STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume ..., Number 1, March 20..

Piecewise linear interpolation revisited: BLaC-wavelets

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Abstract. The central issue of the present note is the BLaC operator, a "Blending of Linear and Constant" approach. Several properties are proved, e.g., its positivity and the reproduction of constant functions. Starting from these results, error estimates in terms of ω_1 and ω_2 are given. Furthermore, we present the degree of approximation in the bivariate tensor product case. This is applicable to image compression.

[2000 MSC] 41A15, 41A25, 41A36, 41A63

Dedicated to Professor Dr. Gh. Coman on the occasion of his 70th birthday.

1. Definitions and properties

BLaC-wavelets ("Blending of Linear and Constant wavelets") were introduced by G. P. Bonneau, S. Hahmann and G. Nielson around 1996 and constitute a tool to compromise between the perfect locality of Haar[1] wavelets and the better regularity of linear wavelets. This compromise is realized by means of a parameter $0 < \Delta \leq 1$ that will appear in the sequel. First we introduce some notations. For the real parameter $0 < \Delta \leq 1$ consider the *scaling function* $\varphi_\Delta : \mathbb{R} \to [0,1]$ given by

$$\varphi_\Delta(x) := \begin{cases} \frac{x}{\Delta}, & 0 \leq x < \Delta, \\ 1, & \Delta \leq x < 1, \\ -\frac{1}{\Delta} \cdot (x - 1 - \Delta), & 1 \leq x < 1 + \Delta, \\ 0, & \text{else.} \end{cases}$$

Remark 1.1. The two extreme situations are obtained for $\Delta = 1$ and $\Delta \to 0$, when φ_Δ reduces to B-spline functions of first order, also called *hat-functions*, and to *piecewise constant* functions, respectively. The gap in between can be smoothly covered by letting Δ be in the interval $(0, 1]$.

Furthermore, for $i = -1, \ldots, 2^n - 1,\ n \in \mathbb{N}$, we define by dilatation and translation of φ_Δ the following family of (fundamental) functions:

(1) $$\varphi_i^n(x) := \varphi_\Delta(2^n x - i), \quad x \in [0,1].$$

[1] Alfréd Haar was born in 1885 in Budapest and died 1933 in Szeged. Until after World War I he had also a chair at the University of Cluj (then Kolozsvár). More about his biography can be found on the following site: http://www-history.mcs.st-andrews.ac.uk/Mathematicians .

In Figure 1 the functions $\varphi_i^n,\ i = -1, \ldots, 2^n - 1$, with a parameter $0 < \Delta < 1$ are depicted. Notice that the support of $\varphi_0^n, \ldots, \varphi_{2^n-2}^n$ is fully inside $[0, 1]$, whereas φ_{-1}^n and $\varphi_{2^n-1}^n$ can be viewed as "incomplete".

