Spinodal decomposition in the presence of elastic interactions

Harald Garcke, Stanislaus Maier-Paape, and Ulrich Weikard

[1] Naturwissenschaftliche Fakultät I - Mathematik, Universität Regensburg, D-93040 Regensburg, Germany, harald.garcke@mathematik.uni-regensburg.de
[2] Institut für Mathematik, RWTH Aachen, D-52062 Aachen, Germany, maier@instmath.RWTH-Aachen.DE
[3] Institut für Mathematik, Universität Duisburg, Lotharstraße 63, D-47048 Duisburg, Germany and Institut für Angewandte Mathematik, Universität Bonn, Wegelerstraße 6, 53115 Bonn, Germany, wkd@iam.uni-bonn.de

Summary. Spinodal decomposition, i.e., the separation of a homogeneous mixture into different phases, can be modeled by the Cahn-Hilliard equation - a fourth order semilinear parabolic equation. If elastic stresses due to a lattice misfit become important, the Cahn-Hilliard equation has to be coupled to an elasticity system to take this into account.

It is the goal of this paper to understand how elastic effects influence the formation of patterns during spinodal decomposition and to analyze what kind of morphologies one has to expect. It is shown that with a probability close to one, the dynamics of randomly chosen initial data in the neighborhood of a uniform mixture will be dominated by an invariant manifold which is tangential to the most unstable eigenfunctions of the linearized operator. For example in the case of cubic anisotropy it is shown that the most unstable eigenfunctions reflect the cubic anisotropy and the anisotropy will influence the dynamics quite drastically.

1 Introduction

In this paper we consider spinodal decomposition in binary alloys in the case where elastic effects become important. It is well known that complex patterns may form in the early stages of spinodal decomposition. We are interested to understand the effect elastic interactions may have on the formation of patterns.

The typical scenario of spinodal decomposition is as follows. At temperatures above a certain critical temperature a uniform mixture of the two components in the alloy is stable. After a rapid quenching this state can become unstable and regions where one or the other of the two components dominate occur. It is the goal of this paper to understand how these regions form and to analyze what kind of morphologies one has to expect.

Let c_1, c_2 denote the concentrations of the two alloy components. Since $c_1 + c_2 = 1$ the variable $c := c_1 - c_2$ completely determines the concentrations. Deformations of the reference configuration are described by the displacement field u, i.e., a material point x in the reference configuration will be found at the point $x + u(t, x)$ at time t.

Since displacement gradients in phase separating systems are small, the theory we consider will be based on the linearized strain tensor

$$\mathcal{E}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

where ∇ is the gradient with respect to space and $(\nabla u)^T$ is the transpose of ∇u.

In this paper we consider a generalization of the Cahn-Hilliard model taking elastic effects into account. The model is due to Larché and Cahn [13] and Onuki [18] using ideas introduced by Eshelby [3] and Khachaturyan [12]. The theory is based on a free energy of the form

$$E(c, u) = \int_{\Omega} \left[\frac{\varepsilon^2}{2} |\nabla c|^2 + \psi(c) + W(c, \mathcal{E}(u)) \right] dx$$

where $\Omega \subset \mathbb{R}^d$ is the domain under consideration, $c : \Omega \to \mathbb{R}$ is the concentration difference and $u : \Omega \to \mathbb{R}^d$ is the displacement field. The function $\psi : \mathbb{R} \to \mathbb{R}$ is the free energy density and is assumed here to be a non-convex function of c, e.g. a double well potential of the form $\psi(c) = (c^2 - 1)^2$. Without the term W this type of free energy goes back to van der Waals [21] and was introduced in the theory of spinodal decomposition by Cahn and Hilliard [2]. The third term is the elastic energy density $W : \mathbb{R} \times \mathbb{R}^{d \times d} \to \mathbb{R}$ which is chosen to be

$$W(c, \mathcal{E}) = \frac{1}{2}(\mathcal{E} - \tilde{\mathcal{E}}(c)) : C[\mathcal{E} - \tilde{\mathcal{E}}(c)]$$

(see Eshelby [3], Khachaturyan [12], Larché and Cahn [13], Fratzl, Penrose and Lebowitz [5]). Here, $\tilde{\mathcal{E}}(c) \in \mathbb{R}^{d \times d}$ is the stress free strain at concentration c, C is a fourth-rank elasticity tensor and the :–product of two matrices $A = (A_{ij})_{i,j=1,\dots,d}$ and $B = (B_{ij})_{i,j=1,\dots,d}$ is given by

$$A : B = \sum_{i,j=1}^{d} A_{ij} B_{ij} \quad \text{and} \quad |A|^2 = A : A \ .$$

Our standing assumptions are:

Assumption 1.1 *The stress free strains depend linearly on the concentration (Vegard's law), i.e.,*

$$\tilde{\mathcal{E}}(c) = c \cdot \mathcal{E}^*$$

with a fixed symmetric matrix

$$\mathcal{E}^* \in \mathbb{R}^{d \times d}.$$

The domain Ω is a assumed to have rectangular shape, i.e.,

$$\Omega = [0, \ell_1] \times \ldots \times [0, \ell_d]$$

with $\ell_1, \ldots, \ell_d > 0$. The elasticity tensor

$$C = (C_{ijmn})_{i,j,m,n=1,\ldots,d}$$

is assumed to be positive definite and to fulfill the symmetry conditions of linear elasticity, i.e.,

$$\begin{aligned} C_{ijmn} &= C_{ijnm} = C_{jimn}, \\ C_{ijmn} &= C_{mnij} \end{aligned} \tag{1}$$

and there exists some $d_0 > 0$, such that

$$\mathcal{E} : C[\mathcal{E}] \geq d_0 |\mathcal{E}|^2 \tag{2}$$

for all symmetric $\mathcal{E} \in \mathbb{R}^{d \times d}$.

We note that the symmetry condition (1) gives for all matrices A and B

$$C[A] : B = A : C[B].$$

Let us give two typical examples for the elasticity tensor C. In the **isotropic case** (cf. [11]) we have

$$C_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \tag{3}$$

where λ, μ are the Lamé constants and δ_{ij} is the Kronecker symbol. This means

$$\begin{aligned} C[\mathcal{E}] = \left(\sum_{m,n=1}^{d} C_{ijmn} \, \mathcal{E}_{mn} \right)_{i,j=1,\ldots,d} &= \lambda (tr\mathcal{E}) \cdot Id + \mu (\mathcal{E} + \mathcal{E}^T) \\ &= \lambda (tr\mathcal{E}) \cdot Id + 2\mu \mathcal{E} \end{aligned}$$

for all symmetric $\mathcal{E} \in \mathbb{R}^{d \times d}$. For many systems it is more realistic to assume **cubic symmetry** (cf. [5], p. 168). In this case there are three degrees of freedom for the elasticity tensor. One usually introduces the notation

$$\begin{aligned} C_{iiii} &:= C_{11} \; i = 1, \ldots, d, \\ C_{iijj} &:= C_{12} \; i \neq j, \\ C_{ijij} &:= C_{44} \; i \neq j \;, \end{aligned}$$

where C_{11}, C_{12}, and C_{44} are given constants. All other entries of C are then either given due to the cubic symmetry, i.e., either there is a symmetry with respect to the coordinate axis or they are set to be zero. Formally this means

$$Q\ C[\mathcal{E}]\ Q^T = C[Q\mathcal{E}Q^T]$$

for all orthogonal $Q \in \mathbb{R}^{d\times d}$ with $\det Q = 1$ which let the d–dimensional cube $[-1,1]^d$ invariant. The elasticity tensor can than be written as

$$\begin{aligned} C_{ijmn} = (C_{11} - C_{12} - 2C_{44})&\delta_{ij}\delta_{jm}\delta_{mn} \\ +C_{12}&\delta_{ij}\delta_{mn} \\ +C_{44}&(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}). \end{aligned} \tag{4}$$

The differential equation for c is given by (see [5, 6, 7, 9])

$$\partial_t c = \Delta w,$$

where w is the chemical potential difference defined as

$$w = \frac{\delta E}{\delta c} = -\varepsilon^2 \Delta c + \psi_{,c}(c) + W_{,c}(c, \mathcal{E}(u))$$

and $W_{,c}(c,\mathcal{E}(u)) = -\mathcal{E}^* :\ C[\mathcal{E}(u) - c\mathcal{E}^*]$.

The system is then completed by the assumption of quasi-static equilibrium for the mechanical part. This is justified since mechanical equilibrium is attained on a much faster time scale than mass diffusion takes place. Therefore, we obtain

$$0 = \frac{\delta E}{\delta u} = -\nabla \cdot W_{,\mathcal{E}}(c, \mathcal{E}(u)) = -\nabla \cdot S,$$

where

$$S = W_{,\mathcal{E}}(c,\mathcal{E}(u)) = C[\mathcal{E}(u) - c\mathcal{E}^*]$$

is the stress tensor and $\nabla\cdot$ is the divergence operator acting on rows. Altogether we obtain a system of a scalar and a vector-valued equation

$$\begin{aligned} \partial_t c &= \Delta(-\varepsilon^2 \Delta c + \psi_{,c}(c) - \mathcal{E}^* : S), \\ 0 &= \nabla \cdot S = \nabla \cdot C[\mathcal{E}(u) - c\mathcal{E}^*]\ . \end{aligned} \tag{5}$$

For definiteness, we assume periodic boundary conditions for c and u. The above equations imply mass conservation for c, i.e.

$$\int_\Omega c(t,x)\ dx \ =:\ c_m \ =\ \text{const.} \tag{6}$$

Therefore the new variable $v := c - c_m$ satisfies $\int_\Omega v dx = 0$. We set

$$f(c) := -\psi_{,c}(c)$$

and, after replacing v by c again, we arrive at

$$\partial_t c = (-\Delta)(\varepsilon^2 \Delta c + f(c_m + c) + \mathcal{E}^* : S), \tag{7}$$
$$0 = \nabla \cdot C[\mathcal{E}(u) - c\mathcal{E}^*] . \tag{8}$$

The elasticity equation (8) is linear in u and c. In Section 2 it is shown that for all previously described elasticity tensors C equation (8) can be solved using Fourier transformation (see also [12, 17]). We obtain some $u = u(c)$ for any given c with $\int_\Omega c\,dx = 0$.

Secondly, in Section 2 we compute the term $\mathcal{E}^* : S = \mathcal{E}^* : C[\mathcal{E}(u) - c\mathcal{E}^*]$ with $u = u(c)$ which enters the equation (7) for c. This will give an operator $\mathcal{L}$

$$\mathcal{L} : \ X \to X, \ c \mapsto \mathcal{E}^* : S, \text{ where } X := \left\{ c \in L^2(\Omega) : \int_\Omega c dx = 0 \right\} , \tag{9}$$

which is linear in c. With the help of $\mathcal{L}$ we can rewrite the equation for c as follows

$$\partial_t c = (-\Delta)(\varepsilon^2 \Delta c \ + \ f(c_m + c) \ + \ \mathcal{L}(c)) . \tag{10}$$

The understand the behavior of (10) for $c \approx 0$ we linearize at $c = 0$ to obtain

$$\partial_t c = \big(-\Delta\big) \ (\varepsilon^2 \Delta c \ + \ f'(c_m) \ c \ + \ \mathcal{L}(c)\big) . \tag{11}$$

The eigenfunctions of the right hand side in (11) are

$$\varphi_\kappa(x) = e^{\mathbf{i}\kappa \cdot x} ,$$

where $\mathbf{i} = \sqrt{-1}$ and

$$\begin{aligned} \kappa &= (\kappa_1, ..., \kappa_d) = 2\pi(\tfrac{\nu_1}{\ell_1}, \ldots, \tfrac{\nu_d}{\ell_d}), \\ \nu &= (\nu_1, \ldots, \nu_d) \in \mathbb{Z}^d . \end{aligned} \tag{12}$$

The associated eigenvalues are

$$\lambda_{\kappa,\varepsilon} = |\kappa|^2 \Big(-\varepsilon^2 |\kappa|^2 + f'(c_m) + L(\kappa)\Big) , \tag{13}$$

where we use that in our applications the linear operator $\mathcal{L}$ can be written as

$$\mathcal{L}(\varphi_\kappa) = L(\kappa) \cdot \varphi_\kappa \tag{14}$$

with some 0–homogeneous function $L : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$. We set $L_{\max}$ and $L_{\min}$ to be the maximal and minimal value of L.

Now the equilibrium $c = 0$ of (10) is unstable, if the maximal eigenvalue in (13) is positive. Therefore we assume:

Assumption 1.2 *Let the mean value c_m of c (cf. (6)) and the maximum of L from (14) satisfy*

$$f'(c_m) + L_{\max} > 0 .$$

It is easy to see that then the eigenvalues are bounded by

$$\lambda_\varepsilon^{\max} := \frac{\left(f'(c_m) + L_{\max}\right)^2}{4\varepsilon^2} \; .$$

In this situation, we can make an adaption of the theory of Maier–Paape and Wanner on spinodal decomposition for the Cahn–Hilliard equation (cf. [14] and [15]). We will outline this in Section 3. In conclusion, we find that the behavior of (10) near $c = 0$ is dominated by a finite dimensional strongly unstable subspace

$$\mathcal{Y}_\varepsilon^+ \; := \; \text{span} \left\{ \varphi_\kappa : \; \lambda_{\kappa,\varepsilon} > \gamma_0 \cdot \lambda_\varepsilon^{\max} \right\}$$

for some $\gamma_0 < 1$ close to one. It turns out that real parts and imaginary parts of this subspace are just a subset of the dominating subspace occurring for the Cahn–Hilliard equation. Therefore we inherit the small order $O(\varepsilon)$ wavelength estimate for the elements in $\mathcal{Y}_\varepsilon^+$ (cf. [14], Section 4). Figure 1 is a sketch of the situation for the Cahn–Hilliard equation ($\mathcal{L} = 0$) which we give for reference. To the left we have the Fourier vectors κ that are excited the most and to the right we have the nodal domains of a typical element of $\mathcal{Y}_\varepsilon^+$ for that case.

