Discretization and Convergence for harmonic maps into trees

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Abstract

The nonlinear Dirichlet problem is considered for maps from a two dimensional domain into trees with one branch point. The Dirichlet energy is defined using a semigroup approach based on Markov kernels. The problem is discretized using a suitable finite element approach and convergence of a corresponding iterative numerical method is proved. The presented approach integrates stochastic methods on discrete lattices and finite element projection techniques. Finally, a couple of numerical results are presented.

Introduction

A smooth map $f : M \to N$ between Riemannian manifolds is called harmonic if its tension field $\tau(f) := trace \nabla(df)$ vanishes [Jos95]. Well known examples are harmonic functions ($N = \mathbb{R}$), geodesics ($M \subset \mathbb{R}$) and minimal surfaces. Harmonic maps play an important role in many areas of mathematics, see Eells, Lemaire (1978, 1988) for a survey. In the last decade, it was developed the study of maps into more general target spaces, e.g. [GS92], [JY93].
Korevaar, Schoen (1993, 1997) and Jost (1994, 1997) independently began to develop a theory of harmonic maps into metric spaces of nonpositive curvature in the sense of Alexandrov (briefly: NPC spaces). These developments are based on the fact that a canonical extension of the energy functional can be defined for maps with values in NPC spaces. In the approach by Korevaar, Schoen, the domain space is still a Riemannian manifold. In Jost's approach, the domain space is a locally compact metric space equipped with an abstract Dirichlet form, replacing the Riemannian manifold equipped with the classical Dirichlet form. Eells, Fuglede (2001) study harmonic maps between Riemannian polyhedra.

In this work we will study the nonlinear Dirichlet problem for harmonic maps $f : M \to N$ from a measure space (M, m) with a local Dirichlet form on it into trees with one branch point ("spiders"). Besides Riemannian manifolds the most simple NPC spaces are metric trees and in particular spiders. To study and understand the nonlinear effects (e.g. on regularity and stability of harmonic maps) arising from negative curvature one may restrict oneself to these prototypes of NPC spaces. Moreover, the study of the nonlinear Dirichlet problem for maps into spiders yields the main module for the analysis of the Dirichlet problem for maps into graphs. A further step might be then to use harmonic maps into graphs to approximate harmonic maps into Riemannian manifolds or more general spaces.

Let (M, m) be a measure space with a local Dirichlet form $\mathcal{E}$ on it with generator A and semigroup e^{At} given by a semigroup of Markov kernels p_t. Let N be a tree with one branch point and a finite number of edges. Following the approach by Jost (1997), we will define a canonical extension $\mathcal{E}_N$ of the energy $\mathcal{E}$ for maps $v : M \to N$ using the semigroup p_t

$$\mathcal{E}_N(v) := \limsup_{t \to 0} \frac{1}{2t} \int_M \int_M d^2(v(x), v(y)) p_t(x, dy) m(dx).$$

This definition yields the identity

$$\sum \mathcal{E}(v_i) = \mathcal{E}_N(v),$$

whereby $v_i : M \to \mathbb{R}$ is the projection of v on the i-th edge [He03]. If the operator A is the Laplace operator Δ on $\mathbb{R}^n$ then one has

$$\mathcal{E}_N(v) = \sum \int_{\mathbb{R}^n} |\nabla v_i|^2.$$

The nonlinear Dirichlet problem for a given map g with $\mathcal{E}_N(g) < \infty$ and a subset $D \subset M$ is to find a map u with $u = g$ on $M \backslash D$ which minimizes the nonlinear energy $\mathcal{E}_N$ (either on M or, equivalently, on D). Such a map is called harmonic on D.
For the proof of existence and uniqueness of a solution to the nonlinear Dirichlet problem see [He03]. It turns out that the solution u depends only on $g|_{\partial D}$.

In the special case $M = \mathbb{R}^2$, $\mathcal{E}$ being the classical Dirichlet form on $\mathbb{R}^2$, D being a polygonal set we will define a numerical algorithm to solve the nonlinear Dirichlet problem.
Within this case we fix suitable triangulations $\mathcal{T}_h$ of D and define a discrete nonlinear energy $\mathcal{E}_N^h$ for maps $\bar{v}_h : \mathcal{N}_h \to N$, whereby $\mathcal{N}_h$ denotes the set of vertices of the triangulation $\mathcal{T}_h$. This yields a discrete nonlinear Dirichlet problem, i.e., for a map $g : \mathbb{R}^2 \to N$ with $\mathcal{E}_N(g) < \infty$ one searches a map $\bar{u}_h : \mathcal{N}_h \to N$ with $\bar{u} = g$ on $\partial D \cap \mathcal{N}_h$ minimizing the discrete nonlinear energy $\mathcal{E}_N^h$. For the construction of the algorithm solving this problem we mainly use the fact that the maps which minimize the discrete energy can be obtained by iteration of nonlinear Markov operators. The latter are defined as barycenters of discrete probability distributions on the n-spider.
Furthermore we define a prolongation operator J_h which extends maps defined on the vertices to maps defined on the whole domain D in such a way that

$$\mathcal{E}_N(J_h(\bar{u}_h)) \leq \mathcal{E}_N^h(\bar{u}_h) \to \mathcal{E}_N(u) \qquad h \to 0.$$

From this the L^2-convergence of $J_h(\bar{u}_h)$ to the solution u of the nonlinear Dirichlet problem follows as a straightforward consequence.

1 Nonlinear Dirichlet problem

Throughout this paper we fix a σ-finite measure space (M, m) and a Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(M, m)$, that is, $\mathcal{E}$ is a closed bilinear form on $L^2(M, m)$ with dense domain $\mathcal{D}(\mathcal{E})$ with $\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leq \mathcal{E}(u, u)$. Moreover, we assume

(A1) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is local, that is, $v, w \in \mathcal{D}(\mathcal{E}),\ v \cdot w = 0$ a.e. $\quad \Rightarrow \quad \mathcal{E}(v, w) = 0$.

(A2) The semigroup $(T_t)_{t \geq 0}$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is given by a semigroup of Markov kernels $p_t(x, dy)$.

Remark 1.1 *Assumption (A2) is always fulfilled if M is a locally compact separable metric space, and the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular and conservative. In particular, this assumption is fulfilled for $M = \mathbb{R}^2$ with m being the Lebesgue measure λ on $\mathbb{R}^2$, and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ being the classical Dirichlet form. i.e. $\mathcal{E}(u) = \int_{\mathbb{R}^2} |\nabla u|^2 d\lambda$. This special case will be further developed along this work.*

For $I = \mathbb{N}$ or $I = \{1, \ldots, n\}$ we define the I-spider as the metric space (N, d) where

$$N := \{(i, t) : i \in I, t \in \mathbb{R}_0^+\} / \sim$$

with $(i,0) \sim (j,0)$ for every $i,j \in I$. A distance d is defined on N by

$$d((i,s),(j,t)) = \begin{cases} |s-t|, & \text{if } i=j \\ s+t, & \text{otherwise.} \end{cases}$$

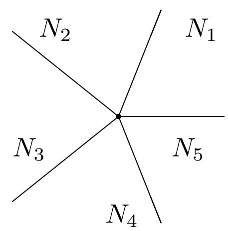

Figure 1: The 5-spider

Additionally, we consider the following functions defined on N by

$$c: N \to I \cup \{0\}, \qquad (i,t) \mapsto \begin{cases} i, & \text{if } t \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

$$\pi: N \to \mathbb{R}_0^+, \qquad (i,t) \mapsto t$$

and

$$\pi_j: N \to \mathbb{R}_0^+, \qquad (i,t) \mapsto \delta_{ij} \cdot t.$$

In the sequel we use the decomposition $\bigcup_{i \in I \cup \{0\}} N_i$ of N, with $N_0 := o := \{(1,0)\}$ and $N_i := \{(i,t) : t \in \mathbb{R}_0^+\}, i \in I$. In this way, to each measurable map $v : M \to N$ one may associate a family of functions $v_i : M \to \mathbb{R}$ ($i \in I$) defined by

$$v_i := \pi_i \circ v.$$

The number $\pi(x)$ plays the role of the modulus of x and $c(x)$ is a generalization of $\operatorname{sgn}(x)$ and interpreted as colour of x.

Remark 1.2 *If $I = \{1,2\}$ then N, N_1 and N_2 can be identified with $\mathbb{R}, \mathbb{R}_0^+$ and $\mathbb{R}_0^-$, resp. Then the functions $c(x), \pi(x), \pi_1(x), \pi_2(x)$ coincide with $\operatorname{sgn}(x), |x|, x_+, x_-$, resp. and $v_1(x), v_2(x)$ coincide with $v_+(x), v_-(x)$.*

Given a measurable map $v : M \to N$ we define the energy function $\mathcal{E}_N$ by

$$\mathcal{E}_N(v) := \limsup_{t \to 0} \frac{1}{2t} \int_M \int_M d^2(v(x),v(y)) p_t(x,dy) m(dx) \tag{1}$$

with $\mathcal{D}(\mathcal{E}_N) := \{v : M \to N \text{ measurable: } \mathcal{E}_N(v) < \infty \text{ and } v_i \in L^2(M,m),\ \forall i \in I\}$.