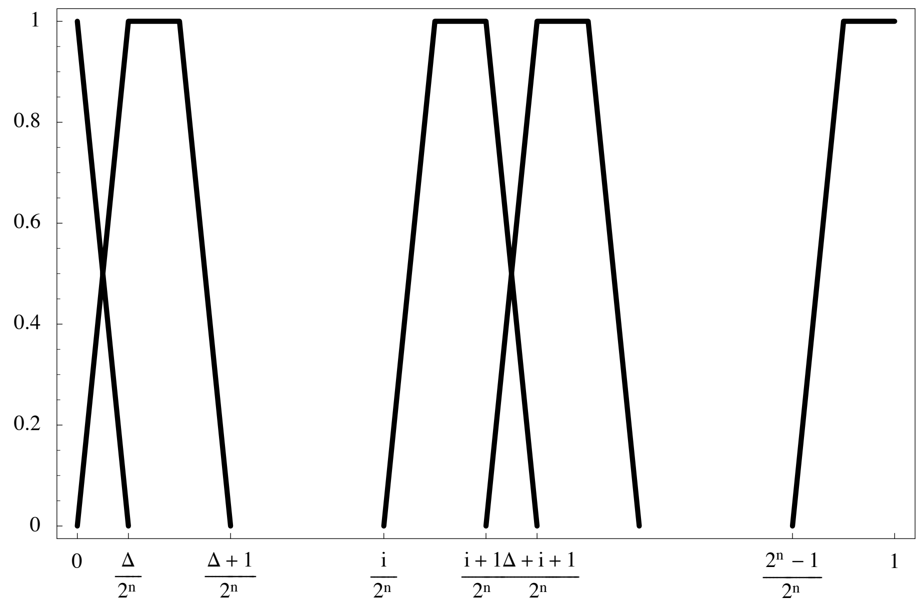

FIGURE 1

Also of great relevance are the midpoints η_i^n of the support line of each φ_i^n. Thus, for $i = 0, \ldots, 2^n - 2$, we have

$$\eta_i^n := \frac{i}{2^n} + \frac{1}{2} \cdot \frac{1 + \Delta}{2^n},$$

and for $i \in \{-1, 2^n - 1\}$ we set

$$\eta_{-1}^n := 0 \text{ and } \eta_{2^n-1}^n := 1.$$

Equipped with these notations we can introduce the following operator.

Definition 1.2. For $f \in C[0, 1]$ and $x \in [0, 1]$ the *BLaC operator* is given by

$$BL_n(f; x) := \sum_{i=-1}^{2^n-1} f(\eta_i^n) \cdot \varphi_i^n(x). \tag{2}$$

(The abbreviation BLaC refers to "Blending of Linear and Constant".)

We first list some elementary facts.

Proposition 1.3.

(i) $BL_n : C[0, 1] \to C[0, 1]$ *is positive and linear;*

(ii) BL_n *interpolates* f *at the points* $\eta_i^n,\ i = -1, \ldots, 2^n - 1$ *(thus also at the endpoints* 0 *and* 1*);*

(iii) $\sum\limits_{i=-1}^{2^n-1} \varphi_i^n(x) = 1$, *i.e.,* BL_n *reproduces constant functions.*
Hence $\|BL_n\| = 1$.

Proof. (i) This is obvious from the definition and the positivity of φ_i^n.

(ii) One can easily observe that $\varphi_i^n(\eta_j^n) = \delta_{i,j}$ (the Kronecker symbol) for $i, j = -1, \dots, 2^n - 1$. Thus $BL_n(f; \eta_j^n) = f(\eta_j^n) \cdot \varphi_j^n(\eta_j^n) = f(\eta_j^n)$, for $j = -1, \dots, 2^n - 1$.

(iii) For $x = 1$ we have $\sum\limits_{i=-1}^{2^n-1} \varphi_i^n(1) = \varphi_{2^n-1}^n(1) = 1$.

Let $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$, $k \in \{0, \dots, 2^n - 1\}$. We discuss separately:
Case 1: For $x \in \left[\frac{k}{2^n}, \frac{k+\Delta}{2^n}\right)$, we have

$$\begin{aligned}\sum_{i=-1}^{2^n-1} \varphi_i^n(x) &= \varphi_{k-1}^n(x) + \varphi_k^n(x) = \varphi_\Delta(2^n x - (k-1)) + \varphi_\Delta(2^n x - k) \\ &= -\frac{1}{\Delta}(2^n x - k - \Delta) + \frac{2^n x - k}{\Delta} = 1.\end{aligned}$$

Case 2: For $x \in \left[\frac{k+\Delta}{2^n}, \frac{k+1}{2^n}\right)$ we get $\sum\limits_{i=-1}^{2^n-1} \varphi_i^n(x) = \varphi_k^n(x) = 1$, due to the definition of φ_Δ.

Hence $\sum\limits_{i=-1}^{2^n-1} \varphi_i^n(x) = 1$ for all $x \in [0, 1]$. □

2. Degree of approximation by the BL_n operator

In the present section we investigate the degree of approximation by the BLaC-operator BL_n. The estimates are given in terms of the first and second order modulus of continuity. We use the following results given by the first author. Here and in the sequel we put $e_1(t) := t$ for $t \in [a, b]$.