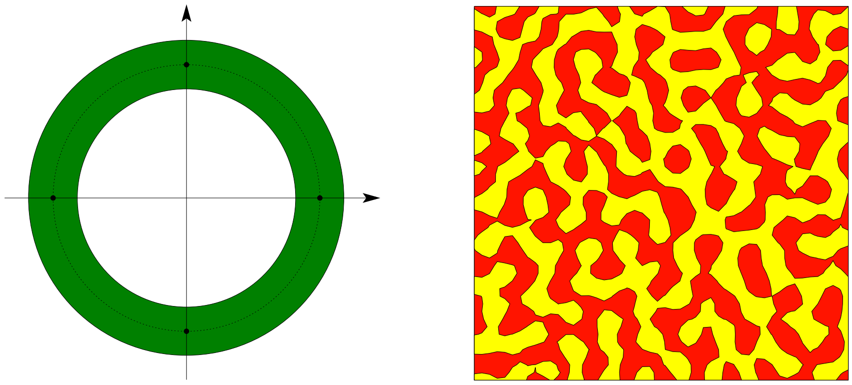

Fig. 1. Cahn–Hilliard situation

The next two situations show the dominant Fourier vectors and typical pattern for the case of cubic symmetry (4), $\mathcal{E}^* = q \cdot Id$ and $\Omega =$ square. We first have negative anisotropy, i.e. $\Delta C = C_{11} - C_{12} - 2C_{44} < 0$, and then in Figure 3 the situation for positive anisotropy, i.e. $\Delta C > 0$.
The last picture, Figure 4, shows the situation for isotropic symmetry (3) and $\mathcal{E}^*$ being not a multiple of the identity matrix (non-dilatational misfit).

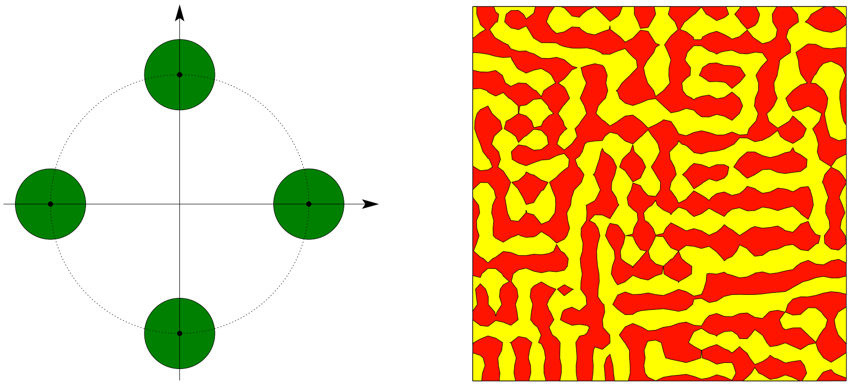

Fig. 2. Cubic symmetry with negative anisotropy

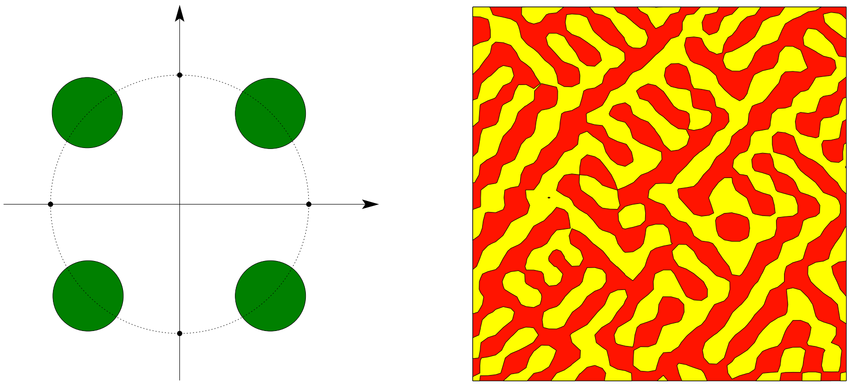

Fig. 3. Cubic symmetry with positive anisotropy

In each of the Figures 2, 3 and 4 we see that certain directions of the nodal domains are selected.

Finally, in Section 4 we present results from numerical simulations of the elastically modified Cahn-Hilliard system which are in agreement with the theoretical predictions of Sections 2 and 3.

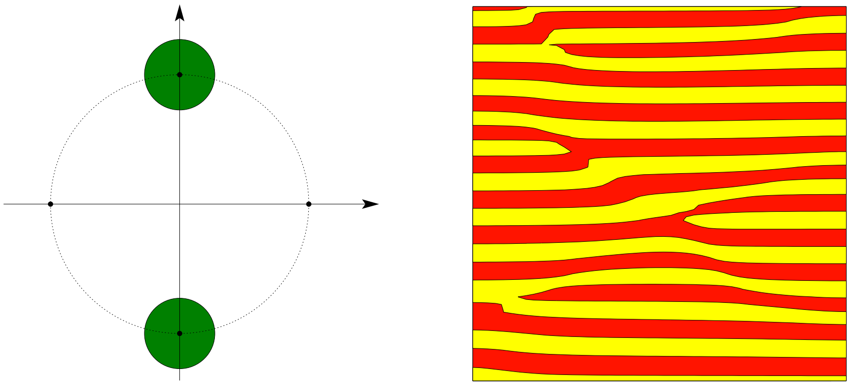

Fig. 4. Isotropic symmetry with anisotropic eigenstrains

2 Solving the elasticity system

We start this section by computing $\mathcal{L}(c) = \mathcal{E}^* : S$, where $S = C[\mathcal{E}(u) - c\mathcal{E}^*]$ and $u = u(c)$ is the unique solution of $\nabla \cdot S = 0$ for given c with $\int_\Omega c dx = 0$. Let us compute $\mathcal{L}$ for the Fourier mode

$$\varphi_\kappa(x) = e^{\mathbf{i}\kappa\cdot x} \ , \tag{15}$$

where $\mathbf{i} = \sqrt{-1}$ and

$$\kappa = (\kappa_1, ..., \kappa_d) = 2\pi(\tfrac{\nu_1}{\ell_1}, \dots, \tfrac{\nu_d}{\ell_d}),$$
$$\nu = (\nu_1, \dots, \nu_d) \in \mathbb{Z}^d \ .$$

For the solution u of (8) with $c = \varphi_\kappa$ as in (15) we make the ansatz

$$u(x) = (u_1(x), ..., u_d(x)) = \tilde{u}e^{\mathbf{i}\kappa\cdot x} \tag{16}$$

with $\tilde{u} \in \mathbb{R}^d$. We obtain

$$\partial_m u_n(x) = \tilde{u}_n \mathbf{i}\kappa_m e^{\mathbf{i}\kappa\cdot x}$$

and

$$\mathcal{E}(u) = \frac{1}{2}\mathbf{i}(\kappa \otimes \tilde{u} + \tilde{u} \otimes \kappa)e^{\mathbf{i}\kappa\cdot x}, \tag{17}$$

where $a \otimes b := ab^T \in \mathbb{R}^{d\times d}$ for all $a, b \in \mathbb{R}^d$. We calculate,

$$\begin{aligned} \nabla \cdot C[\varphi_\kappa \mathcal{E}^*] &= \nabla \cdot C[\mathcal{E}^* e^{\mathbf{i}\kappa\cdot x}] \\ &= (C[\mathcal{E}^*])\mathbf{i}\kappa e^{\mathbf{i}\kappa\cdot x} \\ &= S^* \mathbf{i}\kappa e^{\mathbf{i}\kappa\cdot x} \in \mathbb{R}^d \ , \end{aligned}$$

where $S^* := C[\mathcal{E}^*]$, i.e., S^* is the stress induced by $\mathcal{E}^*$. In addition we compute

$$\begin{aligned}\nabla \cdot C[\mathcal{E}(u)] &= \nabla \cdot (C[\tfrac{1}{2}\mathbf{i}(\kappa \otimes \tilde{u} + \tilde{u} \otimes \kappa)]e^{\mathbf{i}\kappa \cdot x}) \\ &= -\tfrac{1}{2}e^{\mathbf{i}\kappa \cdot x}C[\kappa \otimes \tilde{u} + \tilde{u} \otimes \kappa]\kappa \ .\end{aligned}$$

The symmetries of the elasticity tensor $C_{ijmn} = C_{ijnm}$ (cf. (1)) can be used to obtain

$$C[\kappa \otimes \tilde{u}] = C[\tilde{u} \otimes \kappa]$$

and therefore

$$\nabla \cdot C[\mathcal{E}(u)] = -e^{\mathbf{i}\kappa \cdot x}C[\kappa \otimes \tilde{u}]\kappa \ .$$

Introducing the matrix

$$Z^{-1}(\kappa) = \left(\sum_{j,m=1}^{d} C_{ijmn}\kappa_j \kappa_m \right)_{i,n=1,\ldots,d} , \tag{18}$$

we obtain

$$\nabla \cdot C[\mathcal{E}(u)] = -e^{\mathbf{i}\kappa \cdot x} Z^{-1}(\kappa)\tilde{u} \ .$$

The above used matrix $Z^{-1}(\kappa)$ is in fact the inverse of some matrix $Z(\kappa) \in \mathbb{R}^{d \times d}$. This follows from the next lemma.

Lemma 2.1 *The matrix $Z^{-1}(\kappa)$ is strictly positive definite for all $\kappa \in \mathbb{R}^d \setminus \{0\}$ and therefore invertible.*

Proof: Let $\zeta \in \mathbb{R}^d$. Then we obtain using the symmetry properties of C (1) and the fact that C is positive definite (2)

$$\begin{aligned}\zeta^T Z^{-1}(\kappa)\zeta &= \sum_{i,n=1}^{d} \zeta_i \sum_{j,m=1}^{d} C_{ijmn}\kappa_j\kappa_m\zeta_n \\ &= \sum_{i,j,m,n} C_{ijmn}\zeta_i\kappa_j\zeta_n\kappa_m \\ &= \sum_{i,j,m,n} C_{ijnm}\zeta_i\kappa_j\zeta_n\kappa_m \\ &\geq d_0|\zeta \otimes \kappa|^2 = d_0|\zeta|^2|\kappa|^2 \ .\end{aligned}$$

The above inequality shows that $Z^{-1}(\kappa)$ is positive definite whenever $\kappa \neq 0$. □

To obtain u we need to solve

$$\nabla \cdot C[\mathcal{E}(u)] = \nabla \cdot C[\varphi_\kappa \mathcal{E}^*] \ .$$

This is equivalent to

$$-Z^{-1}(\kappa)\tilde{u} = \mathbf{i}S^*\kappa \ .$$

Using that $Z^{-1}(\kappa)$ is invertible, we get

$$\tilde{u} = -\mathbf{i}Z(\kappa)S^*\kappa \ .$$

Hence from (16)

$$u(x) = -(\mathbf{i}Z(\kappa)S^*\kappa)e^{\mathbf{i}\kappa\cdot x} = -\mathbf{i}Z(\kappa)S^*\kappa\varphi_\kappa(x) \ . \tag{19}$$

Lemma 2.2 *For $\kappa \in \mathbb{R}^d \setminus \{0\}$ we obtain*

$$\mathcal{L}(\varphi_\kappa) = \mathcal{E}^* : (C[Z(\kappa)S^*\kappa\kappa^T] - S^*)\varphi_\kappa \ .$$

Proof: Using the formulas for u and $\tilde{u}$ above and (17) we compute

$$\begin{aligned}\mathcal{L}(\varphi_\kappa) = \mathcal{E}^* : S &= \mathcal{E}^* : C[\mathcal{E}(\tilde{u}e^{\mathbf{i}\kappa\cdot x}) - \varphi_\kappa\mathcal{E}^*] \\ &= \mathcal{E}^* : C[\tilde{u}\otimes\kappa\mathbf{i}e^{\mathbf{i}\kappa\cdot x} - \mathcal{E}^*e^{\mathbf{i}\kappa\cdot x}] \\ &= \mathcal{E}^* : C[Z(\kappa)S^*\kappa\kappa^T - \mathcal{E}^*]e^{\mathbf{i}\kappa\cdot x} \\ &= \mathcal{E}^* : (C[Z(\kappa)S^*\kappa\kappa^T] - S^*)e^{\mathbf{i}\kappa\cdot x}.\end{aligned}$$

This shows the lemma.

□

Remark 2.3 *The operator $\mathcal{L}$ can be interpreted as a pseudo-differential operator of order 0. This follows from the fact that*

$$\mathcal{L}(\varphi_\kappa) = L(\kappa)\varphi_\kappa \tag{20}$$

with a function L. The function L is 0-homogeneous because $Z(\kappa)$ is (-2)-homogeneous which follows from (18) and Cramer's rule. With u taken from (19) and after a partial integration we obtain using (8)

$$0 \geq -\int_\Omega W(\varphi_\kappa, \mathcal{E}(u))\,dx = \frac{1}{2}\int_\Omega \varphi_\kappa\mathcal{E}^* : C[\mathcal{E}(u) - \varphi_\kappa\mathcal{E}^*] = \frac{1}{2}L(\kappa)\int_\Omega \varphi_\kappa^2 \ .$$

We conclude that in the elastically modified Cahn-Hilliard equation (10) the $\mathcal{L}$-term has a stabilizing effect.