Theorem 1.3 *For all measurable maps $v : M \to N$ the condition $v \in \mathcal{D}(\mathcal{E}_N)$ is equivalent to*

$$v_i \in \mathcal{D}(\mathcal{E}),\ \forall i \in I \quad \textit{and} \quad \sum_{i \in I} \mathcal{E}(v_i) < \infty.$$

In this situation, for each $v \in \mathcal{D}(\mathcal{E}_N)$ the following equalities hold

$$\begin{aligned}\mathcal{E}_N(v) &= \lim_{t\to 0} \frac{1}{2t} \int_M \int_M d^2(v(x), v(y)) p_t(x, dy) m(dx) \qquad (2)\\ &= \sum_{i\in I} \mathcal{E}(v_i) \qquad (3)\end{aligned}$$

with

$$\mathcal{E}(v_i) = \lim_{t\to 0} \frac{1}{2t} \int_M \int_M |v_i(x) - v_i(y)|^2 p_t(x, dy) m(dx).$$

For a detailed proof see [He03].

Corollary 1.4 *On $\mathbb{R}^k$ with the Lebesgue measure λ, let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the classical Dirichlet form and $I = \{1, \ldots, n\}$. For all $v \in \mathcal{D}(\mathcal{E}_N)$ one has*

$$\mathcal{E}_N(v) = \sum_{i\in I} \int_{\mathbb{R}^k} |\nabla v_i|^2 d\lambda. \qquad (4)$$

Remark 1.5 *In the situation from above, our notion of the nonlinear energy $\mathcal{E}_N$ coincides with the notion of energy introduced by [KS93] and [Jos94]. That is, for all measurable $v : \mathbb{R}^k \to N$ one has*

$$\begin{aligned}\mathcal{E}_N(v) &= \lim_{r\to 0} \frac{c_k}{r^{k+1}} \int_{\mathbb{R}^k} \int_{\partial B_r(x)} d^2(v(x), v(y)) \sigma_{r,x}(dy) \lambda(dx)\\ &= \lim_{r\to 0} \frac{c_k'}{r^{k+1}} \int_{\mathbb{R}^k} \int_{B_r(x)} d^2(v(x), v(y)) \lambda(dy) \lambda(dx).\end{aligned}$$

For details see [He03].

Definition 1.6 (Nonlinear Dirichlet problem) *Given a map $g \in \mathcal{D}(\mathcal{E}_N)$ and a set $D \subset M$, let us define the class of maps*

$$V_N(g) := \{v \in \mathcal{D}(\mathcal{E}_N) : v = g \ \textit{m-a.e. on } M\backslash D\}.$$

A map $u \in V_N(g)$ is called a solution to the nonlinear Dirichlet problem for g whenever

$$\mathcal{E}_N(u) = \min_{v\in V_N(g)} \mathcal{E}_N(v).$$

The next result states a sufficient condition for the existence (and uniqueness) of a solution to a nonlinear Dirichlet problem in terms of the so-called linear spectral bound λ_D of an open set $D \subset M$, that is,

$$\lambda_D := \inf \left\{ \mathcal{E}(v) : v \in L^2_0(D), \int_M v^2 dm = 1 \right\}$$

where $L^2_0(D) := \{v \in L^2(M) : v = 0 \ m\text{-a.e. on } M\backslash D\}$ and $\mathcal{E}(v) := +\infty$ if $v \notin \mathcal{D}(\mathcal{E})$.

Theorem 1.7 *Given a set $D \subset M$ such that $\lambda_D > 0$, there exists a unique solution to the nonlinear Dirichlet problem for any $g \in \mathcal{D}(\mathcal{E}_N)$.*

For a detailed proof see [He03].

Remark: A refined definition of the Dirichlet problem would require to replace the class $V_N(g)$ by $\tilde{V}_N(g) := \{v \in \mathcal{D}(\mathcal{E}_N) : \tilde{v} = \tilde{g}$ quasi everywhere on $M\backslash D\}$ where $\tilde{v}, \tilde{g}$ denote quasi-continuous versions of v and g, resp. This makes sense whenever the Dirichlet form is quasi regular. However in our application both classes coincide since D always will have a "nice" boundary.

2 Discrete nonlinear Dirichlet problem

From now on, we confine with the special case $(M, m) = (\mathbb{R}^2, \lambda)$ with the corresponding classical Dirichlet form $\mathcal{E}$. For the sake of simplicity, we restrict to a polygonal domain $D \subset \mathbb{R}^2$. Let us suppose that an admissible and regular triangulation $\mathcal{T}_h$ of D in the sense of [Cia78] is given. In addition, we suppose the triangles to be "acute". This, means that all interior angles of all triangles of $\mathcal{T}_h$ are less than or equal to $\frac{\pi}{2}$. Finally, we assume that for the map $g \in \mathcal{D}(\mathcal{E}_N)$, specifying the boundary values for the nonlinear Dirichlet problem, $\pi \circ g$ is the modulus of a linear function on the boundary faces of $\mathcal{T}_h$. For this special situation we define a discrete nonlinear Dirichlet problem which unique solution is used to approximate the solution of the continuous nonlinear Dirichlet problem.

However before we start to discuss the nonlinear case, we will have a closer look on the linear case.

In the sequel $\mathcal{N}_h = \{x_1, \dots, x_l\}$ denotes the set of all vertices of the triangulation $\mathcal{T}_h$. We divide $\mathcal{N}_h$ into two disjoint sets

$$\mathring{\mathcal{N}}_h := \mathcal{N}_h \backslash \partial D \quad \text{and} \quad \mathcal{N}_h^{\partial} := \mathcal{N}_h \cap \partial D.$$

Definition 2.1 *We denote by V^h the standard space of piecewise affine finite elements on $\mathcal{T}_h$ and by $\{\phi_h^i, 1 \leq i \leq l\}$ the corresponding nodal basis of V^h, see [Cia78]. Furthermore we define a Markov kernel p on $\mathcal{N}_h$ by*

$$\forall x_i, x_j \in \mathcal{N}_h : \qquad p(x_i, x_j) := \begin{cases} -\frac{(\nabla \phi_h^i, \nabla \phi_h^j)}{(\nabla \phi_h^i, \nabla \phi_h^i)}, & \text{if } x_i \sim x_j, \\ 0, & \text{otherwise,} \end{cases}$$

where $x_i \sim x_j$ means that there is an edge connecting x_i and x_j and we define a measure μ on $\mathcal{N}_h$ by

$$\forall x_i \in \mathcal{N}_h : \qquad \mu(x_i) := (\nabla \phi_h^i, \nabla \phi_h^i).$$

Remark: Due to the assumptions on the triangulations $\mathcal{T}_h$ one has $(\nabla \phi_h^i, \nabla \phi_h^j) \leq 0$. Furthermore, since $\nabla 1 = 0$, it holds $\sum_{x_j \in \mathcal{N}_h} p(x_i, x_j) = 1$ (cf. [Tho97]).

Lemma 2.2 *Given a function $v_h \in V^h$, for all $1 \leq i \leq l$ define $v_h^i := v_h(x_i)$. Then*

$$\int_D |\nabla v_h|^2 \, d\lambda = \frac{1}{2} \sum_{i=1}^{l} \sum_{j=1}^{l} (v_h^i - v_h^j)^2 p(x_i, x_j) \mu(x_i) \tag{5}$$

and, moreover,

$$\int_D |\nabla v_h|^2 d\lambda = - \sum_{T \in \mathcal{T}_h} \sum_{\substack{i,j=0 \\ i<j}}^{2} (v_h(x_i^T) - v_h(x_j^T))^2 \int_T \nabla \phi_h^{i,T} \nabla \phi_h^{j,T} \, d\lambda, \tag{6}$$

whereby $x_0^T, x_1^T, x_2^T \in \mathcal{N}_h$ denote the vertices of a triangle $T \in \mathcal{T}_h$ and $\phi_h^{i,T}$ denote the corresponding elements of the standard basis.

The difference between formulas (5) and (6) is that in (5) we sum over all vertices of the triangulation and in (6) we sum over all triangles.

Proof: Identity $v_h(x) = \sum_{i=1}^{l} v_h^i \phi_h^i(x)$ leads to

$$
\begin{aligned}
& \frac{1}{2}\sum_{i=1}^{l}\sum_{j=1}^{l}(v_h^i - v_h^j)^2 p(x_i,x_j)\mu(x_i) \\
= \; & \frac{1}{2}\left(2\sum_{i=1}^{l}(v_h^i)^2 \sum_{\substack{j=1\\ j\neq i}}^{l}[-(\nabla\phi_h^i,\nabla\phi_h^j)] + 2\sum_{i=1}^{l}\sum_{\substack{j=1\\ j\neq i}}^{l} v_h^i v_h^j(\nabla\phi_h^i,\nabla\phi_h^j)\right) \\
= \; & \sum_{i=1}^{l}\sum_{j=1}^{l} v_h^i v_h^j (\nabla\phi_h^i,\nabla\phi_h^j) \\
= \; & \int_D |\nabla v_h|^2 d\lambda.
\end{aligned}
$$

A similar procedure shows equation (6). □

Now we are going to extend our frame from functions $v : M \to \mathbb{R}$ to maps $v : M \to N$ where N is the spider with n edges.

