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ε-DIMENSION IN INFINITE DIMENSIONAL HYPERBOLIC CROSS APPROXIMATION AND APPLICATION TO PARAMETRIC ELLIPTIC PDES

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Abstract. In this article, we present a cost-benefit analysis of the approximation in tensor products of Hilbert spaces of Sobolev-analytic type. The Sobolev part is defined on a finite dimensional domain, whereas the analytical space is defined on an infinite dimensional domain. As main mathematical tool, we use the ε-dimension of a subset in a Hilbert space. The ε-dimension gives the lowest number of linear information that is needed to approximate an element from the set in the norm of the Hilbert space up to an accuracy $\varepsilon > 0$. From a practical point of view this means that we a priori fix an accuracy and ask for the amount of information to achieve this accuracy. Such an analysis usually requires sharp estimates on the cardinality of certain index sets which are in our case infinite-dimensional hyperbolic crosses. As main result, we obtain sharp bounds of the ε-dimension of the Sobolev-analytic-type function classes which depend only on the smoothness differences in the Sobolev spaces and the dimension of the finite dimensional domain where these spaces are defined. This implies in particular that, up to constants, the costs of the infinite dimensional (analytical) approximation problem is dominated by the finite-variate Sobolev approximation problem. We demonstrate this procedure with examples of functions spaces stemming from the regularity theory of parametric partial differential equations.

Keywords: infinite-dimensional hyperbolic cross approximation, mixed Sobolev-analytic-type smoothness, ε-dimension, Kolmogorov n-width, linear information, parametric elliptic PDEs, collective Galerkin approximation.

1. Introduction. The main emphasis of this paper lies on the cost-benefit ratio of the approximation for a class of functions stemming from an anisotropic tensor product of smoothness spaces. Let X be a Hilbert space and $W \subset X$ a subset of X. Since we are interested in the cost-benefit ratio of the approximation, we focus on the so-called ε-dimension $n_\varepsilon = n_\varepsilon(W, X)$. It is defined as

$$n_\varepsilon(W, X) := \inf \left\{ n : \exists\, M_n : \dim M_n \le n, \sup_{w \in W} \inf_{v \in M_n} \|w - v\|_X \le \varepsilon \right\},$$

where $M_n \subset X$ is a linear manifold in X of dimension $\le n$. Hence, $n_\varepsilon(W, X)$ is the smallest number of linear functionals that are needed by an algorithm to give for all $f \in W$ an approximation with an error of at most ε. The important concept here is the fact that an approximation quality $\varepsilon > 0$ is a priori fixed and the smallest dimension n of any approximation space realizing the approximation error is sought after. This is the inverse

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of the usual Kolmogorov n-width $d_n(W,X)$ [10] which is given by

$$d_n(W,X) := \inf_{M_n} \sup_{w\in W} \inf_{v\in M_n} \|w-v\|_X,$$

where the outer infimum is taken over all linear manifolds M_n in X of dimension at most n.[1] For a survey and a bibliography on computational complexity see the monographs [13, 14].

To be more specific, we deal with functions defined on a product domain $\mathbb{T}^m \times \mathbb{I}^\infty$, where $\mathbb{T}^m := [0,1]^m$ is the m-dimensional torus with $m < \infty$, and $\mathbb{I}^\infty := [-1,1]^\infty$ is infinite dimensional. Denote by $\mathbb{F}$ the set of all sequences $\mathbf{s} := \{s_j\}_{j\in\mathbb{N}}$ of nonnegative integers such that $\mathrm{supp}(\mathbf{s})$ is finite, where $\mathrm{supp}(\mathbf{s})$ is the support of $\mathbf{s}$, that is the set of all $j \in \mathbb{N}$ such that $s_j \neq 0$. The fundamental space is the set of functions v on $\mathbb{T}^m \times \mathbb{I}^\infty$ which is defined as

$$\mathcal{L} := \left\{ v := \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} v_{\mathbf{k},\mathbf{s}}\phi_{\mathbf{k},\mathbf{s}} \quad \text{such that} \quad \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} |v_{\mathbf{k},\mathbf{s}}|^2 < \infty \right\},$$

where $\{\phi_{\mathbf{k},\mathbf{s}}\}_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}}$ denotes an orthonormal system with respect to the inner product

$$(v,w)_{\mathcal{L}} = \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} v_{\mathbf{k},\mathbf{s}}\bar{w}_{\mathbf{k},\mathbf{s}}.$$