2.1 The isotropic case

In this subsection we assume that

$$C[\mathcal{E}] = \lambda(tr\mathcal{E})Id + \mu(\mathcal{E} + \mathcal{E}^T)$$

with given $\lambda, \mu \in \mathbb{R}$ and for some $q \in \mathbb{R}$

$$\mathcal{E}^* = qId.$$

Lemma 2.4 *Under the above assumption on* C *and* $\mathcal{E}^*$ *we obtain*

$$\mathcal{L} = q^2(d\lambda + 2\mu)\frac{2\mu(1-d)}{2\mu+\lambda} \cdot Id.$$

Proof: Let $\kappa \in \mathbb{R}^d \setminus \{0\}$ be given. First we compute

$$\begin{aligned}
Z^{-1}(\kappa) &= \lambda \left(\sum_{j,m=1}^{d} \delta_{ij}\delta_{mn}\kappa_j\kappa_m \right)_{i,n=1,\ldots,d} \\
&\quad + \mu \left(\sum_{j,m=1}^{d} (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\kappa_j\kappa_m \right)_{i,n=1,\ldots,d} \\
&= \lambda\kappa\kappa^T + \mu(\kappa\kappa^T + |\kappa|^2 Id) \\
&= (\lambda+\mu)\kappa\kappa^T + \mu|\kappa|^2 Id \ .
\end{aligned}$$

We can easily invert $Z^{-1}(\kappa)$ to obtain $Z(\kappa)$ defined as follows

$$Z(\kappa)\kappa = \frac{1}{(2\mu+\lambda)}\frac{1}{|\kappa|^2}\kappa$$

and

$$Z(\kappa)e = \frac{1}{\mu|\kappa|^2}e \quad \text{for all } e \text{ with } \quad \kappa^T \cdot e = 0 \ .$$

Furthermore

$$S^* = C[\mathcal{E}^*] = qCId = q(d\lambda Id + 2\mu Id) = q(d\lambda + 2\mu)Id$$

and

$$Z(\kappa)S^*\kappa\kappa^T = \frac{q(d\lambda+2\mu)}{(2\mu+\lambda)}\ \frac{1}{|\kappa|^2}\ \kappa\kappa^T \ .$$

To compute

$$\begin{aligned}
\mathcal{E}^* : (C[Z(\kappa)S^*\kappa\kappa^T] - S^*) &= q\, Id : (C[Z(\kappa)S^*\kappa\kappa^T] - S^*) \\
&= q \cdot tr(C[Z(\kappa)S^*\kappa\kappa^T] - S^*) \ ,
\end{aligned}$$

we calculate

$$\begin{aligned}
trC[\kappa\kappa^T] &= \sum_i \sum_{m,n} (\lambda\delta_{ii}\delta_{mn} + \mu(\delta_{im}\delta_{in} + \delta_{in}\delta_{im}))\kappa_m \cdot \kappa_n \\
&= (d\lambda + 2\mu)|\kappa|^2.
\end{aligned}$$

This gives

$$\begin{aligned}
\mathcal{E}^* : (C[Z(\kappa)S^*\kappa\kappa^T] - S^*) &= q \cdot tr(C[Z(\kappa)S^*\kappa\kappa^T] - S^*) \\
&= q^2(d\lambda + 2\mu)\left(\tfrac{d\lambda+2\mu}{(2\mu+\lambda)} - d\right) \\
&= q^2(d\lambda + 2\mu)\tfrac{2\mu(1-d)}{2\mu+\lambda} \ ,
\end{aligned}$$

which implies

$$\mathcal{L}(\varphi_\kappa) = q^2(d\lambda + 2\mu)\ \frac{2\mu(1-d)}{2\mu+\lambda}\varphi_\kappa$$

for all $\kappa \neq 0$. Therefore the claim is proved.

□

Remark 2.5 *We have shown that in the isotropic case, the operator $\mathcal{L}$ is a multiple of the identity. C as in (3) is positive definite if and only if*

$$\mu > 0 \quad and \quad 2\mu + d\lambda > 0.$$

Then in particular $2\mu + \lambda > 0$. This implies that $\mathcal{L}$ is a negative multiple of the identity for $q \neq 0$ and $d > 1$. For the evolution equation we observe again that the operator $\mathcal{L}$ has a stabilizing effect when compared to the case without elasticity.

2.2 A cubic elasticity tensor

In this subsection we consider the case $\mathcal{E}^* = q \cdot Id$ and we assume cubic symmetry for the elasticity tensor C (cf. (4)), i.e.,

$$\begin{aligned} C_{ijmn} = (&C_{11} - C_{12} - 2C_{44})\delta_{ij}\delta_{jm}\delta_{mn} \\ &+C_{12}\delta_{ij}\delta_{mn} \\ &+C_{44}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}). \end{aligned}$$

We remark that C is positive definite if and only if $C_{44} > 0$ and

$$\begin{array}{ll} C_{11} > (d-1)|C_{12}| \ , & \text{in case } C_{12} < 0 \\ C_{11} > C_{12} \ , & \text{in case } C_{12} \geq 0 \ . \end{array}$$

The following lemma is easy to verify.

Lemma 2.6 *The matrix $Z^{-1}(\kappa)$ has the entries*

$$\begin{array}{lll} (Z^{-1}(\kappa))_{ii} &= (C_{11} - C_{44})\kappa_i^2 + C_{44}|\kappa|^2, & i = 1, \ldots, d \ , \\ (Z^{-1}(\kappa))_{in} &= (C_{12} + C_{44})\kappa_i\kappa_n & i \neq n \ . \end{array}$$

Lemma 2.7 *With cubic symmetry for C and $\mathcal{E}^* = q \cdot Id$ we obtain*

$$\mathcal{L}(\varphi_\kappa) = q^2(C_{11}+(d-1)C_{12}) \left[(C_{11} + (d-1)C_{12}) \ \left(\sum_{m,p=1}^{d} Z_{mp}(\kappa)\kappa_p\kappa_m \right) - d \right] \varphi_\kappa \ .$$

Proof: According to Lemma 2.2 we have to compute

$$\mathcal{L}(\varphi_\kappa) = q \cdot tr(C[Z(\kappa)S^*\kappa\kappa^T] - S^*)\varphi_\kappa \ .$$

Under the assumptions on C and $\mathcal{E}^*$ we calculate

$$\begin{aligned}(S^*)_{ij} &= (C[\mathcal{E}^*])_{ij} = q(C[Id])_{ij} \\ &= q\left(\sum_{m=1}^{d} C_{ijmm}\right)_{ij} \\ &= q((C_{11} - C_{12} - 2C_{44} + dC_{12} + 2C_{44})\delta_{ij})_{ij}.\end{aligned}$$

This implies

$$S^* = q(C_{11} + (d-1)C_{12})Id$$

and

$$trS^* = q\, d(C_{11} + (d-1)C_{12}).$$

Furthermore, we have

$$\begin{aligned}&tr(C[Z(\kappa)S^*\kappa\kappa^T]) = \\ &= q(C_{11} + (d-1)C_{12})tr(C[Z(\kappa)\kappa\kappa^T]) \\ &= q(C_{11} + (d-1)C_{12}) \sum_{i,m,n} C_{iimn} \sum_p Z_{mp}(\kappa)\kappa_p\kappa_n \\ &= q(C_{11} + (d-1)C_{12}) \left(\sum_{i,m,p} C_{iimm} Z_{mp}(\kappa)\kappa_p\kappa_m\right) \\ &= q(C_{11} + (d-1)C_{12}) \left(\sum_i (C_{11} - C_{12}) \sum_p Z_{ip}(\kappa)\kappa_p\kappa_i + C_{12} d \sum_{m,p} Z_{mp}(\kappa)\kappa_p\kappa_m\right) \\ &= q(C_{11} + (d-1)C_{12})^2 \sum_{m,p} Z_{mp}(\kappa)\kappa_p\kappa_m.\end{aligned}$$

□

For simplicity we only give the formulas for $\mathcal{L}$ in the physically interesting cases $d = 2, 3$.

Lemma 2.8 *(d=2) In two space dimensions we obtain*

$$\mathcal{L}(\varphi_\kappa) = -\frac{q^2 C_{44}(C_{11}^2 - C_{12}^2)|\kappa|^4}{C_{44}C_{11}|\kappa|^4 + (C_{11} + C_{12})(C_{11} - C_{12} - 2C_{44})\kappa_1^2\kappa_2^2}\varphi_\kappa \ .$$

Proof: For $d = 2$ we obtain the inverse of $Z^{-1}(\kappa)$ as

$$Z(\kappa) = \frac{1}{\det Z^{-1}(\kappa)} \begin{pmatrix} |\kappa|^2 C_{44} + (C_{11} - C_{44})\kappa_2^2 & -(C_{12} + C_{44})\kappa_1\kappa_2 \\ -(C_{12} + C_{44})\kappa_1\kappa_2 & |\kappa|^2 C_{44} + (C_{11} - C_{44})\kappa_1^2 \end{pmatrix}$$

with

$$\det Z^{-1}(\kappa) = C_{11}C_{44}|\kappa|^4 + [(C_{11} - C_{44})^2 - (C_{12} + C_{44})^2]\kappa_1^2\kappa_2^2 \ .$$

Hence,

$$\sum_{m,p=1}^{2} Z_{mp}(\kappa)\kappa_p\kappa_m = \frac{|\kappa|^4 C_{44} + 2(C_{11} - C_{12} - 2C_{44})\kappa_1^2\kappa_2^2}{\det Z^{-1}(\kappa)} \ .$$

Altogether, we obtain

$$\mathcal{L}(\varphi_\kappa) = q^2(C_{11} + C_{12}) \left[\frac{(C_{11} + C_{12}) \cdot [|\kappa|^4 C_{44} + 2(C_{11} - C_{12} - 2C_{44})\kappa_1^2\kappa_2^2]}{|\kappa|^4 C_{44}C_{11} + (C_{11} + C_{12})(C_{11} - C_{12} - 2C_{44})\kappa_1^2\kappa_2^2} - 2 \right] \varphi_\kappa$$

$$= -\frac{q^2 C_{44}(C_{11}^2 - C_{12}^2)|\kappa|^4}{C_{44}C_{11}|\kappa|^4 + (C_{11} + C_{12})(C_{11} - C_{12} - 2C_{44})\kappa_1^2\kappa_2^2}\varphi_\kappa \ .$$

Remark 2.9 *In the situation of Lemma 2.8 the 0–homogeneous function L is given by*

$$L(\kappa) = -\frac{q^2 C_{44}(C_{11}^2 - C_{12}^2)|\kappa|^4}{C_{44}C_{11}|\kappa|^4 + (C_{11} + C_{12})(C_{11} - C_{12} - 2C_{44})\kappa_1^2\kappa_2^2} \ .$$

Let us compute now for which directions $\kappa \in \mathbb{R}^d \setminus \{0\}$ $L(\kappa)$ becomes maximal. First of all we note that the fact that C is positive definite implies

$$C_{11} > |C_{12}| \quad \textit{and} \quad C_{44} > 0 \ .$$

Hence we obtain:

i) In the case of positive anisotropy, i.e., $\Delta C := C_{11} - C_{12} - 2C_{44} > 0$:

$$L(\kappa) \quad \textit{is maximal if} \quad \kappa_1^2 = \kappa_2^2.$$

ii) In the case of negative anisotropy, i.e., $\Delta C < 0$:

$$L(\kappa) \quad \textit{is maximal if either } \kappa_1 \textit{ or } \kappa_2 \textit{ is equal to zero.}$$

For the evolution problem this will imply that elastic interactions will amplify wave numbers κ lying in segments around the coordinate axes stronger than other segments in the case that the anisotropy is negative. For positive anisotropy instead the diagonal directions will be stronger amplified.
In the case of three space dimensions we obtain:

Lemma 2.10 *For cubic elasticity and $\mathcal{E}^* = q \cdot Id$ we obtain in three space dimensions for all $\varphi_\kappa(x) = e^{\mathbf{i}\kappa\cdot x}$ with $\kappa \in \mathbb{R}^3 \setminus \{0\}$*

$$\mathcal{L}(\varphi_\kappa) = q^2(C_{11}+2C_{12})\Big[\tfrac{(C_{11}+2C_{12})}{det Z^{-1}(\kappa)}\Big(C_{44}^2|\kappa|^6 + 2C_{44}(C_{11}-C_{12}-2C_{44})|\kappa|^2 (\kappa_1^2\kappa_2^2+\kappa_2^2\kappa_3^2+\kappa_3^2\kappa_1^2) + 3(C_{11}-C_{12}-2C_{44})^2\kappa_1^2\kappa_2^2\kappa_3^2\Big) - 3\Big]\varphi_\kappa$$

where

$$det Z^{-1}(\kappa) = C_{44}^2C_{11}|\kappa|^6 + C_{44}(C_{11}-C_{12}-2C_{44})(C_{12}+C_{11})|\kappa|^2 (\kappa_1^2\kappa_2^2+\kappa_2^2\kappa_3^2+\kappa_3^2\kappa_1^2) + (C_{11}-C_{12}-2C_{44})^2 (2C_{12}+C_{44}+C_{11})\kappa_1^2\kappa_2^2\kappa_3^2.$$

Proof: A straightforward computation using Lemma 2.6 shows

$$Z_{ii}(\kappa) = \tfrac{1}{\det Z^{-1}(\kappa)}\big(C_{44}^2|\kappa|^4 + (C_{11}-C_{44})C_{44}|\kappa|^2(\kappa_j^2+\kappa_\ell^2) + (C_{11}-C_{12}-2C_{44})(C_{12}+C_{11})\kappa_j^2\kappa_\ell^2\big)$$

for $i = 1,2,3$ with j and ℓ such that $j \neq i$, $j \neq \ell$, $\ell \neq i$ and

$$Z_{ij}(\kappa) = \frac{-1}{\det Z^{-1}(\kappa)}(C_{12}+C_{44})\kappa_i\kappa_j(C_{44}|\kappa|^2 + (C_{11}-C_{12}-2C_{44})\kappa_\ell^2)$$

for $i,j = 1,2,3$, $i \neq j$ and ℓ such that $\ell \neq j$ and $\ell \neq i$. We also have

$$\det Z^{-1}(\kappa) = C_{44}^2C_{11}|\kappa|^6 + C_{44}(C_{11}-C_{12}-2C_{44})(C_{12}+C_{11})|\kappa|^2 (\kappa_1^2\kappa_2^2+\kappa_2^2\kappa_3^2+\kappa_3^2\kappa_1^2) + (C_{11}-C_{12}-2C_{44})^2(2C_{12}+C_{44}+C_{11})\kappa_1^2\kappa_2^2\kappa_3^2 \,.$$

Having computed $Z(\kappa)$ we obtain

$$\begin{aligned}\sum_{m,p} Z_{mp}(\kappa)\kappa_p\kappa_m = \tfrac{1}{\det Z^{-1}(\kappa)}\Big[&C_{44}^2|\kappa|^6 \\ &+2|\kappa|^2C_{44}(C_{11}-C_{44})(\kappa_1^2\kappa_2^2+\kappa_2^2\kappa_3^2+\kappa_3^2\kappa_1^2) \\ &+3(C_{11}-C_{12}-2C_{44})(C_{12}+C_{11})\kappa_1^2\kappa_2^2\kappa_3^2 \\ &-2|\kappa|^2C_{44}(C_{12}+C_{44})(\kappa_1^2\kappa_2^2+\kappa_2^2\kappa_3^2+\kappa_3^2\kappa_1^2) \\ &-6(C_{12}+C_{44})(C_{11}-C_{12}-2C_{44})\kappa_1^2\kappa_2^2\kappa_3^2\Big] \\ = \tfrac{1}{\det Z^{-1}(\kappa)}\Big[&C_{44}^2|\kappa|^6 + 2C_{44}(C_{11}-C_{12}-2C_{44})|\kappa|^2(\kappa_1^2\kappa_2^2+\kappa_2^2\kappa_3^2+\kappa_3^2\kappa_1^2) \\ &+3(C_{11}-C_{12}-2C_{44})^2\kappa_1^2\kappa_2^2\kappa_3^2\Big]\end{aligned}$$

Now Lemma 2.7 gives the assertion.

