2\,b(\mu(R_{\epsilon,\delta}))\\
&\qquad + C_{\gamma(\frac{\epsilon_T}{2})} \sup_{k\to\infty} \int_{\Omega\setminus R_{\epsilon,\delta}} \left\|\nabla T\circ\phi - \nabla T\circ\phi^k\right\| \left(1 + \left\|\mathrm{Cof}\, D\phi^k\right\|_2^m\right)\,\mathrm{d}\mu\\
&\leq \liminf_{k\to\infty} \lambda_i^k E[\phi^i] + 2\,b(\mu(R_{\epsilon,\delta})) + \epsilon\, b(\mu(\Omega))\\
&\leq \underline{E} + \epsilon + 2\,b(\mu(R_{\epsilon,\delta})) + \epsilon\, b(\mu(\Omega))\,.
\end{aligned}
$$

Finally, for given $\bar{\epsilon}$ we choose ϵ and then δ and the dependent ϵ_T small enough and $k(\bar{\epsilon})$ large enough to ensure that

$$
\epsilon + 2\,b(\mu(R_{\epsilon,\delta})) + \epsilon\, b(\mu(\Omega)) \leq \bar{\epsilon}\,.
$$

and get $E[\phi] \leq \underline{E} + \bar{\epsilon}$. This holds true for an arbitrary choice of $\bar{\epsilon}$. Thus we conclude

$$
E[\phi] \leq \underline{E} = \inf_{\phi\in\mathcal{A}} E[\phi]\,,
$$

which is the desired result. $\square$

REMARK 4.3. *From the proof we have seen that the assumptions on the reference image could be weakened considerably compared to the template image. With respect to the applications we do not detail this difference here.*

EXAMPLE 4.4. *Let us consider*

$$
g(v, w, A) = \|(\mathbb{1} - v\otimes v)\cdot Aw\|^\gamma\,, \tag{4.5}
$$

for $1 \leq \gamma < q$: Hence, we obtain an admissible matching energy

$$
E_m[\phi] = \int_\Omega \|(\mathbb{1} - (N_T\circ\phi)\otimes(N_T\circ\phi))\cdot \mathrm{Cof}\, D\phi\, N_R\|^\gamma
$$

(Cf. example 2.1). Applying Theorem 4.2 the existence of a minimizing deformation can be established. Recalling Remark 4.1, we recognize that scaling the original energy density by an additional factor $\|\mathrm{Cof}\, D\phi N_R\|^\gamma$ turns the minimization task into a feasible problem. This corresponds to a modification of the area element on the level sets $\mathcal{M}_c^R$. Indeed, $\|\mathrm{Cof}\, D\phi N_R\|$ is the change of the area element at a position x on $\mathcal{M}_{R(x)}^R$ under the deformation.

5. An additional feature based registration energy. As the energy $E_m[\phi]$ depends on the directions of the image normals only, its minimization will lead to an alignment of the level sets of the two images. However, the alignment of significant level sets, which correspond to significant features is not taken into account by the energy. In medical applications such features may be boundaries of organs, bones or tissue structures. Hence, we will incorporate an additional energy which measures the quality of the match of certain clearly detectable features. Suppose $\mathcal{F}_T$ and $\mathcal{F}_R$ are corresponding selected feature sets in the images T and R. These feature sets, may be computed in a previous segmentation step applying for instance an active contour algorithm [9, 20]. We are aiming to penalize a non-proper match of these two sets by a suitable energy. They would be ideally matched, if

$$\mathcal{F}_T = \phi(\mathcal{F}_R)$$

The following energy measures the matching quality for a general deformation ϕ:

$$E_f[\phi] = \int_\Omega |d(\phi(\cdot), \mathcal{F}_T) - d(\cdot, \mathcal{F}_R)|^2 \,\mathrm{d}\mu\,, \tag{5.1}$$

where $d(x, A) := \hat{d} \circ \operatorname{dist}(x, A)$ is a function $\hat{d}$ of the distance of a point x from a set $A \subset \Omega$. We suppose $\hat{d} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ to be monotone and $\hat{d}(0) = 0$. In particular E_f vanishes in case of a perfect match. A suitable choice is $\hat{d}(s) = \alpha s^\delta$ with $0 < \delta \leq 1$ and $\alpha > 0$. We use this energy as a third term in the regularized problem (3.1).