Definition 2.3 (Discrete nonlinear Dirichlet problem) *Given a map $g : \partial D \to N$ let us define*

$$\bar{V}_N^h(g) := \left\{\bar{v}_h : \mathcal{N}_h \to N : \bar{v}_h(x) = \bar{g}_h(x) \quad \forall x \in \mathcal{N}_h^\partial\right\}$$

with $\bar{g}_h(x) := g(x), \forall x \in \mathcal{N}_h^\partial$. A map $\bar{u}_h : \mathcal{N}_h \to N$ is called a solution to the discrete nonlinear Dirichlet problem for g whenever $\bar{u}_h$ fulfills the following two conditions:

1. $\bar{u}_h \in \bar{V}_N^h(g)$
2. $\mathcal{E}_N^h(\bar{u}_h) = \min\limits_{\bar{v}_h \in \bar{V}_N^h(g)} \mathcal{E}_N^h(\bar{v}_h)$, *where*

$$\mathcal{E}_N^h(\bar{v}_h) := \frac{1}{2}\sum_{x_i,x_j\in\mathcal{N}_h} d^2(\bar{v}_h(x_i),\bar{v}_h(x_j))p(x_i,x_j)\mu(x_i) \tag{7}$$

is called the discrete energy corresponding to $\mathcal{T}_h$.

According to [Stu01] we have the following result.

Proposition 2.4 *For each $g : \partial D \to N$ there is a unique solution to the discrete nonlinear Dirichlet problem for g.*

Given the Markov operator p from Definition 2.1 we define another Markov operator $p_{\mathcal{N}_h}$ on $\mathcal{N}_h$ by

$$p_{\mathcal{N}_h}(x,y) \quad := \quad 1\!\!1_{\mathring{\mathcal{N}}_h}(x)p(x,y) + 1\!\!1_{\mathcal{N}_h^\partial}(x)\delta_{\{x\}}(y),$$

where $\mathbb{1}_\cdot$ denotes the indicator function of a set and $\delta_{\{x\}}$ is the Dirac measure with mass at x.

In the sequel, for a given Markov operator q on $\mathcal{N}_h$, we denote by q^N the associated nonlinear Markov operator acting on each map $\bar{v} : \mathcal{N}_h \to N$ by

$$q^N \bar{v}(x) = \operatorname*{argmin}_{z \in N} \sum_{y \in \mathcal{N}_h} d^2(z, \bar{v}(y)) q(x, y),$$

see [Stu01]. In other words, if $(X_n, \mathbb{P}_x)$ is a random walk with transition probability q then

$$q^N \bar{v}(x) = \operatorname*{argmin}_{z \in N} \mathbb{E}_x d^2(z, \bar{v}(X_1)).$$

Proposition 2.5 *For each $\bar{v}_h \in \bar{V}_N^h(g)$ the following two conditions are equivalent:*

1. $p^N_{\mathcal{N}_h} \bar{v}_h = \bar{v}_h$
2. *$\bar{v}_h$ is a solution to the discrete nonlinear Dirichlet problem for g.*

The proof follows closely the arguments used in [Stu01].

Remark:

1. In the linear case (i.e. $N = \mathbb{R}$), the matrix A with components $A_{ij} = \mu(x_i)(\delta_{ij} - p(x_i, x_j))$ is the well–known stiffness matrix and $\bar{u}_h$ solves a corresponding linear system of equations. Furthermore the matrix Q with entries $Q_{ij} = p(x_i, x_j)$ is the iteration matrix of the Jacobi algorithm. Thus the algorithm itself coincides with the corresponding Markov process (see below).
2. If $\bar{v}_h : \mathcal{N}_h \to N$ is a map such that $\bar{v}_h = p^N_{\mathcal{N}_h} \bar{v}_h$, then on $\mathring{\mathcal{N}}_h$ the map $\bar{v}_h$ is given by

$$\bar{v}_h(x) = \operatorname*{argmin}_{z \in N} \left\{ \sum_{y \in \mathcal{N}_h} d^2(z, \bar{v}_h(y)) p(x, y) \right\}, \qquad x \in \mathring{\mathcal{N}}_h.$$

To solve the discrete nonlinear Dirichlet problem, we construct a nonlinear Markov operator Q in such a way that for each $\bar{v}_h \in \bar{V}_N^h(g)$ one has

$$\lim_{n \to \infty} Q^n \bar{v}_h = \bar{u}_h.$$

In order to define this nonlinear Markov operator Q, let us first define the following Markov operators $p_1, \ldots, p_k, k := \#\mathring{\mathcal{N}}_h$, and q:

$$p_i(x, y) \quad := \quad \begin{cases} p(x, y), & \text{if } x = x_i \text{ and } x \sim y \\ 1, & \text{if } x \neq x_i \text{ and } x = y \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, \ldots, k$$

$$q(x, y) \quad := \quad p_k \circ \cdots \circ p_1(x, y).$$

Lemma 2.6 *There exists an exponent $r \in \mathbb{N}$ such that*

$$||q^r||_{\infty,\infty} := \sup\left\{||q^r v||_\infty : ||v||_\infty = 1, v = 0 \text{ on } \mathcal{N}_h^\partial\right\} < 1.$$

Proof: At first, consider $v(x_i) = v^+(x_i) = 1$ for every interior nodes x_i. In each step at least one nodal value of an interior node decreases. Indeed, this is due to the averaging effect of the application of $p_i(\cdot,\cdot)$ over neighbouring nodes. But there is only a finite number of nodes. Hence, there exists a number of iterations $r \leq k$ after which the initial value 1 on every node has been decreased. Furthermore we observe that $v \leq v^+$ implies $q^r v \leq q^r v^+$. Hence, we are done. $\square$

Remark: Based on an ordering of the nodes $x \in \mathring{\mathcal{N}}_h$ with increasing graph distance from the boundary nodes on the edge graph of the triangulation we can achieve $r = 1$ in Lemma 2.6.

Definition 2.7 *To each $i = 1, \ldots, k$ let p_i^N be the nonlinear Markov operator associated to p_i. We define the nonlinear Markov operator Q by*

$$Q := p_k^N \circ \cdots \circ p_1^N.$$

Proposition 2.8 *For each map $\bar{v}_h \in \bar{V}_N^h(g)$ such that $\bar{v}_h = Q\bar{v}_h$, one has*

$$\bar{v}_h(x) = \operatorname*{argmin}_{z\in N}\left\{\sum_{y\in\mathcal{N}_h} d^2(z, \bar{v}_h(y))p(x,y)\right\}, \qquad \forall x \in \mathring{\mathcal{N}}_h.$$

Proof: By construction of each p_i, it follows that

$$p_1^N \bar{v}_h(x) = \begin{cases} \operatorname{argmin}_{z\in N}\{\sum_{y\in\mathcal{N}_h} d^2(z, \bar{v}_h(y))p(x,y)\}, & \text{if } x = x_1 \\ \bar{v}_h(x), & \text{if } x \neq x_1. \end{cases}$$

and

$$p_i^N \bar{v}_h(x_1) = \bar{v}_h(x_1) \qquad i = 2, \ldots, k$$

for all $\bar{v}_h : \mathcal{N}_h \to N$. The equation $Q\bar{v}_h = \bar{v}_h$ leads to

$$p_1^N \bar{v}_h(x_1) = \bar{v}_h(x_1)$$

and the assertion follows for $x_1 \in \mathring{\mathcal{N}}_h$. For $x_i \in \mathring{\mathcal{N}}_h, i > 1$, the proof is analogue. $\square$

Proposition 2.9 *Let $\bar{u}_h$ be the solution to the discrete nonlinear Dirichlet problem for g. Then for each $\bar{v}_h \in \bar{V}_N^h(g)$ one has*

$$\lim_{n\to\infty} d_\infty(Q^n \bar{v}_h, \bar{u}_h) = 0, \quad \textit{where } d_\infty(\bar{v}_h, \bar{w}_h) := \sup_{x\in M} d(\bar{v}_h(x), \bar{w}_h(x)).$$

Proof: According to Theorem 5.2 in [Stu01] and Lemma 2.6

$$d_\infty(Q^r \bar{v}_h, Q^r \bar{w}_h) \leq ||q^r(d(\bar{v}_h, \bar{w}_h))||_\infty \leq ||q^r||_{\infty,\infty} \cdot d_\infty(\bar{v}_h, \bar{w}_h)$$

for all $\bar{v}_h, \bar{w}_h \in \bar{V}_N^h(g)$. Hence there exists a map $\bar{w}_h \in \bar{V}_N^h(g)$ such that $\bar{w}_h = Q\bar{w}_h$ and for all $\bar{v}_h \in \bar{V}_N^h(g)$ it holds

$$d_\infty(Q^n \bar{v}_h, \bar{w}_h) \to 0 \qquad n \to \infty$$

(cf. proof of Theorem 6.4 in [Stu01]). Therefore by Propositions 2.4, 2.5, and 2.8, one obtains $\bar{w}_h = \bar{u}_h$. $\square$

Remark: The previous construction combined with Proposition 2.9 yields the following algorithm:

$\bar{v}_h = g|_{\mathcal{N}_h}$
do
  $\bar{w}_h = \bar{v}_h$
  for $j = 1$ to k
    $\bar{v}_h(x_j) = p_j^N \bar{v}_h(x_j) = \mathrm{argmin}_{z\in N}\{\sum_{y\in\mathcal{N}_h} d^2(z, \bar{v}_h(y))p(x_j, y)\}$
until $(\max_{x_j\in\mathcal{N}_h} d(\bar{v}_h(x_j), \bar{w}_h(x_j)) \leq EPS)$.