Theorem 2.1. *For a positive linear operator* $L : C[a, b] \to B(Y)$, $Y \subseteq [a, b]$ *that reproduces constant functions the following inequality holds:*

$$|L(f; x) - f(x)| \le \max\left\{1, \frac{1}{\delta} \cdot L(|e_1 - x|; x)\right\} \cdot \tilde{\omega}_1(f; \delta)$$

for all $f \in C[a, b]$, $x \in Y$ *and* $\delta > 0$.
Here $\tilde{\omega}_1(f; \cdot)$ *denotes the least concave majorant of the (classical) first order modulus of continuity of* $f \in C[a, b]$.

The above theorem can be formulated for general compact spaces, this version can be found in [5] (see also [6]).
We also have

Corollary 2.2. *Under the assumptions of Theorem 2.1 there holds*

$$|L(f; x) - f(x)| \le 2 \cdot \max\left\{1, \frac{1}{\delta} \cdot L(|e_1 - x|; x)\right\} \cdot \omega_1(f; \delta),$$

where $f \in C[a,b],\ x \in Y$ *and* $\delta > 0$.

We recall here also a general quantitative result involving ω_2; such estimates were first established by H. Gonska (see [6]) and later refined by R. Păltănea (see [8] or [9]). Păltănea's result reads as follows.

Theorem 2.3. *If* Y *is a subset of* $[a,b]$, *and if* $L : C[a,b] \to B(Y)$ *is a positive linear operator satisfying* $L(e_0;x) = 1$ *for all* $x \in Y$, *then for* $f \in C[a,b],\ x \in Y$ *and* $0 < \delta < \frac{b-a}{2}$ *one has*

$$|L(f;x) - f(x)| \le |L(e_1;x) - x| \cdot \frac{1}{\delta} \cdot \omega_1(f;\delta) + \left(1 + \frac{1}{2} \cdot \frac{1}{\delta^2} L((e_1 - x)^2;x)\right) \cdot \omega_2(f;\delta).$$

We establish next two quantitative statements, one in terms of ω_1, the second one involving both ω_1 and ω_2.

Proposition 2.4. *For any* $f \in C[0,1] \to C[0,1]$ *and* $x \in [0,1]$ *there holds*

$$|BL_n(f;x) - f(x)| \le 2 \cdot \omega_1\left(f; \frac{1}{2^n}\right). \tag{3}$$

Proof. First we prove that

$$|BL_n(|e_1 - x|;x)| \le \frac{1}{2^n}, \text{ for all } x \in [0,1].$$

We have $BL_n(|e_1 - x|;x) = \sum_{i=-1}^{2^n-1} |\eta_i^n - x| \cdot \varphi_i^n(x)$. We suppose that $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right),\ k \in \{0, \ldots, 2^n - 1\}$. This excludes only $x = 1$ in which case we have $BL_n(|e_1 - 1|;1) = 0$.
Case 1: For $x \in \left[\frac{k}{2^n}, \frac{k+\Delta}{2^n}\right)$, we get

$$\begin{aligned} BL_n(|e_1 - x|;x) &= (x - \eta_{k-1}^n) \cdot \varphi_{k-1}^n(x) + (\eta_k^n - x) \cdot \varphi_k^n(x) \\ &= (x - \eta_{k-1}^n) \cdot \varphi_{k-1}^n(x) + (\eta_k^n - x) \cdot (1 - \varphi_{k-1}^n(x)) \\ &\le \max\{\eta_k^n - x, x - \eta_{k-1}^n\} \le (\eta_k^n - x + x - \eta_{k-1}^n) = \eta_k^n - \eta_{k-1}^n. \end{aligned}$$