Corollary 5.1 (Existence of minimizers in presence of a feature matching energy). *Suppose the assumptions of Theorem 4.2 hold. Furthermore let $\hat{d} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be continuous and consider*

$$E[\phi] = E_m[\phi] + E_{reg}[\phi] + E_f[\phi]\,. \tag{5.2}$$

Then $E[\cdot]$ attains its minimum over all deformations $\phi \in \mathcal{A}$ and the minimizing deformation ϕ is a homeomorphism and furthermore $\det D\phi > 0$ *a.e. in* Ω.

Proof. Due to the Lipschitz continuity of $\operatorname{dist}(\cdot, A)$ for arbitrary sets $A \subset \Omega$ with $A \neq \emptyset$ and the continuity of $\hat{d}$, the integrand of E_f is uniformly continuous in ϕ. Hence, the proof of Theorem 4.2 can easily be generalized. □

The overall energy will therefore not only align the directions correctly, but also penalize displaced features. In this setting it is thus possible to incorporate some *a priori* knowledge to improve the matching results. Let us emphasize that the morphological registration provides good results if the morphologies encoded by the normal fields of the two images actually coincide up to a deformation. But in cases where the images of different modalities reveal similar but different geometrical structures, which are not strictly equivalent in terms of mathematical morphology, a weak form of “landmarks” is recommendable to support the matching.

6. Generalized boundary conditions. So far we have imposed boundary conditions of Dirichlet type on $\partial\Omega$ for the deformation. This might be a reasonable assumptions in case of objects located in the center of the image with a considerable distance from the boundary (cf. Figure 3.1). If the objects cover the whole image domain we can not assume that the requested deformation obeys these artificial boundary conditions. In fact, structures visible close to the boundary in the reference image R will not be present in the template image T and vice versa. Hence we ask for more general boundary conditions. These applications in mind we have to tolerate

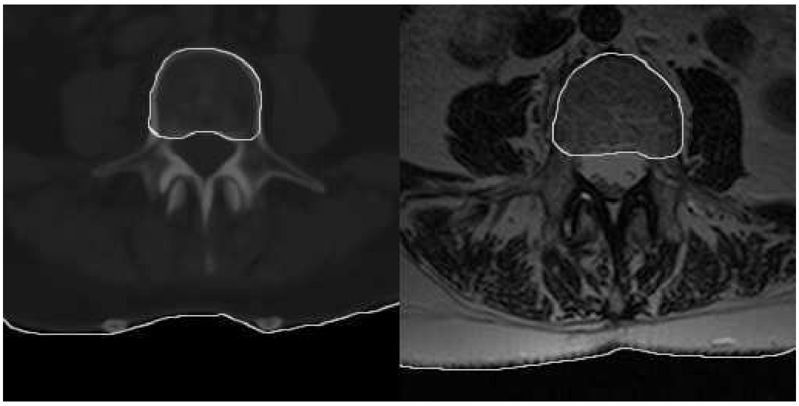


FIG. 6.1. *Feature sets $\mathcal{F}_R$ and $\mathcal{F}_T$ superimposed on darkened corresponding images for better visibility (cf. Figure 7.2 for registration results.)*

deformations $\phi(\Omega) \not\subset \Omega$ in the admissible set of deformations. But the integrand of the matching energy is only defined on $\phi^{-1}(\mathrm{Im}\,(\phi) \cap \Omega)$. Hence, we replace $\int_\Omega g_0(\cdot)$ by $\int_{\Omega^\phi} g_0(\cdot)$, where $\Omega^\phi := \{x \in \Omega \,|\, \phi(x) \in \Omega\}$ and obtain the new matching energy

$$\tilde{E}_m[\phi] := \int_{\Omega^\phi} g_0(\nabla T \circ \phi, \nabla R, \mathrm{Cof}\, D\phi)\, \mathrm{d}\mu \tag{6.1}$$