Here EPS is a user prescribed threshold value. This algorithm provides an approximation to the exact solution $\bar{u}_h$ of the discrete nonlinear Dirichlet problem for the boundary value function g.

3 Extending maps on vertices to maps on the domain

By means of a proper prolongation procedure, to each map in $\bar{V}_N^h(g)$ we are going to associate a map in $V_N(g)$. In other words, each map $\bar{v}_h$ which is defined on the vertices of the triangulation $\mathcal{T}_h$ will be extended to a map v_h, defined on the whole domain D, with almost the same energy, i. e., for each $\bar{v}_h \in \bar{V}_N^h(g)$ we will verify that

$$\mathcal{E}_N(v_h) \leq \mathcal{E}_N^h(\bar{v}_h) + R_{g,D},$$

with a nonnegative constant $R_{g,D}$ only depending on the polygonal domain D, the regularity of the triangulation $\mathcal{T}_h$, and the map g.

As before let us consider the sets $D, \mathcal{T}_h, \mathcal{N}_h = \{x_1, \ldots, x_l\}$, and a map $g \in \mathcal{D}(\mathcal{E}_N)$. Given a vector $\bar{v}_h \in N^l$ our aim is to construct a continuous map $v_h : \overline{D} \to N$, affine on each triangle $T \in \mathcal{T}_h$, such that $v_h^i := v_h(x_i) = \bar{v}_h(x_i)$ for all $i = 1, \ldots, l$. Hence, we will define v_h on each triangle $T \in \mathcal{T}_h$ separately. Let $T \in \mathcal{T}_h$ be given with vertices a_0, a_1, a_2. To define $v_h|_T$ we have to distinguish the following cases:

(i) $\#(\{c(\bar{v}_h(a_j))\}_{j\in\{0,1,2\}}) = 1$

(ii) $\#(\{c(\bar{v}_h(a_j))\}_{j\in\{0,1,2\}}) = 2$ and $\exists j \in \{0,1,2\} : c(\bar{v}_h(a_j)) = 0$

(iii) $\#(\{c(\bar{v}_h(a_j))\}_{j\in\{0,1,2\}}) = 2$ and $\forall j \in \{0,1,2\} : c(\bar{v}_h(a_j)) > 0$

(iv) $\#(\{c(\bar{v}_h(a_j))\}_{j\in\{0,1,2\}}) = 3$ and $\exists j \in \{0,1,2\} : c(\bar{v}_h(a_j)) = 0$

(v) $\#(\{c(\bar{v}_h(a_j))\}_{j\in\{0,1,2\}}) = 3$ and $\forall j \in \{0,1,2\} : c(\bar{v}_h(a_j)) > 0$

case (i):
We define an affine function $l : T \to \mathbb{R}$ with $l(a_j) = \pi(\bar{v}_h(a_j)), j = 0, 1, 2$ and for each $x \in T$ we set $v_h|_T(x) := (c(\bar{v}_h(a_0)), l(x))$.

case (ii):
Without loss of generality we may assume that $c(\bar{v}_h(a_0)) > 0$. Then we define an affine function $l : T \to \mathbb{R}$ by $l(a_j) := \pi(\bar{v}_h(a_j)), j = 0, 1, 2$ and for each $x \in T$ we set $v_h|_T(x) := (c(\bar{v}_h(a_0)), l(x))$.

case (iii):
Without loss of generality we may assume that $c(\bar{v}_h(a_0)) = c(\bar{v}_h(a_2))$. Then we define the points $a_{0,1}$ and $a_{1,2}$ by

$$a_{i-1,i} = \gamma_{i-1,i} a_i + (1-\gamma_{i-1,i}) a_{i-1} \qquad \text{where} \qquad \gamma_{i-1,i} = \frac{\pi(\bar{v}_h(a_{i-1}))}{\pi(\bar{v}_h(a_i)) + \pi(\bar{v}_h(a_{i-1}))} \qquad i \in \{1,2\}$$

In addition on the triangle $T_1 := \Delta a_{0,1} a_1 a_{1,2}$ we define an affine function $l : T_1 \to \mathbb{R}$ by $l(a_1) := \pi(\bar{v}_h(a_1)), l(a_{0,1}) := l(a_{1,2}) := 0$ and on $R_{0,2} := T \backslash T_1$ we define a bilinear function $b : R_{0,2} \to \mathbb{R}$ by $b(a_0) := \pi(\bar{v}_h(a_0)), b(a_2) := \pi(\bar{v}_h(a_2)), b(a_{0,1}) := b(a_{1,2}) := 0$. Then we set

$$v_h|_T(x) := \begin{cases} (c(\bar{v}_h(a_1)), l(x)), & \text{if } x \in T_1 \\ (c(\bar{v}_h(a_0)), b(x)), & \text{if } x \in R_{0,2}. \end{cases}$$

case (iv):
Without loss of generality we may assume that $c(\bar{v}_h(a_1)) = 0$. Then we define the point $a_{0,2}$ by

$$a_{0,2} = \gamma_{0,2} a_0 + (1-\gamma_{0,2}) a_2 \qquad \text{where} \qquad \gamma_{0,2} = \frac{\pi(\bar{v}_h(a_2))}{\pi(\bar{v}_h(a_0)) + \pi(\bar{v}_h(a_2))}$$

and we construct on the triangles $T_0 := \Delta a_0 a_1 a_{0,2}$ and $T_2 := \Delta a_{0,2} a_1 a_2$ two affine functions $l_0 : T_0 \to \mathbb{R}$ by $l(a_0) := \pi(\bar{v}_h(a_0)), l(a_1) := l(a_{0,2}) := 0$ and $l_2 : T_2 \to \mathbb{R}$ by $l(a_2) := \pi(\bar{v}_h(a_2)), l(a_1) := l(a_{0,2}) := 0$. Then we define

$$v_h|_T(x) := \begin{cases} (c(\bar{v}_h(a_0)), l_0(x)), & \text{if } x \in T_0 \\ (c(\bar{v}_h(a_2)), l_2(x)), & \text{if } x \in T_2. \end{cases}$$

case (v):
In the sequel we interpret all the indices i as i mod (3).
We define the points $a_{i,i+1}, i \in \{0,1,2\}$ by

$$a_{i,i+1} = \gamma_{i,i+1} a_i + (1-\gamma_{i,i+1}) a_{i+1} \qquad \text{where} \qquad \gamma_{i,i+1} = \frac{\pi(\bar{v}_h(a_{i+1}))}{\pi(\bar{v}_h(a_i)) + \pi(\bar{v}_h(a_{i+1}))} \qquad i \in \{0,1,2\}$$

and on the triangles $T_i := \Delta a_i a_{i,i+1} a_{i,i+2}, i \in \{0,1,2\}$ we define the affine functions $l_i : T_i \to \mathbb{R}$, $l_i(a_i) := \pi(\bar{v}_h(a_i)), l_j(a_{i,i+1}) := l_j(a_{i,i+2}) := 0$, for $i \in \{0,1,2\}$.
Moreover we define $T_{0,1,2} := \Delta a_{0,1} a_{0,2} a_{1,2}$ and we set

$$v_h|_T(x) := \begin{cases} (c(\bar{v}_h(a_i)), l_i(x)), & \text{if } x \in T_i \quad i \in \{0,1,2\} \\ (1,0), & \text{if } x \in T_{0,1,2}. \end{cases}$$

The five cases described above are graphically summarized in the following figures. In all these cases, points of the spider are described by a colour ($\stackrel{\triangle}{=}$ axis) and a height ($\stackrel{\triangle}{=}$ distance from origin). The black colour describes the origin.

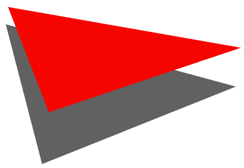

Figure 2: case (i)

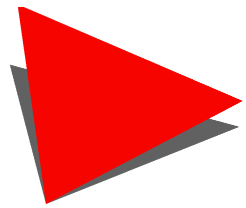

Figure 3: case (ii)

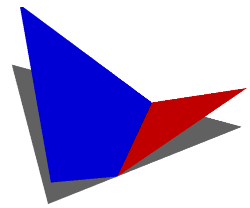

Figure 4: case (iii)

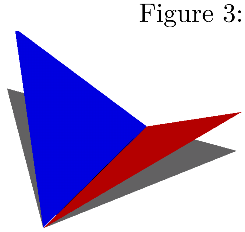

Figure 5: case (iv)

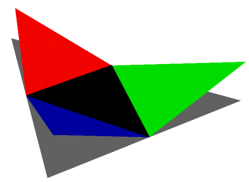

Figure 6: case (v)

Definition 3.1 *We define an injective mapping $J_h : \bar{V}_N^h(g) \to V_N(g)$ by*

$$J_h(\bar{v}_h)(x) := \begin{cases} v_h(x), & \text{if } x \in D \\ g(x), & \text{otherwise,} \end{cases}$$

for $\bar{v}_h \in \bar{V}_N^h(g)$. In the sequel we will denote the prolongation $J_h(\bar{v}_h)$ of $\bar{v}_h$ just by v_h.