Thus, for $k = 0$ we have $BL_n(|e_1 - x|;x) \le \eta_0^n - \eta_{-1}^n = \frac{1}{2} \cdot \frac{1+\Delta}{2^n} \le \frac{1}{2^n}$. For $k > 0$ we get $BL_n(|e_1 - x|;x) \le \eta_k^n - \eta_{k-1}^n = \frac{k}{2^n} - \frac{k-1}{2^n} = \frac{1}{2^n}$.
Case 2: $x \in \left[\frac{k+\Delta}{2^n}, \frac{k+1}{2^n}\right)$. Then

$$BL_n(|e_1 - x|;x) = |\eta_k^n - x| \cdot \varphi_k^n(x) = |\eta_k^n - x| \le \frac{1-\Delta}{2^{n+1}} \le \frac{1}{2^n}.$$

Thus $BL_n(|e_1 - x|;x) \le \frac{1}{2^n}$, for all $x \in [0,1]$. Applying Corollary 2.2 with $\delta = \frac{1}{2^n}$ yields the estimate (3). □

Proposition 2.5. *For any* $f \in C[0,1] \to C[0,1]$, all $x \in [0,1]$ *and* $0 < \delta < \frac{1}{2}$ *the following inequality holds:*

$$|BL_n(f;x) - f(x)| \le \frac{1-\Delta}{2^{n+1}} \cdot \frac{1}{\delta} \cdot \omega_1(f;\delta) + \left[1 + \frac{1}{2 \cdot \delta^2} \cdot \frac{1}{2^{2n}}\right] \cdot \omega_2(f;\delta). \tag{4}$$

Proof. In order to apply Theorem 2.3 we have to find suitable upper bounds for $BL_n(e_1 - x; x)$ and for $BL_n((e_1 - x)^2; x)$. In both cases the approach is the same as for $BL_n(|e_1 - x|; x)$. First note that $BL_n(e_1 - 1; 1) = 0$ and $BL_n((e_1 - 1)^2; 1) = 0$. We consider again two cases:
Case 1: $x \in \left[\frac{k}{2^n}, \frac{k+\Delta}{2^n}\right),\ k \in \{0, \ldots, 2^n - 1\}$.
First we deal with the case $k = 0$. Here we have

$$BL_n(e_1 - x; x) = (\eta_{-1}^n - x) \cdot \varphi_{-1}^n(x) + (\eta_0^n(x) - x) \cdot \varphi_0^n(x)$$

and after some elementary computations we obtain in this case

$$BL_n(e_1 - x; x) = \frac{x(1-\Delta)}{2\Delta} \leq \frac{\Delta}{2^n} \cdot \frac{1-\Delta}{2\Delta} = \frac{1-\Delta}{2^{n+1}}.$$

For $1 \leq k \leq 2^n - 1$ we write successively:

$$\begin{aligned}
BL_n(e_1 - x; x) &= (\eta_{k-1}^n - x) \cdot \varphi_{k-1}^n(x) + (\eta_k^n - x) \cdot \varphi_k^n(x) \\
&= \frac{1}{2^{n+1}} \cdot \frac{1}{\Delta} \left[(2k - 1 + \Delta - 2^{n+1}x)(-2^n x + k + \Delta)\right. \\
&+ \left.(2k + 1 + \Delta - 2^{n+1}x) \cdot (2^n x - k)\right] \\
&= \frac{1}{2^{n+1}} \cdot \frac{1}{\Delta} \left[(2^n x - k) \cdot (2 - 2\Delta) + \Delta(-1 + \Delta)\right] \\
&= \frac{1}{2^{n+1}} \cdot \frac{1-\Delta}{\Delta} \left[2(2^n x - k) - \Delta\right] \\
&\leq \frac{1}{2^{n+1}} \cdot \frac{1-\Delta}{\Delta} \left[2\left(2^n \cdot \frac{k+\Delta}{2^n} - k\right) - \Delta\right] = \frac{1-\Delta}{2^{n+1}}.
\end{aligned}$$