□

We now discuss the result of Lemma 2.10. As in (20) we can write $\mathcal{L}$ with the help of a 0–homogeneous function L. We want to determine directions for which L becomes maximal.

We obtain the following result.

Lemma 2.11 *In the situation of Lemma 2.10 with $\mathcal{L}(\varphi_\kappa) = L(\kappa) \cdot \varphi_\kappa$ we find for L:*
Case A**:** *Positive anisotropy, i.e., $\Delta C > 0$. Here L is maximal in directions given by the eight points*

$$\Big(\pm \frac{1}{\sqrt{3}} \; , \; \pm \frac{1}{\sqrt{3}} \; , \; \pm \frac{1}{\sqrt{3}} \Big) \; . \tag{21}$$

Case B**:** *Negative anisotropy, i.e., $\Delta C < 0$. Here L is maximal in directions given by the six points*

$$(\pm\, 1, 0, 0) \; , \; (0, \pm\, 1, 0), \quad \textit{and} \quad (0, 0, \pm\, 1) \; . \tag{22}$$

Proof: For negative anisotropy we can use the identity (which can be verified after a tedious but straightforward computation)

$$\tfrac{C_{11}}{q^2(C_{11}+2C_{12})^2} L(\kappa) + 2\tfrac{C_{11}-C_{12}}{C_{11}+2C_{12}} = \tfrac{\Delta C}{\det Z^{-1}(\kappa)} \Big[C_{44}\left(C_{11}-C_{12}\right) \textstyle\sum_{i,j,\, i\neq j} \kappa_i^4 \kappa_j^2 + \\ 2 \Big(\left(C_{11}-C_{12}-C_{44}\right)^2 + C_{44}\left(C_{11}-C_{12}\right) \Big) \kappa_1^2 \kappa_2^2 \kappa_3^2 \Big]$$

to conclude that that L is maximal in the directions corresponding to the coordinate axes. For κ on the coordinate axes the right hand side of the above identity is zero whereas for all other directions the right hand side is strictly negative. Here we use the fact that C is positive definite which yields $C_{44} > 0$, $C_{11} - C_{12} > 0$, $C_{11} > 0$, $C_{11} + 2C_{12} > 0$ and $det Z^{-1}(\kappa) > 0$.

To discuss the case of positive anisotropy, i.e., $\Delta C > 0$, we consider

$$M(\kappa) = \frac{\alpha}{\Delta C} \left(\frac{C_{11}}{q^2(C_{11}+2C_{12})^2} L(\kappa) + 2 \frac{C_{11}-C_{12}}{C_{11}+2C_{12}} \right) - \beta \; ,$$

where

$$\begin{aligned} \beta &= 9C_{44}(C_{11}-C_{12}) + \gamma_1, \\ \alpha &= 27C_{44}^2 C_{11} + 9\gamma_2 + \gamma_3, \\ \gamma_1 &= (\Delta C)\left(2C_{11} - 2C_{12} - C_{44}\right), \\ \gamma_2 &= (\Delta C) C_{44} (C_{11} + C_{12}), \\ \gamma_3 &= (\Delta C)^2 \left(2C_{12} + C_{44} + C_{11}\right) \end{aligned}$$

which are all positive constants. It now holds

$$M(\kappa) \\ = \frac{1}{\det Z^{-1}(\kappa)} \Big[-(9a+b)|\kappa|^6 + (27a-c)|\kappa|^2 \sum_{i,j\, i<j} \kappa_i^2\kappa_j^2 + (27b+9c)\kappa_1^2\kappa_2^2\kappa_3^2 \Big] \ , \tag{23}$$

with

$$\begin{aligned} a &= C_{44}(C_{11}-C_{12})C_{44}^2 C_{11}, \\ b &= \gamma_1 C_{44}^2 C_{11}, \\ c &= (\Delta C)^3 C_{44} C_{11} \ . \end{aligned}$$

Again the verification of the above identities needs some patience but is straightforward. We remark that $a, b, c > 0$.

The above relation between L and M and the following statement proves the lemma.

Claim: $M(\kappa) \le 0$ *and* $M(\kappa) = 0$ *if and only if* $\kappa_1^2 = \kappa_2^2 = \kappa_3^2$.

We have:

$$\begin{aligned} &\left(\det Z^{-1}(\kappa)\right) M(\kappa) \\ &\quad = a\, 9 \left(3|\kappa|^2 \sum_{i,j\, i<j} \kappa_i^2\kappa_j^2 - |\kappa|^6\right) \\ &\quad +b\left(27\kappa_1^2\kappa_2^2\kappa_3^2 - |\kappa|^6\right) \\ &\quad +c\left(9\kappa_1^2\kappa_2^2\kappa_3^2 - |\kappa|^2 \sum_{i,j\, i<j} \kappa_i^2\kappa_j^2 - |\kappa|^6\right) . \end{aligned}$$

To show the claim we are going to show that the term on the right hand side in the above identity is is nonnegative for κ with $|\kappa| = 1$. For the first two terms it is easy to verify that the maximum on S^2 is attained if and only if $\kappa_1^2 = \kappa_2^2 = \kappa_3^2$ and in this case the terms are zero. The last term is a bit more difficult to handle. W.l.o.g. we assume $\kappa_1^2 \le \kappa_2^2 \le \kappa_3^2$. Then we obtain

$$\begin{aligned} &9\kappa_1^2\kappa_2^2\kappa_3^2 - |\kappa|^2 \sum_{i,j\, i<j} \kappa_i^2\kappa_j^2 - |\kappa|^6 \\ &\quad = \kappa_1^2\kappa_2^2(2\kappa_3^2 - \kappa_1^2 - \kappa_2^2) + \kappa_2^2\kappa_3^2(2\kappa_1^2 - \kappa_2^2 - \kappa_3^2) + \kappa_1^2\kappa_3^2(2\kappa_2^2 - \kappa_1^2 - \kappa_3^2) \\ &\quad \le \kappa_1^2\kappa_3^2(2\kappa_3^2 - \kappa_1^2 - \kappa_2^2) + \kappa_1^2\kappa_3^2(2\kappa_1^2 - \kappa_2^2 - \kappa_3^2) + \kappa_1^2\kappa_3^2(2\kappa_2^2 - \kappa_1^2 - \kappa_3^2) \\ &\quad = 0 \end{aligned}$$

and this proves the lemma.

□

2.3 Anisotropic eigenstrains

In this section we consider the case that the eigenstrain $\mathcal{E}^*$ is not a multiple of the identity matrix. We restrict ourselves to isotropic elasticity in two dimensions and eigenstrains of the form

$$\mathcal{E}^* = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{R} \quad \text{with } a \cdot b > 0 \ .$$

Lemma 2.12 *For an isotropic elasticity tensor C and $\mathcal{E}^*$ as above we obtain for all $\kappa \in \mathbb{R}^2 \setminus \{0\}$*

$$\mathcal{L}(\varphi_\kappa) = L(\kappa)\varphi_\kappa$$

with

$$\begin{aligned} L(\kappa) = \tfrac{1}{\mu(\lambda+2\mu)|\kappa|^4} \Big[& \mu e^2 \kappa_1^4 + \mu f^2 \kappa_2^4 \\ & + \left[(\lambda+\mu)(e-f)^2 + \mu(e^2+f^2)\right] \kappa_1^2\kappa_2^2 \Big] \\ & - \Big[\lambda(a+b)^2 + 2\mu(a^2+b^2)\Big] \end{aligned}$$

where $e = \lambda(a+b) + 2\mu a$ and $f = \lambda(a+b) + 2\mu b$.

Proof: For all $\kappa \neq 0$ we have to compute

$$L(\kappa) = \mathcal{E}^* : (C[Z(\kappa)S^*\kappa\kappa^T] - S^*).$$

Using the symmetry of C and the definition $S^* = C[\mathcal{E}^*]$ we obtain

$$L(\kappa) = S^* : (Z(\kappa)S^*\kappa\kappa^T - \mathcal{E}^*).$$

Furthermore, we have

$$\begin{aligned} S^* = C[\mathcal{E}^*] &= \lambda tr\mathcal{E}^* Id + 2\mu\mathcal{E}^* \\ &= \lambda(a+b)Id + 2\mu \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \\ &= \begin{pmatrix} \lambda(a+b) + 2\mu a & 0 \\ 0 & \lambda(a+b) + 2\mu b \end{pmatrix}. \end{aligned}$$

We have (cf. the proof of Lemma 2.8 with $C_{12} = \lambda$, $C_{44} = \mu$ and $C_{11} = \lambda + 2\mu$)

$$Z(\kappa) = \frac{1}{\det Z^{-1}(\kappa)} \begin{pmatrix} \mu|\kappa|^2 + (\lambda+\mu)\kappa_2^2 & -(\lambda+\mu)\kappa_1\kappa_2 \\ -(\lambda+\mu)\kappa_1\kappa_2 & \mu|\kappa|^2 + (\lambda+\mu)\kappa_1^2 \end{pmatrix}$$

with

$$\det Z^{-1}(\kappa) = \mu(\lambda+2\mu)|\kappa|^4.$$

Therefore,

$$S^* Z(\kappa) S^* = \\ \begin{pmatrix} (\lambda(a+b)+2\mu a)^2 Z_{11}(\kappa) & (\lambda(a+b)+2\mu a)(\lambda(a+b)+2\mu b) Z_{21}(\kappa) \\ (\lambda(a+b)+2\mu a)(\lambda(a+b)+2\mu b) Z_{12}(\kappa) & (\lambda(a+b)+2\mu b)^2 Z_{22}(\kappa) \end{pmatrix}.$$

Altogether we obtain

$$
\begin{aligned}
&tr(S^* Z(\kappa) S^* \kappa\kappa^T) \\
&= \tfrac{1}{\mu(\lambda+2\mu)|\kappa|^4} \Big[(\lambda(a+b)+2\mu a)^2(|\kappa|^2\mu + (\lambda+\mu)\kappa_2^2)\kappa_1^2 \\
&\quad -2(\lambda(a+b)+2\mu a)(\lambda(a+b)+2\mu b)(\lambda+\mu)\kappa_1^2\kappa_2^2 \\
&\quad +(\lambda(a+b)+2\mu b)^2(|\kappa|^2\mu + (\lambda+\mu)\kappa_1^2)\kappa_2^2\Big] \\
&= \tfrac{1}{\mu(\lambda+2\mu)|\kappa|^4} \Big[\mu(\lambda(a+b)+2\mu a)^2\kappa_1^4 + \mu(\lambda(a+b)+2\mu b)^2\kappa_2^4 \\
&\quad + \big[(\lambda+2\mu)(\lambda(a+b)+2\mu a)^2 - 2(\lambda+\mu)(\lambda(a+b)+2\mu a) \\
&\quad (\lambda(a+b)+2\mu b) + (\lambda(a+b)+2\mu b)^2(\lambda+2\mu)\big]\,\kappa_1^2\kappa_2^2\Big] \\
&= \tfrac{1}{\mu(\lambda+2\mu)|\kappa|^4} \Big[\mu(\lambda(a+b)+2\mu a)^2\kappa_1^4 + \mu(\lambda(a+b)+2\mu b)^2\kappa_2^4 \\
&\quad + \big[(\lambda+\mu)4\mu^2(a-b)^2 + \mu((\lambda(a+b)+2\mu a)^2 + (\lambda(a+b)+2\mu b)^2)\big]\,\kappa_1^2\kappa_2^2\Big] \;.
\end{aligned}
$$

Also we compute

$$
\begin{aligned}
S^* : \mathcal{E}^* &= (\lambda(a+b)+2\mu a)a + (\lambda(a+b)+2\mu b)b \\
&= \lambda(a+b)^2 + 2\mu(a^2+b^2) \;.
\end{aligned}
$$

This proves the claim.

□

Remark 2.13 *We want to determine the maximum of L on S^1. Setting $z = \kappa_1^2$ (i.e. $1 - z = \kappa_2^2$) an analysis of the function*

$$
g(z) = \mu e^2 z^2 + \mu f^2(1-z)^2 + \big[(\lambda+\mu)(e-f)^2 + \mu(e^2+f^2)\big]\,z(1-z)
$$

shows that g can attain its maximum only for $z = 0, 1$. This means that L attains its maximum on the coordinate axes. Here one obtains that the maximum is attained on the x_1-axis if and only if $a^2 > b^2$ and on the x_2-axis if and only if $b^2 > a^2$.

The wave numbers κ for which L attains its maximum are amplified most strongly by elastic interactions. In this case only wave numbers lying on one particular coordinate axis are most strongly amplified.