Remark: Note that for each $\bar{v}_h \in \bar{V}_N^h(g)$ one has

$$\int_D |\nabla(\pi_i(v_h))|^2 d\lambda < \infty, \qquad \forall i \in \{1, \ldots, n\}$$

and

$$v_h(x) = g(x), \qquad \forall x \in \mathbb{R}^2 \backslash D.$$

Therefore v_h is well defined as an element of the space $V_N(g)$. In fact, according to Corollary 1.4 one has

$$\mathcal{E}_N(v_h) = \sum_{j=1}^n \left[\int_D |\nabla(\pi_j(v_h))|^2 d\lambda + \int_{\mathbb{R}^2 \backslash D} |\nabla(\pi_j(g))|^2 d\lambda \right]$$

Proposition 3.2 *For every $\bar{v}_h \in V_N^h(g)$ one has*

$$\mathcal{E}_N(v_h) \leq \mathcal{E}_N^h(\bar{v}_h) + R_{g,D} \tag{8}$$

where

$$R_{g,D} := \sum_{i=1}^n \int_{\mathbb{R}^2 \backslash D} |\nabla(\pi_i(g))|^2 d\lambda. \tag{9}$$

Proof: Observe that due to (6) the discrete nonlinear energy $\mathcal{E}_N^h(\bar{v}_h)$ may be rewritten as

$$\mathcal{E}_N^h(\bar{v}_h) = -\frac{1}{2} \sum_{T \in \mathcal{T}_h} \sum_{x_i, x_j \in \mathcal{N}_h} d^2(\bar{v}_h(x_i), \bar{v}_h(x_j)) \int_T \nabla \phi_h^{i,T} \nabla \phi_h^{j,T} \, d\lambda.$$

By the definition of J_h and Corollary 1.4,

$$\begin{aligned}\mathcal{E}_N(v_h) &= \sum_{i=1}^{n}\left[\int_{\mathbb{R}^2\setminus D}|\nabla(\pi_i(v_h))|^2 + \sum_{T\in\mathcal{T}_h}\int_T |\nabla(\pi_i(v_h))|^2\right] \\ &= R_{g,D} + \sum_{T\in\mathcal{T}_h}\sum_{i=1}^{n}\int_T |\nabla(\pi_i(v_h))|^2.\end{aligned}$$

Thus the rest of the proof amounts to show that for each $T\in\mathcal{T}_h$ with vertices a_0,a_1,a_2 with $v_h^i := v_h(a_i), i\in\{0,1,2\}$, the following inequality holds:

$$\begin{aligned}\sum_{j=1}^{n}\int_T |\nabla\pi_j(v_h)|^2\,d\lambda \;&\le\; -d^2(v_h^0,v_h^1)\int_T \nabla\phi_h^{0,T}\nabla\phi_h^{1,T}\,d\lambda \\ &-d^2(v_h^1,v_h^2)\int_T \nabla\phi_h^{1,T}\nabla\phi_h^{2,T}\,d\lambda \;-\; d^2(v_h^0,v_h^2)\int_T \nabla\phi_h^{0,T}\nabla\phi_h^{2,T}\,d\lambda. \qquad (10)\end{aligned}$$

By the definition of J_h, to each $\bar{v}_h\in\bar{V}_N^h$ one has to prove (10) for the five different cases described at the beginning of this section. The cases $(i)-(iv)$ can be reduced to the well known linear case, holding the equality in (10). Indeed if at most two colours are involved we can apply the identification discussed in Remark 1.2. To treat the case (v), let us introduce the notation $\alpha_i = c(v_h^i), i\in\{0,1,2\}$. We obtain

$$\sum_{j=1}^{n}\int_T |\nabla\pi_j(v_h)|^2\,d\lambda = \sum_{i=0}^{2}\int_{T_i}|\nabla\pi_{\alpha_i}(v_h)|^2\,d\lambda.$$

For $i=0,1,2$ one obtains $\nabla\pi_{\alpha_i}(v_h)\equiv\beta_i$ for some constant β_i. Hence

$$\int_{T_i}|\nabla\pi_{\alpha_i}(v_h)|^2\,d\lambda = \frac{\lambda(T_i)}{\lambda(T)}\int_T \beta_i.$$

Furthermore $\beta_i = \nabla w_h^i$, where w_h^i is affine on T with nodal values $w_h^i(v_h^i) = \pi_{\alpha_i}(v_h^i)$ and $w_h^i(v_h^{i\pm1}) = -\pi_{\alpha_{i\pm1}}(v_h^{i\pm1})$, again due to the identification in Remark 1.2 on distinct edges. Hence by formula (6) we obtain

$$\begin{aligned}&\int_{T_i}|\nabla\pi_{\alpha_i}(v_h)|^2\,d\lambda = \frac{\lambda(T_i)}{\lambda(T)}\int_T \left|\nabla w_h^i\right|^2 d\lambda \\ &= -\left[d^2(v_h^i,v_h^{i+1})\int_T \nabla\phi_h^{i,T}\nabla\phi_h^{i+1,T}d\lambda \;+\; d^2(v_h^{i+1},v_h^{i+2})\int_T \nabla\phi_h^{i+1,T}\nabla\phi_h^{i+2,T}d\lambda \right.\\ &\qquad \left. +d^2(v_h^i,v_h^{i+2})\int_T \nabla\phi_h^{i,T}\nabla\phi_h^{i+2,T}d\lambda\right]\cdot\lambda(T_i)/\lambda(T), \qquad i\in\{0,1,2\},\end{aligned}$$

which completes the proof, since $\lambda(T_0\cup T_1\cup T_2)\le\lambda(T)$. □

4 Convergence

In what follows we will consider a sequence of successively refined, regular triangulations $\mathcal{T}_h$ and ask for the convergence of the resulting discrete harmonic maps $u_h\in V_N(g)$ to the solution u of the continuous problem for $h\to 0$. For the ease of presentation we here restrict to homogeneously refined meshes, i.e. we assume

$$\min_{T\in\mathcal{T}_h} h(T) \ge c\,\max_{T\in\mathcal{T}_h} h(T)$$

with $h(T) = \mathrm{diam}(T)$. In our applications we generate the sequence of triangulation applying an iterative subdivision of triangles into four congruent triangles [Bra92]. In the sequel p resp. μ denote the Markov kernel resp. the measure defined in Section 2 corresponding to the current triangulation $\mathcal{T}_h$. Furthermore we will use a generic constant C.

Theorem 4.1 *Let $\bar{u}_h$ be the solution to the discrete nonlinear Dirichlet problem for a map g as described above and let $J_h : \bar{V}_N^h(g) \to V_N(g)$ be the mapping defined in Section 3. Then*

$$\lim_{h\to 0} \mathcal{E}_N(u_h) = \mathcal{E}_N(u). \tag{11}$$

For the proof of Theorem 4.1 we need a couple of preliminary definitions and lemmata.

Definition 4.2 *For a triangulation $\mathcal{T}_h$ we define the set*

$$S_i := \cup\{T \in \mathcal{T}_h : x_i \in T\}, \quad x_i \in \mathcal{N}_h$$

called the patch for the vertex x_i.

Definition 4.3 *Given a function $v \in H^{1,2}(D)$, let p_i be the local L^2-projection of v on S_i on the set $\mathcal{P}_1(S_i)$ of all polynomials on S_i of degree ≤ 1 with real coefficients. The corresponding Clement interpolation operator $\mathcal{I}_h$ is defined by*

$$\mathcal{I}_h v := \sum_{i=1}^{l} p_i(x_i)\phi_h^i\,.$$

In [Cle75] this interpolation operator is discussed and interpolation error estimates in are proved in Sobolev norms. In what follows we require interpolation error estimates in Hölder norms given in the following Lemma.

Lemma 4.4 *Suppose v is a Hölder continuous function on $\overline{D}$, i.e. for some $0 < \alpha < 1$ the estimate $|v(x) - v(y)| \leq C_\alpha |x-y|^\alpha$ holds for all $x, y \in \overline{D}$, then there is a constant $C_I > 0$ independent of h such that*

$$|\mathcal{I}_h v(x) - v(x)| \leq C_I \cdot h^\alpha, \qquad \forall x \in \overline{D}.$$

Proof: At first we show that for every S_i the local L^2 projection p_i defined above is Hölder continuous with respect to the Hölder exponent α. Indeed, let us first fix a set S_i and consider candidates $q \in \mathcal{P}_1$ for the best L^2 projection p_i on S_i. We observe that if $\|\nabla q\| \geq C \max_{x,y\in S_i} |v(x) - v(y)|$ for C large enough, then the constant function $\tilde{q} := |S_i|^{-1} \int_{S_i} v$ leads to a smaller projection error. Hence, we immediately observe that $\|\nabla p_i\| \geq Ch^\alpha$. Due to the regularity of the triangulation the constant C can be chosen independent of S_i and i. Next, we observe that by the mean value theorem there is a point $y_i \in S_i$ such that $p_i(y_i) = v(y_i)$. Thus, we get

$$|p_i(x) - v(x)| \leq |p_i(x) - p_i(y_i)| + |v(y_i) - v(x)| \leq C\,|x - y_i|^\alpha \leq C\,h^\alpha\,.$$

Finally on each triangle $T \in \mathcal{T}_h$ the operator $\mathcal{I}_h$ is a convex combination of p_i values. Thus, we obtain the desired result. □

Due to our homogeneity assumption we obtain

Lemma 4.5 *The total number n_h of triangles $T \in \mathcal{T}_h$ with $T \cap \partial D \neq \emptyset$ may be bounded by*

$$n_h \leq ch^{-1}$$

with a constant c independent of the triangulations.