In order to study approximation numbers such as $n_\varepsilon(W,X)$, we need to define the smoothness space X and the smoothness class W as well. Smoothness spaces are modeled here by general sequences of scalars $\lambda := \{\lambda(\mathbf{k},\mathbf{s})\}_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}}$ with $\lambda(\mathbf{k},\mathbf{s}) \neq 0$. Then, we define the associated space (see (3.2))

$$\mathcal{L}^\lambda := \left\{ v \in \mathcal{L} \ : \ \tilde{v} := \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} \lambda(\mathbf{k},\mathbf{s})\, v_{\mathbf{k},\mathbf{s}}\phi_{\mathbf{k},\mathbf{s}} \in \mathcal{L} \right\}. \tag{1.1}$$

For $v \in \mathcal{L}^\lambda$, we define

$$\|v\|^2_{\mathcal{L}^\lambda} := \|\tilde{v}\|^2_{\mathcal{L}} = \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} |\lambda(\mathbf{k},\mathbf{s})|^2\, |v_{\mathbf{k},\mathbf{s}}|^2,$$

where $\tilde{v}$ is defined in (1.1) (see (3.3)). Let us assume to have two such sequences λ and ν with $\nu \le \lambda$ in the point-wise sense. Then we can chose

$$X = \mathcal{L}^\nu \quad \text{and} \quad W = \mathcal{U}^\lambda,$$

where $\mathcal{U}^\lambda$ denotes the unit ball in $\mathcal{L}^\lambda$. Hence, we are left with estimating $n_\varepsilon(\mathcal{U}^\lambda, \mathcal{L}^\nu)$. To account for the fact that we work on a product domain $\mathbb{T}^m \times \mathbb{I}^\infty$, the concrete smoothness spaces are parametrized by a number a and a positive sequence $\mathbf{b} = (b_j)_{j\in\mathbb{N}}$ such that $\rho_{a,\mathbf{b}}(\mathbf{k},\mathbf{s})$ are tensor product and order dependent weights (see also (4.1))

$$\rho_{a,\mathbf{b}}(\mathbf{k},\mathbf{s}) := |\mathbf{k}|^a_\infty \frac{\mathbf{s}!}{|\mathbf{s}|_1!}\mathbf{b}^{-\mathbf{s}}, \tag{1.2}$$

where $|\mathbf{k}|_\infty := \max_{1\le j\le m} |k_j|$ and for $\mathbf{s} \in \mathbb{F}$,

$$\mathbf{s}! := \prod_{j=1}^\infty s_j!, \quad |\mathbf{s}|_1 := \sum_{j=1}^\infty s_j, \quad \text{and} \quad \mathbf{b}^{-\mathbf{s}} := \prod_{j=1}^\infty b_j^{-s_j}.$$

[1] A different worst-case setting is represented by the linear n-width $\lambda_n(W,X)$ [15]. This corresponds to a characterization of the best linear approximation error, see, e.g., [8] for definitions. Since X is here a Hilbert space, both concepts coincide, i.e., we have $d_n(W,X) = \lambda_n(W,X)$.