Remark 2.14 *In Figure 5 we show the graph of $\lambda_{\kappa,\varepsilon}$ in the two dimensional case as a function of κ_1 and κ_2 given by equation (13). On the left hand we consider the case of an elasticity tensor with positive anisotropy (cf. Lemma 2.8) whereas on the right the elasticity tensor is isotropic but the eigenstrain is not (cf. Lemma 2.12).*

3 Spinodal decomposition and elastic interactions

In this section we will apply the abstract results of Section 2 of [15] to the Cahn-Hilliard equation with elasticity. This application is very much along

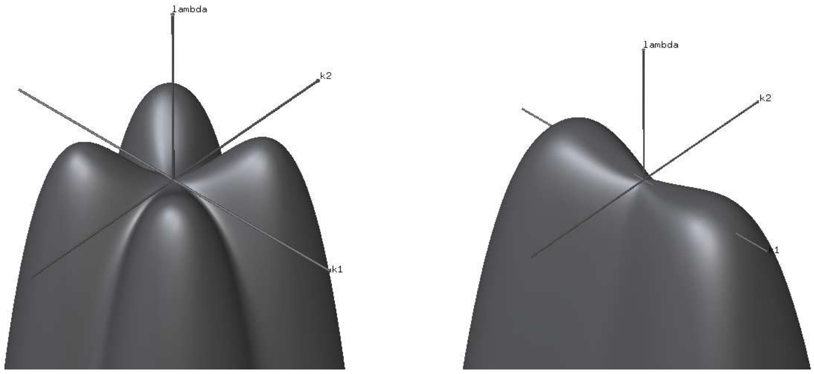

Fig. 5. $\lambda_{\kappa,\varepsilon}$ in the two dimensional case as a function of κ_1 and κ_2; on the left: with positive anisotropy of the elasticity tensor, on the right: with anisotropic eigenstrains.

the lines of the application of this theory given in Section 3 of [15] and Section 2 of [16], where the binary and multicomponent Cahn–Hilliard model were discussed. We consider the equation (10), i.e.,

$$\begin{aligned} \partial_t c \;&=\; (-\Delta)(\varepsilon^2 \Delta c \;+\; f(c_m + c) \;+\; \mathcal{L}(c)) \text{ in } \Omega \;, \\ & c \text{ is periodic}, \\ & \int_\Omega c dx \;=\; 0 \;, \end{aligned} \qquad (24)$$

where $\varepsilon > 0$ is a small parameter. We specify our assumptions as follows.

(A1) On top of our standing Assumption 1.1 we let Ω be of rectangular shape in $\mathbb{R}^d$, where $d \in \{1, 2, 3\}$.
(A2) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function.

In this situation we defined the linear operator $\mathcal{L} : \;\; X \to X$ in Section 1 (cf. (9)), which, as we saw in Section 2, always has a representation on the eigenfunctions $\varphi_\kappa = e^{\mathbf{i}\kappa x}$ as $\mathcal{L}(\varphi_\kappa) = \;L(\kappa)\; \varphi_\kappa$, with some 0–homogeneous function L (cf. (14)). Our last assumption is:

(A3) Let c_m and L satisfy Assumption 1.2 to make the equilibrium c_m unstable.

To rewrite (24) we set

$$\tilde{f}(c) := f(c_m + c) - f'(c_m)c - f(c_m) \;, \qquad (25)$$

so that $\tilde{f}(0) = \tilde{f}'(0) = 0$. Furthermore, let

$$A_\varepsilon c := (-\Delta)(\varepsilon^2 \Delta c + f'(c_m)c) \ , \ B_\varepsilon c := A_\varepsilon c + (-\Delta) \ \mathcal{L}(c) \ ,$$

and

$$F(c) := (-\Delta)\tilde{f}(c) \ . \tag{26}$$

Then formally the first equation in (24) is of the form

$$\partial_t c = B_\varepsilon c + F(c) \ . \tag{27}$$

The evolution equation (27) is of the same form as the one used in the binary Cahn–Hilliard case considered in [15]. Thus, we can use the abstract theory developed there to prove the dominance properties mentioned in the introduction. For this we only have to verify hypotheses (H1) through (H3) in [15]. This will be done in the following subsections. Basically, we have to verify the following three claims.

(H1) The operator $-B_\varepsilon$ is sectorial in the Hilbert space X.

(H2) There exists a decomposition $X = X^{--} \oplus X^{-} \oplus X^{+} \oplus X^{++}$ into pairwise orthogonal subspaces, such that all subspaces are finite-dimensional except X^{--}, and such that the linear semigroup corresponding to $\partial_t c = B_\varepsilon c$ satisfies several dichotomy estimates, see Lemma 3.8(b) below.

(H3) The nonlinear mapping $F : X^\alpha \to X$ is C^1 with $F(0) = 0$ and $DF(0) = 0$. Furthermore, it satisfies a global Lipschitz condition with constant L_F, i.e., for all $c, \tilde{c} \in X^\alpha$ we have

$$\|F(c) - F(\tilde{c})\|_X \le L_F \|c - \tilde{c}\|_{X^\alpha} \ .$$

Here X^α denotes the fractional power space corresponding to B_ε and $\alpha \in (0,1)$.

Hypothesis (H3) was already developed in [15], Section 3.3, because the nonlinearity in (24) is exactly the same as for the Cahn–Hilliard equation. In the course of verifying the hypotheses (H1) and (H2), we also calculate several constants introduced in [15], which in turn furnish an upper bound on the Lipschitz constant L_F. (Notice that since the nonlinearity in our example (see (26)) does not satisfy a global Lipschitz condition, we have to employ a standard cut-off technique.) This will eventually determine the size of the neighborhood on which our results are valid.

3.1 Spectral properties

In the following lemma we collect several properties of the linear operator B_ε which will be needed later. These results are the obvious generalization of Lemma 3.1 in [15], where the operator A_ε with Neumann type boundary conditions was discussed. We will again use the eigenfunctions $\varphi_\kappa = e^{\mathbf{i}\kappa x}$ of $-\Delta$ and their eigenvalues $|\kappa|^2$.

Lemma 3.1 *Assume that (A1), (A2), and (A3) are satisfied, and let X be defined as in (9). Define the operator $B_\varepsilon : X \to X$ by*

$$B_\varepsilon c = (-\Delta)\left(\varepsilon^2 \Delta c + f'(c_m)c + \mathcal{L}(c)\right) \ ,$$

with domain

$$D(B_\varepsilon) = \left\{c \in X \cap H^4(\Omega) : \ c \ \text{ is periodic}\right\} \ .$$

Then the following assertions hold.

(a) The spectrum of the operator $-\Delta : X \to X$ with domain

$$D(-\Delta) = \{c \in X \cap H^2(\Omega) : \ c \ \text{ is periodic}\}$$

consists of all $|\kappa|^2$, where $\kappa \in \mathbb{R}^d \setminus \{0\}$ satisfies (12). The corresponding normalized eigenfunctions $\tilde{\varphi}_\kappa = \varphi_\kappa / \|\varphi_\kappa\|$ form a complete $L^2(\Omega)$-orthonormal set in X. Furthermore, if $N_d(\mu)$ denotes the number of eigenvalues less than $\mu \in \mathbb{R}$ (counting multiplicities), then we have that

$$N_d(\mu) \sim \mu^{d/2} \ \text{as} \ \mu \to \infty \ , \tag{28}$$

where the proportionality constant depends only on Ω.

(b) The operator $-B_\varepsilon$ defined above is selfadjoint and sectorial. The spectrum of B_ε consists of real eigenvalues $\lambda_{\kappa,\varepsilon}$ given in (13) with corresponding eigenfunctions φ_κ. Moreover, the largest eigenvalue is for small $\varepsilon > 0$ of the order

$$\lambda_\varepsilon^{\max} := \frac{\left(f'(c_m) + L_{\max}\right)^2}{4\varepsilon^2} \ , \quad \text{and bounded by} \quad \lambda_\varepsilon^{\max} \ . \tag{29}$$

For later reference let us also introduce the following notation.

Definition 3.2 *We denote all κ related to eigenvalues of $-\Delta$ by*

$$\Lambda := \left\{\kappa \in \mathbb{R}^d \setminus \{0\} : \ \kappa \text{ is of the form (12)}\right\}$$

and $P\Lambda = \left\{\theta \in S^{d-1} : \ \kappa/|\kappa| = \theta \text{ for some } \kappa \in \Lambda\right\}$.
Furthermore, for given $\theta \in P\Lambda$ let

$$\Lambda(\theta) = \{\kappa \in \Lambda : \ \kappa/|\kappa| = \theta\} \ \subset \ \mathbb{R}^d \setminus \{0\}$$

be all κ corresponding to eigenvalues of $-\Delta$ lying on the same half ray as θ. Any $\theta \in P\Lambda$ satisfying $L(\theta) = L_{\max}$ is called $\theta_{\max}$.

Due to the above lemma, B_ε generates an analytic semigroup $S_\varepsilon(t)$ on X. Furthermore, for every B_ε the fractional power space $X^{1/2,\varepsilon} \subset X$ is defined; cf. the discussion in Subsection 2.1 of [15] following (H1). Although formally this fractional power space depends on ε, it will turn out in the next lemma that *algebraically*, i.e., as a vector space, we have $X^{1/2,\varepsilon} = H^2_{av}(\Omega)$, where

$$H^2_{av}(\Omega) := \left\{ c \in H^2(\Omega) \, : \, \int\limits_\Omega c dx = 0 \quad \text{and} \quad c \;\text{ is periodic} \right\} .$$

Only the *topological* structure given by the norm $|| \cdot ||_{1/2,\varepsilon}$ on $X^{1/2,\varepsilon}$ will depend on the parameter ε. Fortunately, the norms $|| \cdot ||_{1/2,\varepsilon}$ will turn out to be equivalent to the standard $H^2(\Omega)$-norm on $H^2_{av}(\Omega)$. More precisely, we have the following result.

Lemma 3.3 *Assume that (A1) and (A3) are satisfied, and let* $||\cdot||_{1/2,\varepsilon}$ *denote the norm on* $X^{1/2,\varepsilon}$ *defined by*

$$||c||_{1/2,\varepsilon} := \left|\left|(-B_\varepsilon + b_\varepsilon I)^{1/2} c\right|\right| \quad \textit{for all} \quad c \in X^{1/2,\varepsilon} ,$$

where $b_\varepsilon = \left(f'(c_m) + L_{\max}\right)^2 / \varepsilon^2$ *and* $||\cdot|| = ||\cdot||_{L^2(\Omega)}$. *Moreover, let* $||\cdot||_{H^2(\Omega)}$ *denote the standard* $H^2(\Omega)$*-norm, and define another norm* $|| \cdot ||_*$ *on* $H^2(\Omega)$ *by*

$$||c||_* := \sqrt{||c||^2 + ||\Delta c||^2} \quad \textit{for all} \quad c \in H^2(\Omega) .$$

For $\varepsilon^2 > 0$ *we assume* $\varepsilon^2 \le f'(c_m) + L_{\max}$ *and, in case* $f'(c_m) + L_{\min} < 0$, *we assume additionally* $\varepsilon^2 < |f'(c_m) + L_{\min}|$. *Then we have*

$$X^{1/2,\varepsilon} = H^2_{av}(\Omega) ,$$

and the norm $||\cdot||_{1/2,\varepsilon}$ *is equivalent to both* $||\cdot||_{H^2(\Omega)}$ *and* $||\cdot||_*$. *More precisely, there exists a (*ε*-independent) constant* C *depending only on the domain* Ω *such that for all* $c \in H^2_{av}(\Omega)$ *the estimates*

$$\frac{\varepsilon}{\sqrt{2}} \cdot ||c||_* \le ||c||_{1/2,\varepsilon} \le \frac{\hat{C}}{\varepsilon} \cdot ||c||_* \tag{30}$$

and

$$\frac{\varepsilon}{C \cdot \sqrt{2}} \cdot ||c||_{H^2(\Omega)} \le ||c||_{1/2,\varepsilon} \le \frac{\hat{C} \cdot C}{\varepsilon} \cdot ||c||_{H^2(\Omega)}$$

hold. Here we set

$$\hat{C} := \begin{cases} f'(c_m) + L_{\max} \, , & \textit{in case} \;\; f'(c_m) + L_{\min} \ge 0 \\ \sqrt{\left(f'(c_m) + L_{\max}\right)^2 + \frac{1}{2} \, \left(f'(c_m) + L_{\min}\right)^2} \, , & \textit{otherwise} \, . \end{cases}$$

Proof: As already pointed out in Lemma 3.2 in [15] we have that the norm $|| \cdot ||_*$ is an equivalent norm on $H^2(\Omega)$, i.e., there is a constant C such that for all $c \in H^2(\Omega)$ the estimate

$$\frac{1}{C} \cdot ||c||_{H^2(\Omega)} \leq ||c||_* \leq C \cdot ||c||_{H^2(\Omega)} \tag{31}$$

holds, cf. Temam [20, p. 150] and Fife, Kielhöfer, Maier-Paape, Wanner [4, Lemma 6.1] for the necessary elliptic regularity results.