Proof of Theorem 4.1:
Since g is Lipschitz continuous one has that the solution to the nonlinear Dirichlet problem u is Hölder continuous with $\alpha > \log_4 3$ (cf. [Ser94] and Remark 1.5). In the following we will denote the Hölder constant of the map u by C_α. Now we define

$$N_0 := \{x \in D : u(x) = o\}$$

and

$$N_0^h := \{y \in D : \mathrm{dist}(y, N_0) \leq \gamma \cdot h\}$$

for a constant $\gamma > 0$. Then

$$(\pi_i(u) - \delta_h)^+(x) = 0 \qquad \forall x \in N_0^h$$

holds for all $i \in \{1, \ldots, n\}$ with $\delta_h := C_\alpha \gamma^\alpha \cdot h^\alpha$.
By this construction we ensure that the black region ($\pi \equiv 0$) is a fat strip which is of the minimal width $2\gamma \cdot h$. Hence, choosing γ large enough we are able to avoid an interference of the involved local L^2 projections in the construction of a comparison function.
For each $i \in \{1, \ldots, n\}$ we define $\mathcal{I}_{h,i}^\delta(u) := \mathcal{I}_h((\pi_i(u) - \delta_h)^+)$. Due to Theorem 1.1 we have

$$||\mathcal{I}_{h,i}^\delta(u) - (\pi_i(u) - \delta_h)^+||_{1,2} = \nu(h) \xrightarrow{h \to 0} 0 \quad \forall i \in \{1, \ldots, \}$$

(cf. [Cle75], Corollary 1.4). Moreover, one has

$$\left| \int_D |\nabla((\pi_i(u) - \delta_h)^+)|^2 d\lambda - \int_D |\nabla(\pi_i(u))|^2 d\lambda \right| \to 0 \qquad h \to 0.$$

Thus, it follows

$$\int_D |\nabla(\mathcal{I}_{h,i}^\delta(u))|^2 d\lambda \leq \int_D |\nabla(\pi_i(u))|^2 d\lambda + \beta(h) \tag{12}$$

where $\beta(h)$ is converging to 0 for $h \to 0$.

Observe that the functions $(\pi_i(u) - \delta_h)^+, 1 \leq i \leq n$, are Hölder-continuous with the same constants α and C_α as u. Hence, according to Lemma 4.4, the following inequalities hold for each $i \in \{1, \ldots, n\}$:

$$\begin{aligned} |\mathcal{I}_{h,i}^\delta(u)(x) - \mathcal{I}_{h,i}^\delta(u)(y)| &\leq |\mathcal{I}_{h,i}^\delta(u)(x) - (\pi_i(u) - \delta_h)^+(x)| \\ &\quad + |(\pi_i(u) - \delta_h)^+(x) - (\pi_i(u) - \delta_h)^+(y)| \\ &\quad + |(\pi_i(u) - \delta_h)^+(y) - \mathcal{I}_{h,i}^\delta(u)(y)| \\ &\leq (2\, C_I + C_\alpha) \cdot h^\alpha \end{aligned}$$

and

$$\begin{aligned} |\mathcal{I}_{h,i}^\delta(u)(x) - (\pi_i(u))(y)| &\leq |\mathcal{I}_{h,i}^\delta(u)(x) - \mathcal{I}_{h,i}^\delta(u)(y)| + |\mathcal{I}_{h,i}^\delta(u)(y) - (\pi_i(u) - \delta_h)^+(y)| \\ &\quad + |(\pi_i(u) - \delta_h)^+(y) - (\pi_i(u))(y)| \\ &\leq |\mathcal{I}_{h,i}^\delta(u)(x) - \mathcal{I}_{h,i}^\delta(u)(y)| + (C_I + C_\alpha \gamma^\alpha) h^\alpha \end{aligned}$$

as well as

$$\begin{aligned} |(\pi_i(u))(x) - (\pi_i(u))(y)| &\leq |(\pi_i(u))(x) - \mathcal{I}_{h,i}^\delta(u)(x)| + |\mathcal{I}_{h,i}^\delta(u)(x) - \mathcal{I}_{h,i}^\delta(u)(y)| \\ &\quad + |\mathcal{I}_{h,i}^\delta(u)(y) - (\pi_i(u))(y)| \\ &\leq |\mathcal{I}_{h,i}^\delta(u)(x) - \mathcal{I}_{h,i}^\delta(u)(y)| + C \cdot h^\alpha. \end{aligned}$$

By means of $\mathcal{I}^\delta_{h,i}(u)$ one can now introduce a piecewise affine function ξ^h_i on $\overline{D}$, which obeys the imposed boundary conditions on the nodes. Thus, we define its nodal values:

$$\xi^h_i(x_j) := \begin{cases} \mathcal{I}^\delta_{h,i}(u)(x_j), & \text{if } x_j \notin \partial D \\ (\pi_i(u))(x_j), & \text{if } x_j \in \partial D \end{cases}$$

for all $x_j \in \mathcal{N}_h$.

To compare the energy of ξ^h_i with the energy of $\mathcal{I}^\delta_{h,i}(u)$ it is sufficient to analyse the differences on "boundary triangles". For a given triangle $T \in \mathcal{T}_h$ with $T \cap \partial D \neq \emptyset$ and $i \in \{1, \ldots, n\}$, we obtain

$$\begin{aligned}
\int_T |\nabla \xi^h_i|^2 d\lambda &\overset{(6)}{=} \sum_{\substack{s,t=0\\ s<t}}^{2} -|\xi^h_i(a_s) - \xi^h_i(a_t)|^2 \cdot \int_T \nabla\phi_h^{s,T}\nabla\phi_h^{t,T} d\lambda \\
&\leq \sum_{\substack{s,t=0\\ s<t}}^{2} -(|\mathcal{I}^\delta_{h,i}(u)(a_s) - \mathcal{I}^\delta_{h,i}(u)(a_t)| + C\cdot h^\alpha)^2 \cdot \int_T \nabla\phi_h^{s,T}\nabla\phi_h^{t,T} d\lambda \\
&\leq \sum_{\substack{s,t=0\\ s<t}}^{2} \Big[-|\mathcal{I}^\delta_{h,i}(u)(a_s) - \mathcal{I}^\delta_{h,i}(u)(a_t)|^2 \cdot \int_T \nabla\phi_h^{s,T}\nabla\phi_h^{t,T} d\lambda \\
&\qquad + 2\,|\mathcal{I}^\delta_{h,i}(u)(a_s) - \mathcal{I}^\delta_{h,i}(u)(a_t)| C\cdot h^\alpha + (C\cdot h^\alpha)^2 \Big] \\
&\leq \int_T |\nabla \mathcal{I}^\delta_{h,i}(u)|^2 + C\cdot h^{2\alpha}
\end{aligned}$$

where we have the scaling behavior of the local stiffness matrix in two dimensions

$$-\int_T \nabla\phi_h^{i,T}\nabla\phi_h^{j,T} \leq C,$$

for all triangles $T \in \mathcal{T}_h$ and nodes $x_i, x_j \in \mathcal{N}_h$. According to Lemma 4.5 we obtain

$$\int_D |\nabla \xi^h_i|^2 d\lambda = \sum_{T\in\mathcal{T}_h} \int_T |\nabla \xi^h_i|^2 d\lambda \leq \sum_{T\in\mathcal{T}_h} \int_T |\nabla(\mathcal{I}^\delta_{h,i}(u))|^2 d\lambda + n_h \cdot C \cdot h^{2\alpha} \tag{13}$$

for all $i \in \{1, \ldots, n\}$. Furthermore, we can estimate $n_h \leq ch^{-1}$ and hence $n_h \cdot C \cdot h^{2\alpha} \leq Ch^{2\alpha-1}$. Finally, we verify that $2\alpha - 1 > 2\log_4 3 - 1 \geq 0.5849..$. Hence, the effect of our correction in the neighbour of the boundary ∂D on the energy tends to zero as $h \to 0$.

Using the functions ξ^h_i our aim is now to construct a map $\bar{v}_h \in \bar{V}^h_N(g)$. For this purpose we will use the fact that the functions $\mathcal{I}^\delta_{h,i}(u)$ are not interfering with each other and that $\xi^h_i(x) = (\pi_i(g))(x)$ for all $x \in \mathcal{N}^\partial_h$. We define the map $\bar{v}_h \in \bar{V}^h_N(g)$ by

$$\bar{v}_h(x) := \begin{cases} (j, \xi^h_j(x)) & , \text{if } \exists\, j \in \{1, \ldots, n\} : \xi^h_j(x) \neq 0 \\ o & , \text{otherwise} \end{cases}$$

for all $x \in \mathcal{N}_h$. We observe that this definition is not ambiguous. Indeed, by construction there is at most one j with $\xi^h_j(x) \neq 0$.