Both $A^{\alpha,\mathbf{b}} := \mathcal{L}^\lambda$ and $K^\beta := \mathcal{L}^\nu$ with $\lambda(\mathbf{k},\mathbf{s}) = \rho_{\alpha,\mathbf{b}}(\mathbf{k},\mathbf{s})$ $(\alpha > 0)$ and $\nu(\mathbf{k},\mathbf{s}) = |\mathbf{k}|_\infty^\beta$, respectively, will be of this specific form. The space $A^{\alpha,\mathbf{b}}$ is of Sobolev-analytic-type, while the space K^β is of Sobolev-type. We provide a motivation for such classes of function spaces by considering the regularity spaces arising in the theory of parametric partial differential equations (PDEs). The simpler case of tensor product weights

$$\tilde{\rho}_{a,\mathbf{b}}(\mathbf{k},\mathbf{s}) := \ |\mathbf{k}|_\infty^a \, \mathbf{b}^{-\mathbf{s}} \tag{1.3}$$

was already treated in [7]. As shown in Section 2 (see (2.4)–(2.5) and Lemmas 2.1 and 2.2), the tensor product and order dependent weights of the form (1.2) are more natural and important than the tensor product weights of the form (1.3), since they are the majorants of the coefficients in Taylor and Legendre expansions of the solution to parametric operator equations and in particular, parametric elliptic PDEs.

The main contribution of this paper is the fact that the ε-dimension in the space K^β of our Sobolev-analytic-type function class $U^{\alpha,\mathbf{b}}$, which is defined as the unit ball in $A^{\alpha,\mathbf{b}}$, depends only on the Sobolev smoothness differences in the spaces $A^{\alpha,\mathbf{b}}$ and K^β, and the dimension of the finite dimensional domain where these spaces are defined. More precisely, let $\alpha > \beta \geq 0$ and $\mathbf{b} = (b_j)_{j\in\mathbb{N}}$ be a positive sequence. Assume that

$$C_p := \ \sum_{\mathbf{s}\in\mathbb{F}} \left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!} \mathbf{b}^{\mathbf{s}} \right)^p < \infty \tag{1.4}$$

for $p = m/(\alpha-\beta)$, where $\mathbf{b}^{\mathbf{s}} := \ \prod_{j=1}^\infty b_j^{s_j}$. Then we have for every $\varepsilon \in (0,1]$,

$$2^m \left(\lfloor \varepsilon^{-1/(\alpha-\beta)} \rfloor - 1 \right)^m \ \leq \ n_\varepsilon(U^{\alpha,\mathbf{b}}, K^\beta) \ \leq \ 3^m C_{m/(\alpha-\beta)} \, \varepsilon^{-m/(\alpha-\beta)}, \tag{1.5}$$

These tight estimates imply in particular that, up to constants, the costs of solving the infinite dimensional (analytical) approximation problem are dominated by the finite-variate Sobolev-smooth approximation problem. The bounds in (1.5) are derived from the inequalities

$$|E_{a,\mathbf{b}}(1/\varepsilon)| - 1 \ \leq \ n_\varepsilon(\mathcal{U}^\lambda, \mathcal{L}^\nu) \ \leq \ |E_{a,\mathbf{b}}(1/\varepsilon)|,$$

and the tight estimates

$$2^m \lfloor T^{1/a} \rfloor^m \ \leq \ |E_{a,\mathbf{b}}(T)| \ \leq \ 3^m C_{m/a} \, T^{m/a} \tag{1.6}$$

for $a = \alpha - \beta$, of the cardinality of infinite-dimensional hyperbolic crosses

$$E_{a,\mathbf{b}}(T) := \ \big\{ (\mathbf{k},\mathbf{s}) \in \mathbb{Z}^m \times \mathbb{F} : 0 < \rho_{a,\mathbf{b}}(\mathbf{k},\mathbf{s}) \leq T \big\}.$$