We proceed in a similar way for the second moments. Hence we get

$$\begin{aligned}
BL_n((e_1 - x)^2; x) &= (x - \eta_{k-1}^n)^2 \cdot \varphi_{k-1}^n(x) + (\eta_k^n - x)^2 \cdot \varphi_k^n(x) \\
&\leq \max\{(x - \eta_{k-1}^n)^2, (\eta_k^n - x)^2\} \leq \left(\max\{(x - \eta_{k-1}^n), (\eta_k^n - x)\}\right)^2 \\
&\leq \left(\frac{1}{2^n}\right)^2 = \frac{1}{2^{2n}}.
\end{aligned}$$

Case 2: $x \in \left[\frac{k+\Delta}{2^n}, \frac{k+1}{2^n}\right),\ k \in \{0, \ldots, 2^n - 1\}$. For the first moment we arrive at

$$|BL_n(e_1 - x; x)| \leq BL_n(|e_1 - x|; x) \leq \frac{1-\Delta}{2^{n+1}},$$

and for the second moment we have

$$BL_n((e_1 - x)^2; x) = (x - \eta_k^n)^2 \cdot \varphi_k^n(x) = (x - \eta_k^n)^2 \cdot 1 \leq \left(\frac{1-\Delta}{2^{n+1}}\right)^2 \leq \frac{1}{2^{2n}}.$$

Thus, we proved that for all $x \in [0, 1]$

$$|BL_n(e_1 - x; x)| \leq \frac{1-\Delta}{2^{n+1}} \text{ and } BL_n((e_1 - x)^2; x) \leq \frac{1}{2^{2n}}.$$

An application of Theorem 2.3 gives the statement (4). $\square$

Proposition 2.6. *For the particular choice* $\delta = \frac{1}{2^n}$, $n \geq 1$, *the estimate (4) becomes*

$$|BL_n(f;x) - f(x)| \leq \frac{(1-\Delta)}{2} \cdot \omega_1\left(f; \frac{1}{2^n}\right) + \frac{3}{2} \cdot \omega_2\left(f; \frac{1}{2^n}\right). \tag{5}$$

Remark 2.7. BL_n is an approximation operator, i.e., BL_nf converges uniformly towards f, $f \in C[0,1]$ as $n \to \infty$, see (5). For $\Delta = 1$, i.e., for *piecewise linear interpolation* at $0, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{2^n-1}{2^n}, 1$ the first term in (5) vanishes and we obtain a well-known inequality for polygonal line interpolation at the knots listed above. In fact, it was our aim to obtain for the first moments of the operator an upper bound involving the term $1 - \Delta$, in order to have it vanish for the piecewise linear interpolators.

3. Multivariate approximation

In the sequel we present statements on the degrees of approximation in the bivariate case. Only the *tensor product* case of the BL_n operators will be discussed here, but similar results can be given for Boolean sums as well. A general background on tensor products of univariate operators is provided by [2], [3] and the references cited therein. For our purposes we employ a convenient inheritance theorem that can be found in [1].

The quantitative results will be given in terms of *partial and total moduli of smoothness* of order r, $r \in \{1,2\}$, defined on compact intervals $I, J \subset \mathbb{R}$, for $f \in C(I \times J)$ and $\delta \geq 0$. We recall here their definitions.

$$\omega_r(f;\delta,0) := \sup\left\{\left|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu} \cdot f(x+\nu h, y)\right| : (x,y),(x+rh,y) \in I \times J, |h| \leq \delta\right\}$$

and

$$\omega_r(f;0,\delta) := \sup\left\{\left|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu} \cdot f(x, y+\nu h)\right| : (x,y),(x,y+rh) \in I \times J, |h| \leq \delta\right\}.$$

The total moduli of smoothness are

$$\omega_r(f;\delta_1,\delta_2) := \sup\left\{\left|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu} \cdot f(x+\nu h_1, y+\nu h_2)\right| : (x,y),(x+rh_1,y+rh_2) \in I \times J, |h_1| \leq \delta_1, |h_2| \leq \delta_2\right\}.$$