Therefore we only have to verify (30). According to Amann [1, Theorem 4.6.7] the fractional power space $X^{1/2,\varepsilon}$ consists of all functions $c \in X$ whose $L^2(\Omega)$-Fourier coefficients $\xi_\kappa = (c, \varphi_\kappa)$, $\kappa \in \Lambda$, satisfy

$$\sum_{\kappa \in \Lambda} (b_\varepsilon - \lambda_{\kappa,\varepsilon}) \xi_\kappa^2 ||\varphi_\kappa||^2 < \infty \ .$$

In that case, the norm $||c||_{1/2,\varepsilon}$ is given by

$$||c||_{1/2,\varepsilon} = \left(\sum_{\kappa \in \Lambda} (b_\varepsilon - \lambda_{\kappa,\varepsilon}) \xi_\kappa^2 ||\varphi_\kappa||^2 \right)^{1/2} \ .$$

On the other hand a function $c \in X$ is in $H^2_{av}(\Omega)$ if and only if

$$||c||_*^2 = \sum_{\kappa \in \Lambda} \left(1 + |\kappa|^4\right) \ \xi_\kappa^2 ||\varphi_\kappa||^2 < \infty \ . \tag{32}$$

These characterizations eventually furnish the stated results. Using the representation (13) of $\lambda_{\kappa,\varepsilon}$ and $L(\kappa) \in [L_{\min}, L_{\max}]$ we obtain

$$|\kappa|^2 \left(-\varepsilon^2 |\kappa|^2 + f'(c_m) + L_{\min}\right) \leq \lambda_{\kappa,\varepsilon} \leq |\kappa|^2 \left(-\varepsilon^2 |\kappa|^2 + f'(c_m) + L_{\max}\right) \ .$$

An easy calculation gives for all $s \geq 0$ and $0 < \varepsilon^2 < f'(c_m) + L_{\max}$

$$\frac{\varepsilon^2}{2} \ (1 + s^2) \leq b_\varepsilon + \varepsilon^2 s^2 - (f'(c_m) + L_{\max})\, s$$

yielding the left hand side of (30). For the right hand side of (30) we observe for $s \geq 0$ and ε as assumed

$$b_\varepsilon + \varepsilon^2 s^2 - (f'(c_m) + L_{\min})\, s \leq \frac{\hat{C}^2}{\varepsilon^2} \ (1 + s^2) \ .$$

These estimates immediately imply the assertions of the lemma.

□

Since according to the above lemma the fractional power space $X^{1/2,\varepsilon}$ algebraically does not depend on ε, we will omit the superscript ε in the following and simply write $X^{1/2} = H^2_{av}(\Omega)$. Also, Lemma 3.3 shows that we may choose either the standard $H^2(\Omega)$-norm or the norm $||\cdot||_*$ on the fractional power space $X^{1/2}$, since both of them are equivalent to the graph norm $||\cdot||_{1/2,\varepsilon}$ of $(-B_\varepsilon + b_\varepsilon I)^{1/2}$. It will turn out to be very convenient to choose the norm $||\cdot||_*$ as a (non-standard) norm on $X^{1/2}$. Thus, from now on we identify the space $(X^{1/2}, ||\cdot||_{1/2,\varepsilon})$ with the space $(H^2_{av}(\Omega), ||\cdot||_*)$.

3.2 Spectral gaps and exponential dichotomy estimates

The following lemma proves the existence of suitable spectral gaps in the spectrum of B_ε. They will be used to define the decomposition of X mentioned in (H2), and therefore eventually furnish the dichotomy estimates, see Lemma 3.8(b). The size of these gaps turns out to be crucial, because it provides a restriction on the possible size of the global Lipschitz constant of the nonlinearity in (H3).

Lemma 3.4 *Assume that (A1) and (A3) are satisfied, so that in particular we have $d \in \{1,2,3\}$. Furthermore, fix two constants $v_* < v^* < 1$ and let $\lambda_\varepsilon^{\max}$ be defined as in (29).*

Then there exist constants $\varepsilon_0, \mu_0 > 0$ depending only on v_*, v^*, Ω, and $f'(c_m) + L_{\max}$ such that for arbitrary $0 < \varepsilon \le \varepsilon_0$ the following holds. The linear operator B_ε has eigenvalues $\lambda_*(\varepsilon)$ and $\lambda^*(\varepsilon)$ satisfying both

$$v_* \cdot \lambda_\varepsilon^{\max} \;\le\; \lambda_*(\varepsilon) \;<\; \lambda^*(\varepsilon) \;\le\; v^* \cdot \lambda_\varepsilon^{\max}$$

and

$$\lambda^*(\varepsilon) - \lambda_*(\varepsilon) \ge \mu_0 \cdot \varepsilon^{d-2} \; .$$

Moreover, the whole interval $(\lambda_*(\varepsilon), \lambda^*(\varepsilon))$ is part of the resolvent set of B_ε. In fact, both the interval $(v_* \cdot \lambda_\varepsilon^{\max}, \lambda_*(\varepsilon))$ and $(\lambda^*(\varepsilon), v^* \cdot \lambda_\varepsilon^{\max})$ contains eigenvalues of the form $\lambda_{\kappa,\varepsilon} \in \Lambda(\theta)$ if

$$\lambda_{\varepsilon,\theta}^{\max} \;:=\; \max\{s(f'(c_m) + L(\theta) - \varepsilon^2 s) \;:\; s \ge 0\} \;\ge\; v^* \cdot \lambda_\varepsilon^{\max} \; .$$

In particular, in the above intervals are always eigenvalues from $\Lambda(\theta_{\max})$.

Proof: Fix two constants $v_* < v_{**} < v^{**} < v^*$. Due to Lemma 3.1, especially (28), we can choose $\varepsilon_0 > 0$ small enough such that for all $0 < \varepsilon \le \varepsilon_0$ the following two assertions hold.

1. Both in the interval $[v^{**}, v^*] \cdot \lambda_\varepsilon^{\max}$ and in $[v_*, v_{**}] \cdot \lambda_\varepsilon^{\max}$ there is at least one eigenvalue of B_ε. Let $\lambda^{**}(\varepsilon)$ denote the smallest eigenvalue in the first interval, and $\lambda_{**}(\varepsilon)$ the largest one in the second interval.
2. The number of eigenvalues of B_ε in the interval $(v_{**}, v^{**}) \cdot \lambda_\varepsilon^{\max}$ is bounded above by $C \cdot \varepsilon^{-d} - 1$, where C depends only on v_*, v^*, Ω, and $f'(c_m) + L_{\max}$.

Let $\mu_0 := (f'(c_m) + L_{\max})^2 (v^{**} - v_{**})/(4C)$ and assume that any two consecutive eigenvalues of the operator B_ε in the interval $[\lambda_{**}(\varepsilon), \lambda^{**}(\varepsilon)]$ are strictly less than $\mu_0 \cdot \varepsilon^{d-2}$ apart. Then we get

$$\lambda^{**}(\varepsilon) - \lambda_{**}(\varepsilon) < \mu_0 \cdot \varepsilon^{d-2} \cdot C \cdot \varepsilon^{-d} = \frac{(f'(c_m) + L_{\max})^2 \, (v^{**} - v_{**})}{4\varepsilon^2} \; .$$

This however contradicts the fact that

$$\lambda^{**}(\varepsilon) - \lambda_{**}(\varepsilon) \ge (v^{**} - v_{**}) \cdot \lambda_\varepsilon^{\max} = \frac{(f'(c_m) + L_{\max})^2 \, (v^{**} - v_{**})}{4\varepsilon^2} \; ,$$

and the proof of the lemma is complete.

□

Loosely speaking, the above lemma states that it is possible to find gaps in the spectrum of B_ε whose size can be controlled as $\varepsilon \to 0$. We will use this fact to obtain the spectral gaps needed for assumption (H2). To this end, choose constants

$$\underline{v}^{--} < \overline{v}^{--} \ll 0 \ll \underline{v}^{-} < \overline{v}^{-} < \underline{v}^{+} < \overline{v}^{+} < 1 , \tag{33}$$

where typically the differences $\overline{v}^{--} - \underline{v}^{--}$, $\overline{v}^{-} - \underline{v}^{-}$, and $\overline{v}^{+} - \underline{v}^{+}$ will be small. Using these constants the following results are an immediate consequence of Lemma 3.4. They give the appropriate choices of the spectral gaps for the operators B_ε needed for our application.

Corollary 3.5 *Assume that all the assumptions of Lemma 3.4 are satisfied. Then with the constants from (33) there exist intervals*

$$\begin{array}{rcccl} \hat{J}_\varepsilon^{--} &:=& [\hat{a}_\varepsilon^{--}, b_\varepsilon^{--}] &\subset& [\underline{v}^{--}, \overline{v}^{--}] \cdot \lambda_\varepsilon^{\max} , \\ J_\varepsilon^{-} &:=& [a_\varepsilon^{-}, b_\varepsilon^{-}] &\subset& [\underline{v}^{-}, \overline{v}^{-}] \cdot \lambda_\varepsilon^{\max} , \\ J_\varepsilon^{+} &:=& [a_\varepsilon^{+}, b_\varepsilon^{+}] &\subset& [\underline{v}^{+}, \overline{v}^{+}] \cdot \lambda_\varepsilon^{\max} \end{array}$$

such that for sufficiently small $\varepsilon > 0$ the following holds.

(a) Each of the intervals $\hat{J}_\varepsilon^{--}$, J_ε^{-}, J_ε^{+} is contained in the resolvent set of B_ε.
(b) If we define $a_\varepsilon^{--} := (\hat{a}_\varepsilon^{--} + b_\varepsilon^{--})/2$ and let

$$J_\varepsilon^{--} := [a_\varepsilon^{--}, b_\varepsilon^{--}] \subset \hat{J}_\varepsilon^{--} \subset [\underline{v}^{--}, \overline{v}^{--}] \cdot \lambda_\varepsilon^{\max} ,$$

then there is an ε-independent constant $\mu_1 > 0$ such that the length of each of the intervals J_ε^{--}, J_ε^{-}, and J_ε^{+} is at least $\mu_1 \cdot \varepsilon^{d-2}$. The constant μ_1 depends only on Ω, $f'(c_m) + L_{\max}$, and the constants in (33).
(c) The interval $[\underline{v}^{--} \cdot \lambda_\varepsilon^{\max}, a_\varepsilon^{--})$ is not contained in the resolvent set of B_ε, i.e., this interval contains at least one eigenvalue of B_ε.

Using this result, we can now define the subspace decomposition of X needed for applying the results of [15].

Definition 3.6 *Using the constants introduced in Corollary 3.5, define the intervals $I_\varepsilon^{--} := (-\infty, a_\varepsilon^{--})$, $I_\varepsilon^{-} := (b_\varepsilon^{--}, a_\varepsilon^{-})$, $I_\varepsilon^{+} := (b_\varepsilon^{-}, a_\varepsilon^{+})$, and $I_\varepsilon^{++} := (b_\varepsilon^{+}, \lambda_\varepsilon^{\max}]$. Furthermore, let X_ε^{--}, X_ε^{-}, X_ε^{+}, and X_ε^{++} denote the sum of all eigenspaces of the operator B_ε corresponding to eigenvalues $\lambda_{\kappa,\varepsilon}$ in I_ε^{--}, I_ε^{-}, I_ε^{+}, and I_ε^{++}, respectively.*

For further reference we define

Definition 3.7 *For given $\theta \in P\Lambda$, $\varepsilon > 0$, we denote by $\lambda_{i,\theta,\varepsilon}$, $i \in \mathbb{N}$ the ordered eigenvalues $\lambda_{\kappa,\varepsilon}$ of B_ε lying in $\Lambda(\theta)$. Hence we have $\lambda_{1,\theta,\varepsilon} \geq \lambda_{2,\theta,\varepsilon} \geq \ldots \to -\infty$. The corresponding eigenfunctions $\psi_{i,\theta,\varepsilon}, i \in \mathbb{N}$ of B_ε are obtained form the eigenfunctions φ_κ through this ordering procedure in the obvious way. Furthermore, denote by $\tilde{\kappa}_{i,\theta,\varepsilon} \in \Lambda, i \in \mathbb{N}$ the related wave vector κ of $\psi_{i,\theta,\varepsilon}$. Note that $|\tilde{\kappa}_{i,\theta,\varepsilon}|$ is monotonically increasing for increasing i in the region $|\tilde{\kappa}_{i,\theta,\varepsilon}|^2 > (f'(c_m) + L(\theta))\,/\varepsilon^2$.*

The restrictions of B_ε or of the corresponding (linear) analytic semigroup $S_\varepsilon(t)$ to each of the subspaces defined above will be denoted by the appropriate superscript. With these definitions we can proceed to verifying the two hypotheses (H1) and (H2) for the linearization. Again we use the non-standard norm $||\cdot||_*$ (which is equivalent to the norm $||\cdot||_{1/2,\varepsilon}$) on $X^{1/2} = H^2_{av}(\Omega)$.

Lemma 3.8 *Assume that (A1) and (A3) hold. Let $-B_\varepsilon : X \to X$ denote the self-adjoint and sectorial operator defined in Lemma 3.1, let $S_\varepsilon(t) : X \to X$, $t \geq 0$, denote the corresponding analytic semigroup, and let $X^{1/2} = H^2_{av}(\Omega)$ denote the fractional power space of Subsection 3.1 with norm $||\cdot||_*$. Furthermore, consider the constants and intervals introduced in Corollary 3.5 and Definition 3.6. Then the following assertions hold for arbitrary $0 < \varepsilon \leq \varepsilon_0$, where ε_0 depends only on the domain Ω, the constant $f'(c_m) + L_{\max}$, and the constants in (33).*

(a) The subspaces X_ε^-, X_ε^+, and X_ε^{++} are finite-dimensional subspaces of $X^{1/2}$, and their dimensions are proportional to ε^{-d}, where d denotes the dimension of the domain Ω. Furthermore, the spaces X_ε^{--}, X_ε^-, X_ε^+, and X_ε^{++} are orthogonal with respect to the $L^2(\Omega)$-scalar product $(\cdot,\cdot)$, and their restrictions to $X^{1/2}$ are orthogonal with respect to $(\cdot,\cdot)_*$.