Taking into account this reformulated matching energy we are basically facing two problems:

(i) Considering a total energy $E[\phi] = \tilde{E}_m[\phi] + E_{reg}[\phi]$ we are lead to irrelevant, trivial solutions. Indeed, taking into account a simple translation ϕ_{trans} with $\phi_{trans}(\Omega) \cap \Omega = \emptyset$, one obtains $\tilde{E}_m[\phi_{trans}] = 0$. Hence, we no longer measure the matching of relevant image features. We propose to avoid this problem by incorporating the above feature energy $E_f[\phi]$. which can be regarded as a weak boundary condition. Indeed, if $\alpha \to \infty$ we enforce an interior boundary condition on the feature sets, i. e. $\phi(\mathcal{F}_R) = \mathcal{F}_T$.

(ii) Injectivity can no longer be expected for a minimizing deformation. It might happen that parts of the domain Ω fold over each other under a deformation ϕ, although ϕ is locally injective, i.e., $\det D\phi > 0$ (cf. the exposition of this problem in [13] Section 7.9). Following Ciarlet and Necas [14] we introduce an additional condition on the set of admissible deformations

$$\int_\Omega \det D\phi \leq \mu(\phi(\Omega))\,.$$

Then, we expect the minimizer of the energy $E = \tilde{E}_m + E_{reg} + E_f$ to be injective on Ω, where as on $\partial\Omega$ we might observe self contact. In the actual applications considered so far we have not detected any lack of global injectivity due to overlapping parts of the deformed domain. Hence, there was no need, to incorporate this nonlinear contact condition in the algorithm.

7. Multiscale-Minimization and Discretization. The total energy is highly nonlinear. Especially the matching energy E_m with the nonlinearity $\nabla T \circ \phi$ depending on the complexity of image data will usually lead to multiple at least local minima. Hence, in order to ensure a robust and efficient minimization, we have to consider a global minimization strategy, which is capable to compute large deformation which minimize the total registration energy. Here we propose a continuous annealing method based on a scale of registration problems

$$\tilde{E}^\sigma[\phi] := \tilde{E}^\sigma_m[\phi] + E_{reg}[\phi] + E_f[\phi]\,,$$

where $\sigma > 0$ is the scale parameter. This enables us to compute global instead of only local deformations and usually avoids a tedious pre registration step. The definition of the energy scales for the matching energy is based on a scale space approach for the underlying images (cf. [2]). We choose

$$E^\sigma_m[\phi] := \int\limits_{\Omega^\phi} g_0(\nabla T^\sigma \circ \phi, \nabla R^\sigma, \mathrm{Cof}\, D\phi)\,\mathrm{d}\mu\,,$$

where $I^\sigma := G^\sigma[I]$ for $I = T, R$ and G^σ denotes the convolution with a "Gaussian" filter of width σ (cf. Figure 7.1 for a multiscale of images and the corresponding effect on the Gauss maps N^σ_T and N^σ_R). In fact, we consider the heat equation semigroup and set $I^\sigma := u(\sigma^2/2)$ where u is the solution of the initial boundary value problem

$$\begin{aligned} \partial_t u - \Delta u &= 0 \qquad && \text{in } \mathbb{R}^+ \times \Omega\,, \\ \partial_\nu u(t,\cdot) &= 0 && \text{on } \mathbb{R}^+ \times \partial\Omega\,, \\ u(0,\cdot) &= I && \text{in } \Omega\,, \end{aligned} \tag{7.1}$$