Remark 3.1. The following relation holds between the two types of moduli

$$\{\omega_r(f;\delta_1,0),\ \omega_r(f;0,\delta_2)\} \leq \omega_r(f;\delta_1,\delta_2). \tag{6}$$

The inheritance principle mentioned involves discretely defined operators $L : C(I) \to C(I')$ and $M : C(J) \to C(J')$, where $I' \subseteq I, J' \subseteq J$ are non-trivial compact intervals of the real axis $\mathbb{R}$, and their *parametric extensions* to $C(I \times J)$. L and M are defined on finitely many, mutually distinct points $x_e,\ e \in E$, and $y_f, f \in F$, (with suitable index sets E and F), and have the form

$$\begin{aligned} L(g;x) &= \sum_{e\in E} g(x_e)\cdot A_e(x), \\ M(h;y) &= \sum_{f\in F} h(y_f)\cdot B_f(y), \end{aligned}$$

with $A_e \in C(I')$ and $B_f \in C(J')$ as fundamental functions. Consequently, their *parametric extensions* to $C(I\times J)$ are given by

$$\begin{aligned} {}_xL(f;x,y) &= L(f_y;x) = \sum_{e\in E} f_y(x_e)\cdot A_e(x) = \sum_{e\in E} f(x_e,y)\cdot A_e(x), \\ {}_yM(f;x,y) &= M(f_x;y) = \sum_{f\in F} f_x(y_f)\cdot B_f(y) = \sum_{f\in F} f(x,y_f)\cdot B_f(y), \end{aligned}$$

with $f \in C(I\times J)$ and $(x,y)\in I\times J$.

For discretely defined operators we have the following representation of the tensor product of L and M

$$({}_xL\circ {}_yM)(f;x,y) = \sum_{e\in E}\sum_{f\in F} f(x_e,y_f)\cdot A_e(x)\cdot B_f(y),\ f\in C(I\times J) \tag{7}$$

(and similarly for ${}_yM\circ {}_xL$).
We use the following general quantitative result regarding tensor products.

Theorem 3.2. *(see [Th. 37, 1]) Let L and M be defined as above and such that for fixed $r,s\in\mathbb{N}_0$*

$$\begin{aligned} |L(g;x)-g(x)| &\le \sum_{\rho=0}^{r}\Gamma_{\rho,L}(x)\cdot\omega_\rho(g;\Lambda_{\rho,L}(x)),\ x\in I', g\in C(I) \text{ and} \\ |M(h;y)-h(y)| &\le \sum_{\gamma=0}^{s}\Gamma_{\gamma,M}(y)\cdot\omega_\gamma(h;\Lambda_{\gamma,M}(y)),\ y\in J', h\in C(J). \end{aligned}$$

Here, Γ and Λ are bounded functions.
(i) Then for $(x,y)\in I'\times J'$ and $f\in C(I\times J)$ the following hold:

$$\begin{aligned} |({}_xL\circ {}_yM)f(x,y)-f(x,y)| &\le \sum_{\rho=0}^{r}\Gamma_{\rho,L}(x)\cdot\omega_\rho(f;\Lambda_{\rho,L}(x),0) \\ &+ \|L\|\cdot\sum_{\gamma=0}^{s}\Gamma_{\gamma,M}(y)\cdot\omega_\gamma(f;0,\Lambda_{\gamma,M}(y)). \end{aligned}$$

(ii) A symmetric upper bound is given by

$$\sum_{\gamma=0}^{s}\Gamma_{\gamma,M}(y)\cdot\omega_\gamma(f;0,\Lambda_{\gamma,M}(y)) + \sum_{\rho=0}^{r}\Gamma_{\rho,L}(x)\cdot\omega_\rho(f;\Lambda_{\rho,L}(x),0).$$