(b) There exists a constant $M_\varepsilon^{--} > 0$ such that for every $c^{++} \in X_\varepsilon^{++}$, $c^+ \in X_\varepsilon^+$, $c^- \in X_\varepsilon^-$, $c_*^{--} \in X_\varepsilon^{--} \cap X^{1/2}$, and $c^{--} \in X_\varepsilon^{--}$ the estimates

$$\begin{aligned}
||S_\varepsilon^{++}(t)c^{++}||_* &\leq e^{b_\varepsilon^+ t} \cdot ||c^{++}||_* && \text{for } t \leq 0\,, \\
||S_\varepsilon^{+}(t)c^{+}||_* &\leq e^{a_\varepsilon^+ t} \cdot ||c^{+}||_* && \text{for } t \geq 0\,, \\
||S_\varepsilon^{+}(t)c^{+}||_* &\leq e^{b_\varepsilon^- t} \cdot ||c^{+}||_* && \text{for } t \leq 0\,, \\
||S_\varepsilon^{-}(t)c^{-}||_* &\leq e^{a_\varepsilon^- t} \cdot ||c^{-}||_* && \text{for } t \geq 0\,, \\
||S_\varepsilon^{-}(t)c^{-}||_* &\leq e^{b_\varepsilon^{--} t} \cdot ||c^{-}||_* && \text{for } t \leq 0\,, \\
||S_\varepsilon^{--}(t)c_*^{--}||_* &\leq e^{a_\varepsilon^{--} t} \cdot ||c_*^{--}||_* && \text{for } t \geq 0\,, \\
||S_\varepsilon^{--}(t)c^{--}||_* &\leq M_\varepsilon^{--} \cdot t^{-1/2} \cdot e^{a_\varepsilon^{--} t} \cdot ||c^{--}|| && \text{for } t > 0
\end{aligned}$$

hold, and

$$M_\varepsilon^{--} \leq C \cdot \varepsilon^{-(1+d/2)} \quad \text{as} \quad \varepsilon \to 0\,,$$

where $C > 0$ depends only on Ω, $f'(c_m) + L_{\max}$, and the constants in (33). Note that due to the finite dimensions of the subspaces X_ε^-, X_ε^+, and X_ε^{++} the semigroups $S_\varepsilon^-(t)$, $S_\varepsilon^+(t)$, and $S_\varepsilon^{++}(t)$ can be extended to groups.

(c) There exists a constant $M_{1/2,\varepsilon} \geq 1$ *such that for all* $c \in X_\varepsilon^- \oplus X_\varepsilon^+ \oplus X_\varepsilon^{++}$ *we have*

$$\frac{1}{M_{1/2,\varepsilon}} \cdot ||c|| \leq ||c|| \leq ||c||_* \leq M_{1/2,\varepsilon} \cdot ||c|| \ ,$$

and

$$M_{1/2,\varepsilon} \cdot \varepsilon^2 \to C \quad \text{as} \quad \varepsilon \to 0 \ ,$$

where $C > 0$ *depends only on* $f'(c_m) + L_{\max}$*, and the constants in (33).*

Proof: The assertions of *(a)* follow easily from Lemma 3.1, (28), Corollary 3.5, and Definition 3.6.
As for the proof of *(b)*, let $c \in X$ be arbitrary, let $\psi_{i,\theta,\varepsilon}$ denote the eigenfunctions of B_ε according to Definition 3.7, and let $c = \sum_{\theta \in P\Lambda} \sum_{i=1}^\infty \xi_{i,\theta,\varepsilon}\psi_{i,\theta,\varepsilon}$ denote the Fourier series representation of c in X, i.e., let $\xi_{i,\theta,\varepsilon} := (c, \psi_{i,\theta,\varepsilon})$, where $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$-scalar product. Then we have an explicit spectral representation of the semigroup $S_\varepsilon(t)$ given by

$$S_\varepsilon(t)c = \sum_{\theta \in P\Lambda} \sum_{i=1}^\infty e^{\lambda_{i,\theta,\varepsilon} \cdot t} \cdot \xi_{i,\theta,\varepsilon} \cdot \psi_{i,\theta,\varepsilon} \qquad \text{for} \qquad t > 0 \ ,$$

and if $c \in X^{1/2}$, then (32) furnishes

$$\|c\|_*^2 = \sum_{\theta \in P\Lambda} \sum_{i=1}^\infty \left(1 + |\tilde{\kappa}_{i,\theta,\varepsilon}|^4\right) \cdot \xi_{i,\theta,\varepsilon}^2 \cdot \|\psi_{i,\theta,\varepsilon}\|^2 < \infty \ .$$

These two identities already imply the first six inequalities in part *(b)*. For example, let $\lambda_{1,\theta,\varepsilon} \geq \ldots \geq \lambda_{n_0(\theta),\theta,\varepsilon}$, for $\theta \in P\Lambda$, denote all eigenvalues of B_ε in the interval I_ε^{++}, where $n_0(\theta) = 0$ is possible. Then an element $c^{++} \in X_\varepsilon^{++}$ has the Fourier series representation $c^{++} = \sum_{\theta \in P\Lambda} \sum_{i=1}^{n_0(\theta)} \xi_{i,\theta,\varepsilon}\psi_{i,\theta,\varepsilon}$, and for every $t \leq 0$ we obtain

$$\begin{aligned} \|S_\varepsilon^{++}(t)c^{++}\|_*^2 &= \sum_{\theta \in P\Lambda} \sum_{i=1}^{n_0(\theta)} \left(1 + |\tilde{\kappa}_{i,\theta,\varepsilon}|^4\right) \cdot e^{2\lambda_{i,\theta,\varepsilon} \cdot t} \cdot \xi_{i,\theta,\varepsilon}^2 \cdot \|\psi_{i,\theta,\varepsilon}\|^2 \\ &\leq \sum_{\theta \in P\Lambda} e^{2\lambda_{n_0(\theta),\theta,\varepsilon} \cdot t} \cdot \sum_{i=1}^{n_0(\theta)} \left(1 + |\tilde{\kappa}_{i,\theta,\varepsilon}|^4\right) \cdot \xi_{i,\theta,\varepsilon}^2 \cdot \|\psi_{i,\theta,\varepsilon}\|^2 \\ &\leq e^{2b_\varepsilon^+ \cdot t} \cdot \|c^{++}\|_*^2 \ , \end{aligned}$$

since $\lambda_{n_0(\theta),\theta,\varepsilon} \in I_\varepsilon^{++} = (b_\varepsilon^+, \lambda_\varepsilon^{\max}]$ according to Definition 3.6. The remaining five of the first six inequalities follow analogously.

In order to prove the seventh inequality, let $c^{--} \in X_\varepsilon^{--}$ be arbitrary. If $n_1(\theta) \geq 1$ is chosen in such a way that $\lambda_{i,\theta,\varepsilon}$ for $i \geq n_1(\theta)$ and $\theta \in P\Lambda$ denote all the eigenvalues of B_ε which are contained in I_ε^{--}, then c^{--} has the Fourier series representation $c^{--} = \sum_{\theta \in P\Lambda} \sum_{i=n_1(\theta)}^\infty \xi_{i,\theta,\varepsilon}\psi_{i,\theta,\varepsilon}$ in X, and for arbitrary $t > 0$ we actually have $S_\varepsilon^{--}(t)c^{--} \in X^{1/2}$. Due to the choice of the interval I_ε^{--} one further obtains $\lambda_{i,\theta,\varepsilon} \le \lambda_{n_1(\theta),\theta,\varepsilon} < 0$ for all $i \ge n_1(\theta)$, and therefore $\tilde{\kappa}_{i,\theta,\varepsilon} > \beta_\theta/\varepsilon^2$ for all $i \ge n_1(\theta)$, where we set $\beta_\theta = f'(c_m) + L(\theta)$. Thus,

$$||S_\varepsilon^{--}(t)c^{--}||_*^2 = \sum_{\theta \in P\Lambda} \sum_{i=n_1(\theta)}^{\infty} \left(1 + |\tilde{\kappa}_{i,\theta,\varepsilon}|^4\right) \cdot e^{2\lambda_{i,\theta,\varepsilon} \cdot t} \cdot \xi_{i,\theta,\varepsilon}^2 \cdot \|\psi_{i,\theta,\varepsilon}\|^2 .$$

Now it is easy to verify that for all $t > 0$, $i \ge n_1(\theta)$, and $\lambda > \lambda_{n_1(\theta),\theta,\varepsilon}$ we have

$$\begin{aligned} \left(1 + |\tilde{\kappa}_{i,\theta,\varepsilon}|^4\right) \cdot e^{2\lambda_{i,\theta,\varepsilon} \cdot t} &\le \frac{1 + |\tilde{\kappa}_{i,\theta,\varepsilon}|^4}{2e(\lambda - \lambda_{i,\theta,\varepsilon})} \cdot t^{-1} \cdot e^{2\lambda \cdot t} \\ &= \frac{1 + |\tilde{\kappa}_{i,\theta,\varepsilon}|^4}{2e\left(\lambda - \beta_\theta |\tilde{\kappa}_{i,\theta,\varepsilon}|^2 + \varepsilon^2 |\tilde{\kappa}_{i,\theta,\varepsilon}|^4\right)} \cdot t^{-1} \cdot e^{2\lambda \cdot t} . \end{aligned}$$

To continue, define the function $h(s) = (1 + s^2) \cdot (\lambda - \beta s + \varepsilon^2 s^2)^{-1}$, where $\lambda = a_\varepsilon^{--} < 0$, $s \in Z := \{s > 0 \ : \ \lambda - \beta s + \varepsilon^2 s^2 > 0\}$, and $\beta = \beta_\theta$ for an arbitrary $\theta \in P\Lambda$. Assume first that θ is such that $\beta_\theta \ge 0$. Then h is monotonically decreasing on Z, and this implies

$$\begin{aligned} C_{i,\theta,\varepsilon}^2 := \frac{1 + |\tilde{\kappa}_{i,\theta,\varepsilon}|^4}{2e(\lambda - \beta_\theta |\tilde{\kappa}_{i,\theta,\varepsilon}|^2 + \varepsilon^2 |\tilde{\kappa}_{i,\theta,\varepsilon}|^4)} &\le \frac{1 + |\tilde{\kappa}_{n_1(\theta),\theta,\varepsilon}|^4}{2e(\lambda - \lambda_{n_1(\theta),\theta,\varepsilon})} \\ \le \frac{1 + |\tilde{\kappa}_{n_1(\theta_{\max}),\theta_{\max},\varepsilon}|^4}{2e(a_\varepsilon^{--} - \hat{a}_\varepsilon^{--})} \quad &=: \ \left(\widetilde{M}_\varepsilon^{--}\right)^2 . \end{aligned}$$

If $\beta_\theta < 0$, then h switches exactly once from decreasing to increasing in Z, and this yields

$$C_{i,\theta,\varepsilon} \le \max\left\{\widetilde{M}_\varepsilon^{--}, \varepsilon^{-1}\right\} =: M_\varepsilon^{--} .$$

This proves the seventh estimate in (b). Finally, using a calculation which was already employed in the proof of Lemma 3.6(b) in [15] and which uses the estimate $\lambda_{n_1(\theta_{\max}),\theta_{\max},\varepsilon} \ge \underline{v}^{--} \cdot \lambda_\varepsilon^{\max} = \underline{v}^{--} \cdot (\beta_{\theta_{\max}})^2/(4\varepsilon^2)$ from Corollary 3.5(c), we obtain

$$M_\varepsilon^{--} \le \varepsilon^{-(1+d/2)} \cdot \max\left\{\varepsilon^{d/2}, \sqrt{\frac{\varepsilon^4 + (\beta_{\theta_{\max}})^2 \cdot (1 + \sqrt{1 - \underline{v}^{--}})^2/4}{\mu_1}}\right\} ,$$

where $\mu_1 > 0$ is defined in Corollary 3.5(b). This finally proves the asymptotic behavior of the constant M_ε^{--} for $\varepsilon \to 0$.

The proof of part *(c)* follows the lines of the proof of Lemma 3.6(c) in [15] and is therefore omitted.

□

According to the above lemma the linear part of the Cahn-Hilliard equation with elasticity (24) satisfies (with respect to the non-standard norm $\|\cdot\|_*$) both (H1) and (H2), as well as (7) from Section 2 in [15]. Moreover, the asymptotic behavior for $\varepsilon \to 0$ of certain spectral gaps in the spectrum of B_ε, and of the constants M_ε^{--} and $M_{1/2,\varepsilon}$ have been obtained. We close this subsection with the following remark.

Remark 3.9 With the results of Corollary 3.5 and Lemma 3.8 we can deduce the asymptotic behavior of certain constants introduced in [15, Section 2] for $\varepsilon \to 0$. Although the specific values of these constants are different if compared to the application in [15, Section 3], their dependence on ε is not. Hence, we obtain exactly the same asymptotics, i.e., $C_\varepsilon^{--} \geq C \cdot \varepsilon^d$ and $C_\varepsilon^{+} \geq C \cdot \varepsilon^{d-2}$ for $\varepsilon \to 0$, see Remark 3.7 in [15].

Even though we did not formally introduce these constants, we want to point out that their asymptotic behavior is used to prove that the abstract theory of Section 2 in [15] can be applied to B_ε and a nonlinear function F, whose Lipschitz constant satisfies $0 \leq L_F \leq C \cdot \varepsilon^d$. For more details see [15, Remark 2.11]. All of the constants C above depend only on Ω, $f'(c_m) + L_{\max}$, and the constants in (33).

3.3 Properties of the nonlinearity

Hypothesis (H3) is valid for some function $\hat{F} : H^2_{av}(\Omega) \to L^2(\Omega)$ which coincides with F from (26) on a certain neighborhood of the origin. In order to obtain a global Lipschitz constant $L_{\hat{F}}$ of the order ε^d (as required by Remark 3.9), the size of this neighborhood has to be proportional to ε^d with respect to the $H^2(\Omega)$-norm. This can be proved by applying the results from [15, Section 3.3] to $\tilde{f}$ and F. This immediately furnishes the following result.

Corollary 3.10 *The nonlinear operator F defined in (26) satisfies (H3) with a Lipschitz constant L_F of the order ε^d on an $H^2(\Omega)$-neighborhood of 0 with size proportional to ε^d.*

3.4 Spinodal decomposition

In the previous subsections we established all properties of (24) which are necessary to apply the abstract results of [15, Section 2] to the Cahn–Hilliard equation with elasticity — and this can be done exactly as in Subsection 3.4 of [15]. Moreover, since the asymptotic behavior of the involved constants remains basically unchanged, we obtain exactly the same result, of course after adopting the new notation (c_m instead of μ, B_ε instead of A_ε and so on).

Therefore, we refrain from presenting our main theorem again in as detailed a form as in the binary case, and state only an intuitive abbreviated version.