and ν denotes the outer normal on $\partial\Omega$. Concerning the spatial discretization we deal with images as piecewise bilinear, continuous functions on a regular quadrilateral grid. We use the same discrete function space to define discrete non-rigid deformations. Energy functionals and their gradients are numerically evaluated using a midpoint quadrature rule on the grid cells. We assume $\Omega = [0,1]^2$, and start with an initial coarse mesh $\mathcal{M}_0 = \{\Omega\}$, which is iteratively refined by uniform subdivision, where each element is divided into 4 squares. This refinement process generates a sequences of nested meshes $\mathcal{M}_l$, with $0 \le l \le l_{\max}$, consisting of quadrilateral elements E^i_l $(0 \le i < 4^l)$ of edge length $h_l = 2^{-l}$. The set of vertices of $\mathcal{M}_l$ is denoted by $\mathcal{N}_l$. Let V_l be the corresponding space of piecewise bilinear, continuous finite element functions. Suppose $\{\Psi^i_l\}_{i\le(2^l+1)^2}$ to be the nodal basis of V_l. The discrete gradient $\mathrm{grad}_{V_l}\tilde{E}^\sigma \in V_l^2$ of E on grid level l for a deformation $\Phi \in V_l^2$ is then defined by

$$(\mathrm{grad}_{V_l}\tilde{E}^\sigma[\Phi], \Psi^j_l e_k)_h = \langle(\tilde{E}^\sigma)'[\Phi], \Psi^j_l e_k\rangle$$

for all $j \le (2^l+1)^2$ and $k = 1, 2$. Here $(\cdot,\cdot)_h$ denotes the usual lumped mass product on V_l and e_k the canonical basis in $\mathbb{R}^2$. Then, on level l the necessary condition for $\Phi^l \in V_l^2$ to be a minimizer of $\tilde{E}^\sigma$ over V_l is given by

$$\mathrm{grad}_{V_l} E^\sigma[\Phi] = 0 \quad \text{ for } \Phi \in V_l$$

Now, we introduce multiple discrete scales. Therefore, we replace the filter G^σ by its discrete counterpart, replacing problem (7.1) by a single implicit Euler time step with time step size $\frac{\sigma^2}{2}$ for a usual finite element discretization with lumped masses (cf. [42]). We denote the corresponding solution operator on the finite element space V_l by $G^\sigma_l : V_l \to V_l$. On each scale, we apply a gradient descent algorithm to minimize the energy. Here we might consider a sequence of scales