From (7) we immediately get the explicit representation of the tensor product of two BLaC operators

$$({}_xBL_n \circ {}_yBL_m)(f;x,y) = \sum_{i=-1}^{2^n-1} \sum_{j=-1}^{2^m-1} f(\eta_i^n, \eta_j^m) \cdot \varphi_i^n(x) \cdot \varphi_j^m(y),$$

and can state

Theorem 3.3. *For $n, m \in \mathbb{N}$ we have*

$$\begin{aligned} \|({}_xBL_n \circ {}_yBL_m)f - f\| &\le (1-\Delta)\omega_1\left(f; \frac{1}{2^n}, 0\right) + \frac{3}{2}\omega_2\left(f; \frac{1}{2^n}, 0\right) \\ &+ (1-\Delta)\omega_1\left(f; 0, \frac{1}{2^m}\right) + \frac{3}{2}\omega_2\left(f; 0, \frac{1}{2^m}\right) \\ &\le 2(1-\Delta)\omega_1\left(f; \frac{1}{2^n}, \frac{1}{2^m}\right) + 3\omega_2\left(f; \frac{1}{2^n}, \frac{1}{2^m}\right). \end{aligned}$$

Proof. The proof is immediate. Take in Theorem 3.2 $r = s = 2$, $\Gamma_0(x) = 0, \Gamma_1(x) = 1 - \Delta, \Gamma_2(x) = \frac{3}{2}$ and $\Lambda_1(x) = \Lambda_2(x) = \frac{1}{2^n}$, make an analogous choice with respect to the variable y and use the relation in Proposition 2.6 twice. For the last inequality use (6). □

Remark 3.4. Similar results can be also achieved for Boolean sums of two BLaC operators, using, for example, Th. 31 from [1].

A practical application of the bivariate case is image compression. In the diploma thesis [7] of the third author a method is implemented that enables us to choose in an appropriate way the parameter Δ for a given picture (part of it). Examples are given to illustrate the fact that in most cases it is better to choose Δ not equal to 0 or 1, in order to obtain a more satisfying picture.

References

[1] L. Beutel, H. Gonska, D. Kacsó & G. Tachev, On variation-diminishing Schoenberg operators: new quantitative statements. In: "Multivariate Approximation and Interpolation with Applications" (ed. by M. Gasca), Monogr. Academia Ciencias de Zaragoza **20** (2002), 9–58.

[2] L. Beutel & H. Gonska, Quantitative inheritance properties for simultaneous approximation by tensor product operators, Numerical Algorithms **33** (2003), 83-92.

[3] L. Beutel & H. Gonska, Quantitative inheritance properties for simultaneous approximation by tensor product operators II: applications. In: "Mathematics and its Applications" (Proc. 17th Scientific Session; ed. by G.V.Orman), 1-28. Braşov: Editura Universităţii "Transilvania" 2003.

[4] G. P. Bonneau, Multiresolution analysis with non-nested spaces. In: "Tutorials on Multiresolution in Geometric Modelling" (Lectures European School on Principles of Multiresolution in Geometric Modelling, Munich University of Technology, August 2001; ed. by A. Iske, E. Quak and M. S. Floater), 147–163. Berlin: Springer 2002.

[5] H. Gonska, On approximation in spaces of continuous functions, Bull. Austral. Math. Soc. **28** (1983), no. 3, 411–432.

[6] H. Gonska, On approximation by linear operators: improved estimates, Anal. Numér. Théor. Approx. **14** (1985), 7–32.

[7] O. Nemitz, BLaC-Wavelets und nicht ineinander geschachtelte Wavelets. Diploma thesis, University of Duisburg-Essen 2003.
[8] R. Păltănea, Best constants in estimates with second order moduli of continuity. In: Approx. Theory, (Proc. Int. Dortmund meeting on Approx. Theory 1995, ed. by M. W. Müller et al.). Berlin: Akad. Verlag 1995, 251–275.
[9] R. Păltănea, "Approximation Theory using Positive Linear Operators". Boston: Birkhäuser 2004.

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