Due to (6), the discrete nonlinear energy $\mathcal{E}^h_N(\bar{w}_h)$ of a map $\bar{w}_h \in \bar{V}^h_N(g)$ can be written as

$$\mathcal{E}^h_N(\bar{w}_h) = \sum_{T\in\mathcal{T}_h} \underbrace{-\frac{1}{2} \sum_{x_i,x_j\in\mathcal{N}_h} d^2(\bar{w}_h(x_i), \bar{w}_h(x_j)) \int_T \nabla\phi_h^{i,T}\nabla\phi_h^{j,T} d\lambda}_{:=E^h_T(\bar{w}_h)}.$$

To obtain an estimate of the discrete nonlinear energy of $\bar{v}_h$ we have to investigate $E_T^h(\bar{v}_h)$ for all $T \in \mathcal{T}_h$. Let us denote by $\mathcal{H}_h$ the set of all triangles $T \in \mathcal{T}_h$ with $T \cap \partial D \neq \emptyset$ and there exist two vertices x, y of the triangle T with $x, y \in \partial D$ such that $0 \neq c(g(x)) \neq c(g(y)) \neq 0$. Due to our assumption on g we know that $\#\mathcal{H}_h \leq C$ independent of h. We observe

$$E_T^h(\bar{v}_h) \leq \begin{cases} \sum_{i=1}^n \int_T |\nabla \xi_i^h|^2 d\lambda, & \text{if } T \in \mathcal{T}_h \backslash \mathcal{H}_h \\ 2 \cdot \sum_{i=1}^n \int_T |\nabla \xi_i^h|^2 d\lambda, & \text{if } T \in \mathcal{H}_h, \end{cases}$$

leading to

$$\mathcal{E}_N^h(\bar{v}_h) \quad \leq \quad \sum_{i=1}^n \sum_{T \in \mathcal{T}_h} \int_T |\nabla \xi_i^h|^2 d\lambda + \sum_{i=1}^n \sum_{T \in \mathcal{H}_h} \int_T |\nabla \xi_i^h|^2 d\lambda. \tag{14}$$

Furthermore we observe that $\mathcal{E}_N^h(\bar{u}_h) \leq \mathcal{E}_N^h(\bar{v}_h)$ because $\bar{u}_h$ is the minimizer of the discrete nonlinear energy $\mathcal{E}_N^h$. Hence, it follows

$$\begin{aligned} \mathcal{E}_N(u_h) &\overset{(8)}{\leq} \mathcal{E}_N^h(\bar{u}_h) + R_{g,D} \\ &\leq \mathcal{E}_N^h(\bar{v}_h) + R_{g,D} \\ &\overset{(14)}{\leq} \sum_{i=1}^n \int_D |\nabla \xi_i^h|^2 d\lambda + \sum_{i=1}^n \sum_{T \in \mathcal{H}_h} \int_T |\nabla \xi_i^h|^2 d\lambda + R_{g,D} \\ &\overset{(13)}{\leq} \sum_{i=1}^n \int_D |\nabla (\mathcal{I}_{h,i}^\delta(u))|^2 d\lambda + \sum_{i=1}^n \sum_{T \in \mathcal{H}_h} \int_T |\nabla \xi_i^h|^2 d\lambda + C \cdot h^{2\alpha - 1} + R_{g,D} \\ &\overset{(12)}{\leq} \sum_{i=1}^n \int_{\mathbb{R}^2} |\nabla (\pi_i(u))|^2 d\lambda + \gamma(h) \\ &\overset{(4)}{=} \mathcal{E}_N(u) + \gamma(h) \end{aligned}$$

where

$$\gamma(h) := \sum_{i=1}^n \sum_{T \in \mathcal{H}_h} \int_T |\nabla \xi_i^h|^2 d\lambda + C \cdot h^{2\alpha - 1} + \beta(h).$$

Obviously, $\gamma(h) \to 0$ as $h \to 0$. This yields the desired result $\lim_{h \to 0} \mathcal{E}_N(u_h) = \mathcal{E}_N(u)$. $\square$

Corollary 4.6 *For $h \to 0$ the discrete finite element solutions u_h converge in L^2 to the solution u of the continuous nonlinear Dirichlet problem.*

Proof: For a polygonal set $D \subset \mathbb{R}^2$ we put $L_0^2(D) := \{v \in L^2(\mathbb{R}^2) : v = 0 \ \lambda\text{-a.e. on } \mathbb{R}^2 \backslash D\}$. For measurable maps $v, \tilde{v} : \mathbb{R}^2 \to N$ we define the (pseudo) distance $d_2(v, \tilde{v}) := ||d(v(\cdot), \tilde{v}(\cdot))||_{L_2}$, where $d(\cdot, \cdot)$ is the distance on N. Furthermore, for a fixed measurable map $g : \mathbb{R}^2 \to N$ we define the space of maps $L^2(D, N, g)$ by

$$L^2(D, N, g) := \{v : \mathbb{R}^2 \to N \text{ measurable} : d(v, g) \in L_0^2(D)\}.$$

It holds $V_N(g) \subset L^2(D, N, g)$. For all $v \in L^2(D, N, g) \backslash V_N(g)$ we put $\mathcal{E}_N(v) := \infty$.
The n-spider (N, d) has nonpositive curvature in the sense of A. D. Alexandrov, that is, for any two points $\gamma_0, \gamma_1 \in N$ and any $t \in [0, 1]$ there exists a point $\gamma_t \in N$ such that for all $z \in N$

$$d^2(z, \gamma_t) \leq (1-t) d^2(z, \gamma_0) + t d^2(z, \gamma_1) - (1-t) t d^2(\gamma_0, \gamma_1).$$

For any two geodesics $\gamma, \varphi : [0,1] \mapsto N$ and any $t \in [0,1]$, the previous inequality leads to

$$d^2(\gamma_t, \varphi_t) \leq (1-t)d^2(\gamma_0, \varphi_0) + td^2(\gamma_1, \varphi_1) - t(1-t)[d(\gamma_0, \gamma_1) - d(\varphi_0, \varphi_1)]^2 \tag{15}$$

(cf. Korevaar/Schoen [KS93], Jost [Jos94]).
The set of maps $V_N(g)$ is convex, whereby the geodesic v_t connecting two maps $v_0, v_1 \in V_N(g)$ is defined pointwise as follows: for each $x \in \mathbb{R}^2$, $t \mapsto v_t(x)$ is the (unique) geodesic (parameterized by arc length) connecting $v_0(x), v_1(x) \in N$.
Now we prove that the energy $\mathcal{E}_N$ is strictly convex on $V_N(g)$.
Given $v_0, v_1 \in V_N(g)$ let v_t be the geodesic connecting v_0 and v_1. Inequality (15) with $\varphi_t = v_t(x)$ and $\gamma_t = v_t(y)$ yields

$$\begin{aligned} d^2(v_t(x), v_t(y)) \quad &\leq \quad (1-t)d^2(v_0(x), v_0(y)) + td^2(v_1(x), v_1(y)) \\ &\quad\quad -t(1-t)[d(v_0(x), v_1(x)) - d(v_0(y), v_1(y))]^2. \end{aligned}$$

Integrating both sides w.r.t. $p_s(x, dy)\lambda(dx)$ gives

$$E_s(v_t) \leq (1-t)E_s(v_0) + tE_s(v_1) - (1-t)tE_s(d(v_0, v_1)) \tag{16}$$

whereby for each $s > 0$

$$E_s(v) := \frac{1}{2s} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} d^2(v(x), v(y)) p_s(x, dy) \lambda(dx).$$

Furthermore, $v, \tilde{v} \in V_N(g)$ implies $d(v, \tilde{v}) \in \mathcal{D}(\mathcal{E})$. Indeed,

$$\mathcal{E}(d(v, \tilde{v})) \leq 2\mathcal{E}_N(v) + 2\mathcal{E}_N(\tilde{v})$$

since

$$|d(v(x), \tilde{v}(x)) - d(v(y), \tilde{v}(y))| \leq d(v(x), \tilde{v}(y)) + d(\tilde{v}(x), \tilde{v}(y)).$$

Taking $\limsup_{t \to 0}$ in (16) yields

$$\mathcal{E}_N(v_t) \leq (1-t)\mathcal{E}_N(v_0) + t\mathcal{E}_N(v_1) - (1-t)t\mathcal{E}(d(v_0, v_1)), \tag{17}$$

because $\mathcal{E}(d(v_0, v_1)) = \lim_{s \to 0} E_s(d(v_0, v_1))$.
On the other hand, by the spectral theory, one has

$$\mathcal{E}(d(v, \tilde{v})) \geq \lambda_D \cdot \int_{\mathbb{R}^2} d^2(v(x), \tilde{v}(x)) \lambda(dx)$$

where $\lambda_D > 0$ by assumption. Thus inequality (17) implies

$$\mathcal{E}_N(v_t) \leq (1-t)\mathcal{E}_N(v_0) + t\mathcal{E}_N(v_1) - (1-t)t\lambda_D \cdot d_2^2(v, \tilde{v}) \tag{18}$$

showing that $\mathcal{E}_N$ is strictly convex on $V_N(g)$.
Let $u_{h,t}$ be the geodesic connecting u and u_h. Then inequality (18) yields

$$\mathcal{E}_N(u) \leq \mathcal{E}_N(u_{h,\frac{1}{2}}) \leq \frac{1}{2}\mathcal{E}_N(u) + \frac{1}{2}\mathcal{E}_N(u_h) - \frac{1}{4}\lambda_D d_2^2(u, u_h),$$

and thus

$$\frac{1}{2}\lambda_D d_2^2(u, u_h) \leq \mathcal{E}_N(u_h) - \mathcal{E}_N(u).$$