$$\sigma^k = 2^{-k}\sigma_0$$

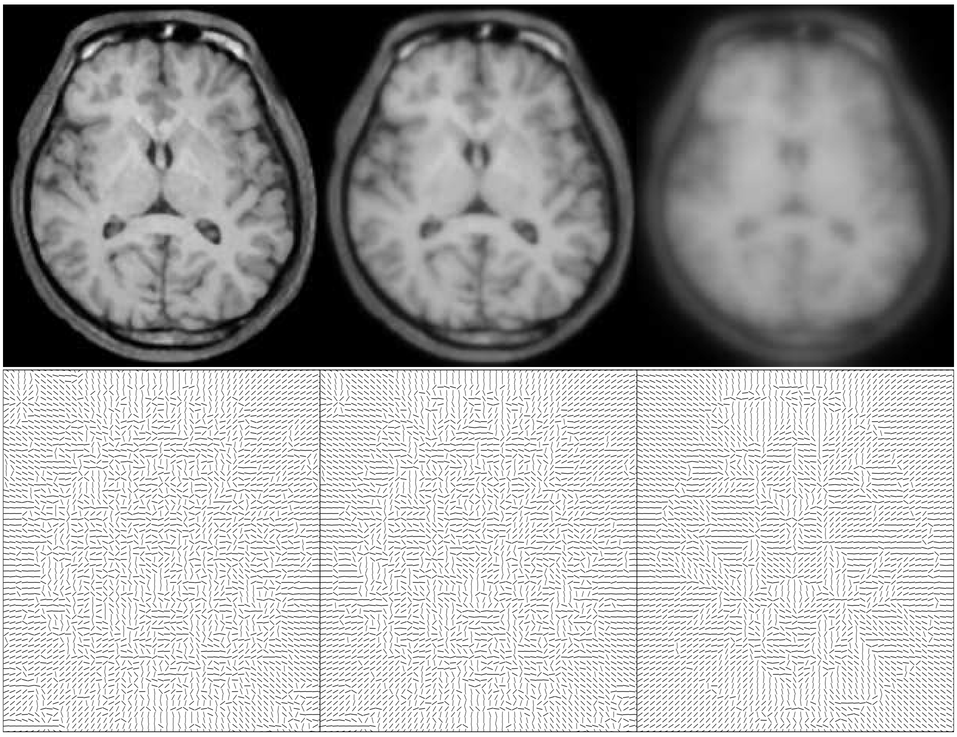


FIG. 7.1. *Image and the Gauss map and the corresponding grid for the brain slice at scales $\sigma = h, 2h, 8h$.*

for $k = 0, \ldots, n$. Obviously, solving a coarse scale minimization process on a fine grid introduces a serious amount of redundancy. It is much more efficient to perform such computations also on coarse grid levels. Thus, we introduce a function l_k which selects for each scale an appropriate grid level. In particular, we choose

$$l_k := \min\{l = 0, \ldots, l_{\max} \,|\, h(l) \leq \gamma\sigma_k\},$$

for a scalar $\gamma > 0$, e. g., $\gamma = 1$, which controls the ratio of the cell size $h(l)$ with respect to a filter width σ_k. On each scale we compute the minimum Φ^k of E^{σ_k} over $V_{l_k}^2$ by a gradient descent method and consider the standard prolongation of $\Phi^{k-1} \in V_{l_{k-1}}$ onto V_{l_k} as the initial value if $l_k \neq l_{k-1}$. It turns out to be suitable to regularize the contribution of the matching energy to the descent direction. Hence, a descent direction $d \in V_{l_k}$ at a position $\Phi \in V_{l_k}^2$ is computed by

$$d := -G_{l_k}^{\alpha}[\mathrm{grad}_{V_{l_k}} E_m^{\sigma_k}[\Phi]] - \mathrm{grad}_{V_{l_k}} \tilde{E}_{reg}[\Phi] - \mathrm{grad}_{V_{l_k}} E_f[\Phi]$$

where $\alpha > 0$ controls the amount of smoothing of the gradient for the registration energy. We have to ensure

$$(d, \mathrm{grad}\tilde{E}^{\sigma_k}[\Phi])_h \leq 0$$

in order to observe stable descent (cf. Table 7.1).

As step size control we consider Armijo's rule [30]. Let us remark that the smoothing by Gaussian convolution is solved efficiently and independently of the filter width σ by a multigrid solver for the heat equation [25]. In the computation for the registration of real MR and CT images of a human spine (cf. Figures 7.2, 7.3, 7.4), we chose the parameter α to be $5h_{l_{\max}}$. Furthermore, concerning the elastic regularization we

scale $\alpha[h_{l_{\max}}]$	0.25	0.5	1.0	2.0	3.0	4.0	5.0
$\gamma_{\min}$	0.9954	0.9585	0.8265	0.6588	0.58481	0.5438	0.5171
γ_{average}	0.9951	0.9556	0.8177	0.6447	0.5586	0.5116	0.4825

TABLE 7.1

To obtain a stable descent in the gradient descent algorithm of the global energy E^σ, the derivative in the direction of the descent direction d, i. e., $(d, gradE^\sigma[\phi])_h$ ought to be ≤ 0 in the scalar product. We have shown the impact of the smoothing parameter α for different scales on $\gamma(-d, gradE^\sigma[\phi])$, where $\gamma(u,v) := \frac{u}{\|u\|} \cdot \frac{v}{\|v\|}$. These values have been determined considering the first 50 steps of the gradient descent of the test example. We list the smallest value $\gamma_{\min}$ and the average value γ_{average}.