Suppose that three constants $0 < r \ll \rho \ll R$ are given. We consider initial conditions from the ball

$$B_r(c_m) \;=\; \{v \in c_m + X_\varepsilon^- \oplus X_\varepsilon^+ \oplus X_\varepsilon^{++} : \; \|v - c_m\|_* < r\} \;\subset\; c_m + H^2_{av}(\Omega)$$

and their evolution under the dynamics of (24). Let M_r denote the set of all initial conditions $v \in B_r(c_m)$ whose corresponding solution of (24) either remains in the larger ball $B_R(c_m)$ for all time, or has distance greater than ρ from $X_\varepsilon^+ \oplus X_\varepsilon^{++}$ upon exiting $B_R(c_m)$. See also Figure 6 in [15]. In other words, the initial conditions in M_r cannot be considered as being dominated by the strongly unstable subspace $\mathcal{Y}_\varepsilon^+ := X_\varepsilon^+ \oplus X_\varepsilon^{++}$.

Our main theorem states that the volume (which is the canonical Lebesgue volume of the finite-dimensional space $X_\varepsilon^- \oplus X_\varepsilon^+ \oplus X_\varepsilon^{++}$) of these "bad" initial conditions compared to the volume of all initial conditions in $B_r(c_m)$ is arbitrarily small, provided the constants $0 < r \ll \rho \ll R$ are chosen proportional to ε^d as $\varepsilon \to 0$.

Theorem 3.11 *We consider solutions of the Cahn–Hilliard equation with elasticity (24) and assume that hypotheses (A1), (A2), and (A3) are satisfied. Then there exists a positive constant ε_0 which depends only on Ω, $f'(c_m) + L_{\max}$, $|f'(c_m) + L_{\min}|$, and the constants in (33), such that for arbitrary $0 < \varepsilon \le \varepsilon_0$ the following holds.*

For every $0 < p \ll 1$ there exist constants $0 < r \ll \rho \ll R$ which depend only on $f'(c_m) + L_{\max}$, Ω, and the constants in (33) (r depends additionally on p) and which are all proportional to ε^d as $\varepsilon \to 0$, such that

$$\frac{\mathrm{vol}(M_r)}{\mathrm{vol}(B_r(c_m))} \;\le\; p\,. \tag{34}$$

Proof: One only has to apply the abstract theory of Section 2 in [15]. Hypotheses (H1) through (H3) have been established in Subsections 3.1 through 3.3, and the necessary constants have been calculated, furnishing the ball with size proportional to ε^d on which the result is valid. □

Remark 3.12 This theorem guarantees that the initial conditions near c_m are dominated by the subspace $\mathcal{Y}_\varepsilon^+$. This is why we call $\mathcal{Y}_\varepsilon^+$ the dominating subspace. Since the nonlinearity F is exactly the same as in [15] we expect that also the results of Sander and Wanner [19] on second phase spinodal decomposition hold true for the Cahn-Hilliard equation with elasticity.

4 Numerical simulations

Finally, we would like to show that the patterns predicted by our analysis in Sections 2 and 3 are in fact observed in typical solutions having initial data close to an unstable homogeneous state c_m, i.e., $\psi_{,cc}(c_m) - L_{\max} = -(f'(c_m) + L_{\max}) < 0$. We made a series of numerical simulations based on a finite element method developed in Garcke, Rumpf, Weikard [10] and in this section we will present typical patterns seen in the numerical experiments. We point out that in [10] optimal error estimates are established, i.e. the method used is very reliable.

Before we present the numerical results, let us remark that the patterns shown are generic in the sense that the initial data have to be degenerate in a certain sense in order not to lead to similar patterns. In all simulations we were choosing $c_m = 0$ and were taking a random perturbation around c_m as initial data. All calculations have been performed on the unit square with $\varepsilon = 10^{-3}$, $\psi(c) = \frac{1}{4}(c^2 - 0.16)^2$ and the solutions are shown at time $t = 0.001$.

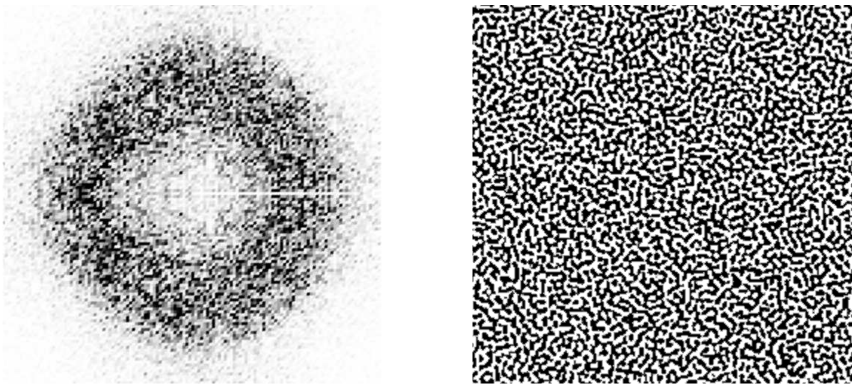

Fig. 6. The case without elasticity; modulus of the Fourier coefficients (left) and sign of the concentration difference c (right)

At first we consider the case without elasticity. The left hand side of Figure 6 shows the modulus of the Fourier coefficients of a solution to the Cahn-Hilliard equation during spinodal decomposition after a fixed time. On the right the sign of the concentration difference c (black denoting positive, white denoting negative values) is shown. The Fourier coefficients were calculated using the FFTW-software package (see http://www.fftw.org for details). Figure 6 corresponds to Figure 1 in that neither in the Fourier coefficients nor in the concentration picture there are any distinguished directions reflecting the isotropy of the Cahn-Hilliard model without elasticity. Essentially the same picture occurs in the case of isotropic elasticity.

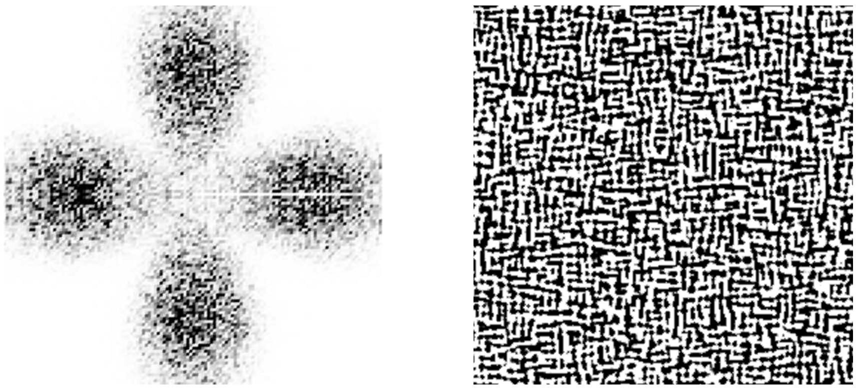

Fig. 7. Negative anisotropy of the elasticity tensor; modulus of the Fourier coefficient (left) and sign of the concentration difference c (right)

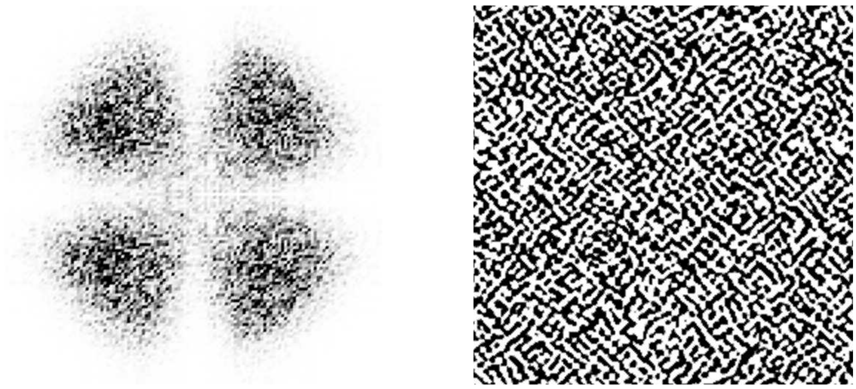

Fig. 8. Positive anisotropy of the elasticity tensor; modulus of the Fourier coefficient (left) and sign of the concentration difference c (right)

In the Figures 7 and 8 we show numerical results with an anisotropic elasticity tensor. Parameters were $C_{11} = 10$, $C_{12} = 1$, $C_{44} = 1$ and $q = \frac{\sqrt{2}}{10}$ (this implies $L_{max} \approx -0.068$, $L_{min} = -0.2$ and $f'(c_m) = 0.16$) for the case with positive anisotropy and $C_{11} = 2$, $C_{12} = 1$, $C_{44} = 100$ and $q = 0.2$ (i.e., $L_{max} = -0.06$, $L_{min} \approx -0.236$ and $f'(c_m) = 0.16$) for the case with negative anisotropy. In both cases the anisotropy is clearly visible in the Fourier coefficients as well as in the concentration.

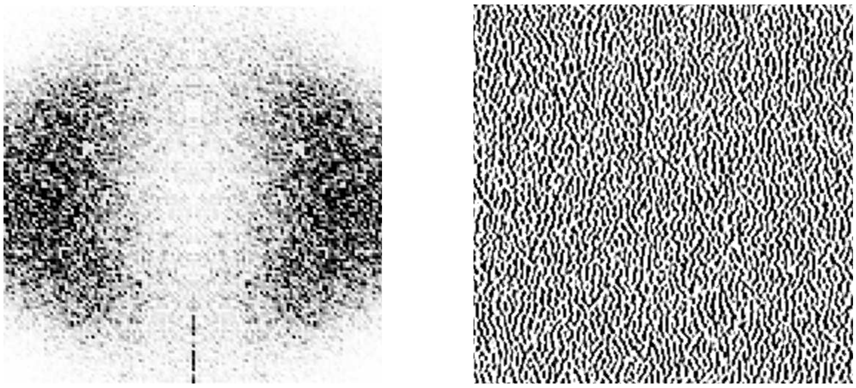

Fig. 9. Anisotropic eigenstrain and isotropic elasticity tensor; modulus of the Fourier coefficient (left) and sign of the concentration difference c (right)

Finally we consider the case with isotropic elasticity tensor but anisotropic eigenstrains (see Figure 9). The parameter were $\lambda = 1$, $\mu = \frac{1}{2}$ and $a = 1$, $b = 0.1$ (i.e., $L_{max} = -0.015$, $L_{min} = -1.5$ and $f'(c_m) = 0.16$).

Acknowledgment

This work has been started while SMP was visiting the University of Bonn as a guest of the Sonderforschungsbereich 256 "Nichtlineare partielle Differentialgleichungen" (SFB256). HG and UW have been members of the SFB256 for many years and all three authors gratefully acknowledge the support of the Sonderforschungsbereich.

References

1. H. Amann, *Linear and Quasilinear Parabolic Problems. Volume I: Abstract Linear Theory,* Birkhäuser, Basel – Boston – Berlin, 1995.
2. J. W. Cahn and J. E. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy,* Journal of Chemical Physics, Vol. 28 (1958), pp. 258–267.
3. J.D. Eshelby, *The continuum theory of lattice defects,* Solid State Physics, 3 (1956), pp. 79-144.
4. P.C. Fife, H. Kielhöfer, S. Maier-Paape, and T. Wanner, *Perturbation of doubly periodic solution branches with applications to the Cahn-Hilliard equation,* Physica D, 100(3-4) (1997), pp. 257-278.
5. P. Fratzl, O. Penrose, and J. L. Lebowitz, *Modelling of phase separation in alloys with coherent elastic misfit,* J. Stat. Physics, 95 5/6 (1999), pp. 1429–1503.
6. H. Garcke, *On mathematical models for phase separation in elastically stressed solids,* habilitation thesis, Bonn, 2000.
7. H. Garcke, *On Cahn–Hilliard systems with elasticity,* submitted.

8. H. Garcke, *The Γ–limit of the elastically modified Ginzburg–Landau energy,* manuscript.
9. H. Garcke, *On a Cahn-Hilliard model for phase separation with elastic misfit,* manuscript.
10. H. Garcke, M. Rumpf, and U. Weikard, *The Cahn-Hilliard equation with elasticity: Finite element approximation and qualitative studies,* Interfaces and Free Boundaries, 3 (2001), pp. 101-118.
11. M.E. Gurtin, *An introduction to Continuum Mechanics*, Academic Press, 1981.
12. A. G. Khachaturyan, *Theory of Structural Transformations in Solids,* Wiley, New York, 1983.
13. F. C. Larché and J. W. Cahn, *The effect of self–stress on diffusion in solids,* Acta Metall., 30 (1982), pp. 1835–1845.
14. S. Maier–Paape and T. Wanner, *Spinodal decomposition for the Cahn-Hilliard equation in higher dimensions. Part I: Probability and wavelength estimate,* Comm. Math. Phys. 195 (1998), pp. 435-464.
15. S. Maier–Paape and T. Wanner, *Spinodal decomposition for the Cahn-Hilliard equation in higher dimensions. Nonlinear dynamics,* Arch. Rat. Mech. Anal., 151 (2000), pp. 187-219.
16. S. Maier-Paape, B. Stoth, and T. Wanner, *Spinodal decomposition for multicomponent Cahn–Hilliard systems,* J. Stat. Phys., Vol. 98, Nos. 3/4, 2000, pp. 871-890.
17. T. Mura, *Micromechanics of Defects in Solids,* Nijhoff, the Hague, 1982.
18. A. Onuki, *Ginzburg–Landau approach to elastic effects in the phase separation of solids*, J. Phys. Soc. Jpn., 58 (1989), pp. 3065–3068.
19. E. Sander and T. Wanner, *Unexpectedly linear behavior for the Cahn–Hilliard equation,* SIAM J. Appl. Math., 60(6) (2000), pp. 2182-2202.
20. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics,* Springer-Verlag, New York – Berlin – Heidelberg, 1988.
21. J. D. van der Waals, *The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density (in Dutch),* Verhaendel. Kronik. Akad. Weten. Amsterdam, Vol. 1, (1893), Engl. translation by J.S. Rowlinson, J. Stat. Phys., 20 (1979), pp. 197–244.