Now, the claimed convergence follows from Theorem 4.1. □

5 Numerical Results

Before we present a couple of numerical results for different boundary data, let us discuss the expected order of convergence of the numerical method. Let us consider the following explicit harmonic map. Let (N, d) be a 3-spider and $D := [-2, 2]^2 \subset \mathbb{R}^2$. Then the map $u : D \to N$ given by

$$u(x, y) = \begin{cases} (1, |x^3 - 3xy^2|/10), & \text{if } -\pi \leq \arctan(x, y) < -4\pi/6 \text{ or } 0 \leq \arctan(x, y) < 2\pi/6 \\ (2, |x^3 - 3xy^2|/10), & \text{if } -4\pi/6 \leq \arctan(x, y) < 0 \text{ or } 2\pi/6 \leq \arctan(x, y) < 4\pi/6 \\ (3, |x^3 - 3xy^2|/10), & \text{otherwise} \end{cases}$$

is a harmonic function on D. Now, we define the boundary data g as a Lagrangian interpolation of $u|_{\partial D}$ onto the piecewise linear and continuous functions on ∂D. In particular we interpolate u at boundary nodes of the triangulations $\mathcal{T}_h$. Next, we have numerically solved the corresponding discrete nonlinear Dirichlet problem and computed the norm of the error $u_h - u$ for a sequence of successively refined grids, with grid sizes $h_k = 0.21, 0.10, 0.06, 0.03$. Finally, we evaluate the experimental order of convergence

$$EOC = \frac{\log \|\pi(u_{h_{k+1}}) - \pi(u)\| - \log \|\pi(u_{h_k}) - \pi(u)\|}{\log h_{k+1} - \log h_k},$$

where we either consider the L^2 or the $H^{1,2}$ norm evaluated via numerical quadrature. The following tables lists the corresponding results

h	$\|\|u - u_h\|\|_{L^2}$	EOC	$\|\|u - u_h\|\|_{H^{1,2}}$	EOC
0.21	6.838e-3	2.0071	3.119e-1	0.5665
0.10	1.620e-4	2.0023	1.066e-2	1.4924
0.06	5.171e-4	2.0004	6.011e-2	1.0049
0.03	1.611e-4	1.9877	3.336e-2	1.0043

Obviously, the EOC reflects a second order convergence in the L^2 norm and a first order convergence in the $H^{1,2}$ norm and thus equals the expected convergence rate of the pure interpolation error. Hence, we observe optimal convergence in the class of piecewise linear approximations.
Figure 7 now shows the numerical results for different boundary data and Figure 8 depicts a couple of intermediate results corresponding to different iteration steps of our numerical method.

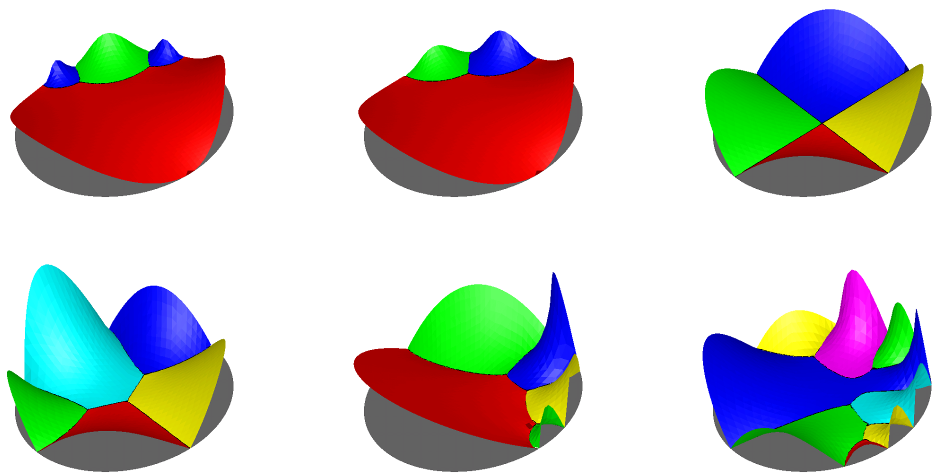

Figure 7: We depict various discrete harmonic maps $v_h \in V_N(g)$ for different boundary data g

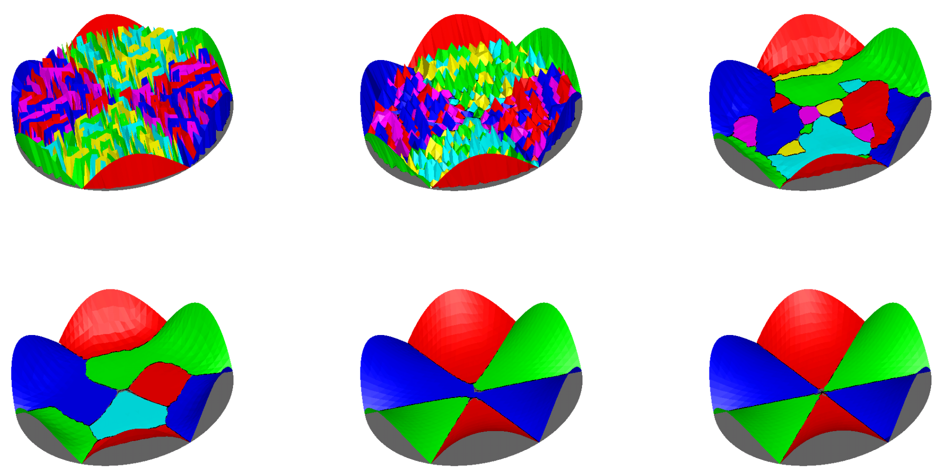

Figure 8: For different steps of our relaxation scheme we show intermediate results (from left to right and from top to bottom the steps 0, 1, 5, 10, 50, 250 are displayed)

References

[Bra92] BRAESS, D. *Theorie, schnelle Löser und Anwendungen in der Elastizitätstheorie.* Springer Verlag, Berlin, 1992.

[Cia78] CIARLET, P. G. *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.

[Cle75] CLEMENT, PH. Approximation by finite element functions using local regularization. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.* **9** (R-2) (1975), 77–84.

[EL78] EELLS, J., LEMAIRE, L. A report on harmonic maps. *Bull. London Math. Soc.* **10** (1) (1978), 1–68.

[EL88] EELLS, J., LEMAIRE, L. Another report on harmonic maps. *Bull. London Math. Soc.* **20** (5) (1988), 385–524.

[FOT94] FUKUSHIMA, M., OSHIMA, Y., TAKEDA, M. *Dirichlet Forms and Symmetrical Markov Processes.* Walter de Gruyter, Berlin, 1994.

[GS92] GROMOV, M., SCHOEN, R. M. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Inst. Hautes Études Sci. Publ. Math.* (76) (1992), 165–246.

[He03] HESSE, M. Harmonic Maps into Trees and Graphs - Analytical, Numerical and Stochastic Aspects. PhD. thesis (in preparation). Bonn, 2003.

[JY93] JOST, J., YAU, S. T. Harmonic maps and superrigidity, differential geometry: partial differential equations on manifolds, Proc. Symp. Pure Math. **54**, Part I, 1993, 245–280.

[Jos94] JOST, J. Equilibrium maps between metric spaces. *Calc. Var. Partial Differential Equations* **2** (2) (1994), 173–204.

[Jos95] JOST, J. *Riemannian Geometry and Geometric Analysis.* Springer Verlag, Berlin, 1995.

[Jos97] JOST, J. Generalized Dirichlet forms and harmonic maps. *Calc. Var. Partial Differential Equations* **5** (1) (1997), 1–19.

[KS93] KOREVAAR, N. J., SCHOEN, R. M. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.* **1** (3–4) (1993), 561–659.

[KS97] KOREVAAR, N. J., SCHOEN, R. M. Global existence theorems for harmonic maps to non-locally compact spaces. *Comm. Anal. Geom.* **5** (2) (1997), 333–387.

[MR92] MA, Z.-M., RÖCKNER, M. *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Springer Verlag, Berlin, 1992.

[Ser94] SERBINOWSKI, T. Boundary regularity of harmonic maps to nonpositively curved metric spaces. *Comm. Anal. Geom.* **2** (1) (1994), 139–153.

[Stu01] STURM, K.-T. Nonlinear Markov operators associated with symmetric Markov kernels and energy minimizing maps between singular spaces. *Calc. Var. Partial Differential Equations* **12** (4) (2001), 317–357.

[Tho97] THOMÉE, V. *Galerkin Finite Element Methods for Parabolic Problems.* Springer Verlag, Berlin, 1997, Springer Series in Computational Mathematics, 25.