so far held on to the Mooney-Rivlin energy, i. e., $p = 2$, q does not have to be specified since the second term of W is redundant in 2 dimensions. The choices for the further parameters are $a = 0.45$, $b = 0.2$, $\gamma = \frac{1}{2}$, $\delta = 1$. To improve the methods performance we first relax the feature based energy $E_f[\cdot] + E_{reg}[\cdot]$ to identify an appropriate initial deformation. Then, we continue with the minimization of the global energy $E[\cdot]$.

Finally, the minimization algorithm can be written in pseudo code as follows:

Algorithm 7.1: multi-scale minimization algorithm

$\Phi_0 := \mathbb{1}$
foreach $k = 0, \dots, n$ **do**
  set level to l_k and grid $\mathcal{M}_{l_k}$
  if $k > 0$ **and** $l_k > l_{k-1}$ **then**
1   prolongate Φ^{k-1} on grid $\mathcal{M}_{l_{k-1}}$ to $\Phi^{k,0}$ on grid $\mathcal{M}_{l_k}$
  end
2 $T_{\sigma_k} := G^{\sigma_k}_{l_k}[T]$, $R_{\sigma_k} := G^{\sigma_k}_{l_k}[R]$,
3 $N_{T_{\sigma_k}} := \frac{\nabla T_{\sigma_k}}{\|\nabla T_{\sigma_k}\|}$, $N_{R_{\sigma_k}} := \frac{\nabla R_{\sigma_k}}{\|\nabla R_{\sigma_k}\|}$,
  $i = 0$
  repeat
4   $d^{k,i} := -G^{\alpha}_{l_k}[\mathrm{grad}_{V_{l_k}} \tilde{E}^{\sigma_k}_m[\Phi^{k,i}]] - \mathrm{grad}_{V_{l_k}} E_{reg}[\Phi^{k,i}] - \mathrm{grad}_{V_{l_k}} E_f[\Phi^{k,i}]$
5   *line-search*: choose step size δ by Armijo's rule
6   $\Phi^{k,i+1} := \Phi^{k,i} + \delta\, d^{k,i}$
   $i := i + 1$
  until $(\|d^{k,i}\| \leq TOL$ **or** $i > MAXITER)$;
  $\Phi^k = \Phi^{k,i}$
end

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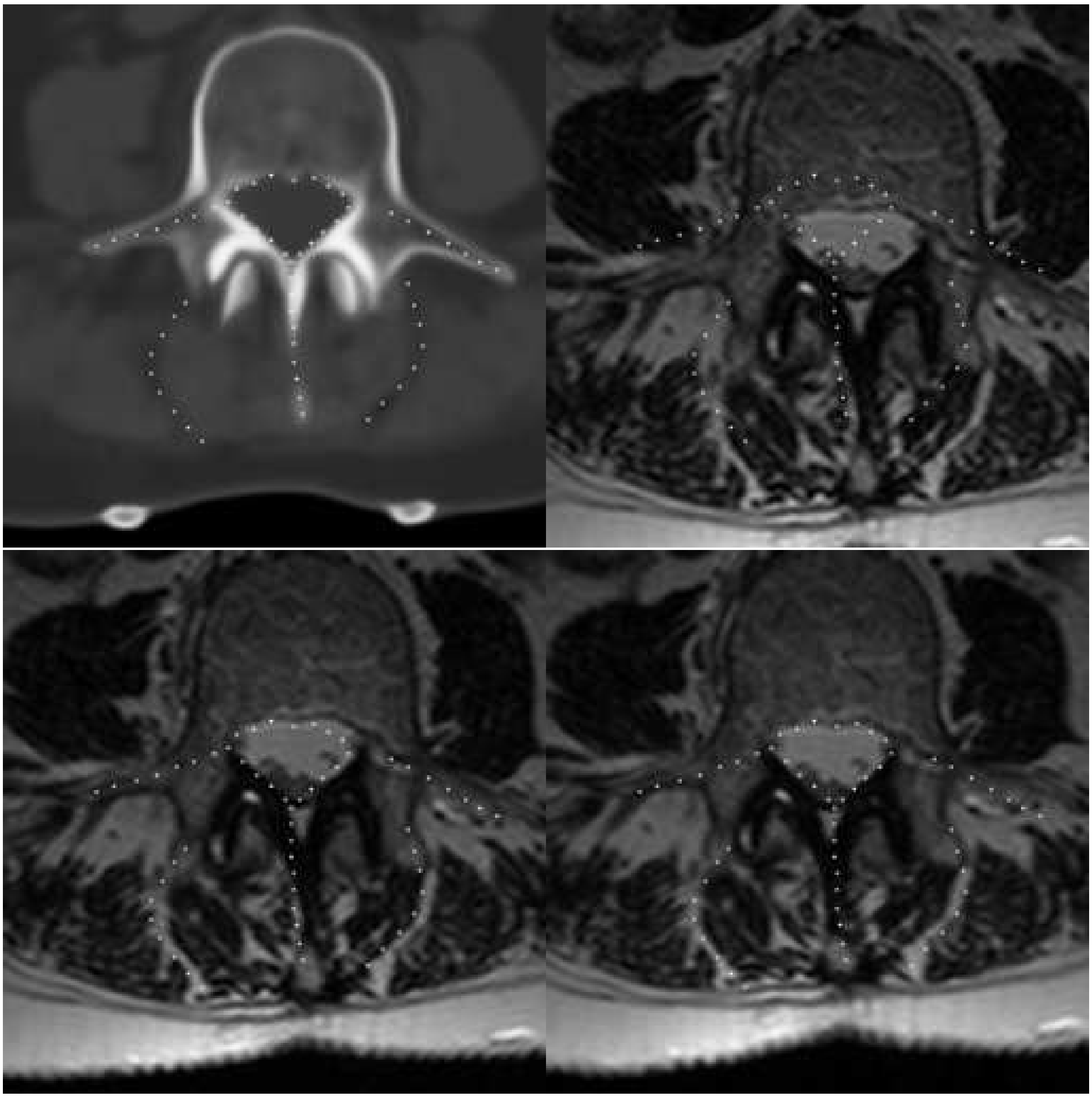

FIG. 7.2. *Sectional morphological registration on a pair of MR and CT images of a human spine. Dotted lines mark certain features visible in the reference image. There are repeatedly drawn at the same position in the other images. Top Left: reference, CT, Top Right: template, MR, with clearly visible misfit of structures marked by the dotted lines. Bottom Left: deformed template after feature based registration* $T\circ\phi_f$*, where* ϕ_f *is the result of a feature based pre-registration (cf. Figure 6.1 for the feature sets used in this example). Bottom Right: deformed template* $T\circ\phi$ *after final registration where the dotted feature lines nicely coincide with the same features in the deformed template MR-image. All images have a resolution of* 257^2.

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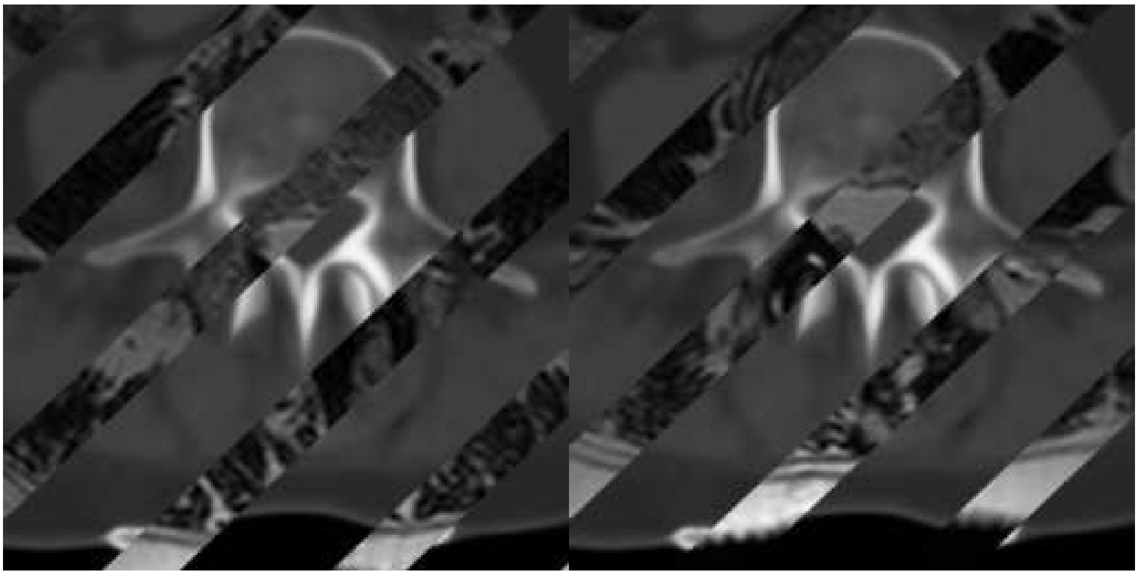

Fig. 7.3. *Comparison of superimposed template and reference before (left) and after (right) registration.*

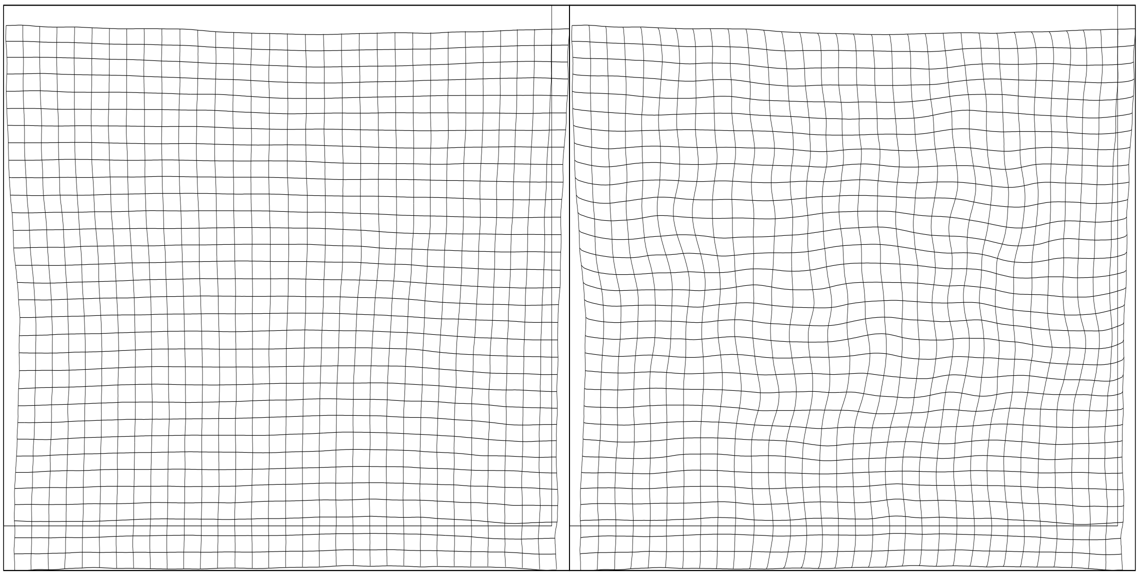

Fig. 7.4. *Left: deformation after the preregistration solely based on the feature energy. Right: final deformation after the registration including feature and morphological matching energy.*

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