The main difficulty in establishing (1.6) is the upper bound which requires a condition for the $\ell_p(\mathbb{F})$-summability of the sequence $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!} \mathbf{b}^{\mathbf{s}} \right)_{\mathbf{s}\in\mathbb{F}}$, i.e., a condition for (1.4), for any p such that $0 < p < \infty$. A necessary and sufficient condition for the case $0 < p \leq 1$ of this $\ell_p(\mathbb{F})$-summability have been proven in [4]. In the present paper, we prove a necessary and sufficient condition for the $\ell_p(\mathbb{F})$-summability of the sequence $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!} \mathbf{b}^{\mathbf{s}} \right)_{\mathbf{s}\in\mathbb{F}}$ for $1 < p < \infty$. This result is novel. The proof is much more technically complicate and based on a different idea. Results similar to (1.5) and (1.6), have been proven in [7] for the situation where tensor product and order dependent weights $\rho_{\alpha,\mathbf{b}}(\mathbf{k},\mathbf{s})$ are replaced by simpler tensor product weights $\tilde{\rho}_{\alpha,\mathbf{b}}(\mathbf{k},\mathbf{s})$. For the last situation the proof of a necessary and sufficient condition for the $\ell_p(\mathbb{F})$-summability of the respective sequence $\left(\mathbf{b}^{\mathbf{s}} \right)_{\mathbf{s}\in\mathbb{F}}$ for $0 < p < \infty$, is a simple task, see Lemma 4.2 and the referred Lemma 7.1 in [4] in its proof.

The above results then are applied to obtain estimates for the error of the Galerkin approximation of a parametrized elliptic Poisson problem. For these parametric elliptic PDEs the solution u belongs to suitable spaces $A^{\alpha,\mathbf{b}}$, and the Galerkin approximation $u_{E_{\alpha-\beta,\mathbf{b}}(T)}$ is from a finite-dimensional subspace of K^β, related to the infinite-dimensional hyperbolic cross index set $E_{\alpha-\beta,\mathbf{b}}(T)$. In particular, with the help of well-known Céa's lemma we derive that for $n := |E_{1,\mathbf{b}}(T)|$, there holds the upper bound $\|u - u_{E_{1,\mathbf{b}}(T)}\|_{L_2(\mathbb{I}^\infty,V,\mu)} \leq B\, n^{1/m}$ with an absolute constant B (see Theorem 6.3 for details).

The remainder of the paper is organized as follows: In Section 2, we consider the general parametrized elliptic Poisson problem and its regularity results both with respect to the spatial and with respect to the infinite-dimensional parametric component. In Section 3, we review the setting of infinite dimensional tensor products of Hilbert spaces and the associated approximation and ε-dimension. In Section 4, we give more details on the applications of the general setting to the smoothness spaces arising in parametric PDEs. The main mathematical results concern the cardinality of the infinite dimensional hyperbolic crosses in Section 5. This section is split into two steps. The first result in Subsection 5.1 addresses the inclusion of the sequence $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)_{\mathbf{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$ with $0 < p < \infty$. The main result in this subsection is Theorem 5.3 which is proven based on a result in Subsection 5.2. A suitable ℓ_p–norm of the sequence $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)_{\mathbf{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$ enters as an absolute constant, c.f. (1.4). In Section 6, we combine our results to derive sharp estimates of the ε-dimension and its inverse, the Kolmogorov n-widths of the Sobolev-analytic-type function classes. These results are then applied to the Galerkin approximation of parametric elliptic PDEs.

Notation. We give a collection of notation (a part of it has been introduced before) which will be used in the present paper: $\mathbb{Z}^m_* := \mathbb{Z}^m \setminus \{\mathbf{0}\}$; $\mathbb{T}^m$ is the m-dimensional torus which is defined as the cross product of m copies of the interval $[0,1]$ with the identification of the end points; $\mathbb{R}^\infty$ is the set of all sequences $\mathbf{y} = (y_j)_{j=1}^\infty$ with $y_j \in \mathbb{R}$; $|\mathbf{k}|_\infty := \max_{1\le j\le m} |k_j|$ for $\mathbf{k} \in \mathbb{Z}^m$. Similarly, we set $\mathbb{I} = [-1,1]$ and $\mathbb{I}^\infty$ is the set of all sequences $\mathbf{y} = (y_j)_{j=1}^\infty$ with $y_j \in \mathbb{I}$. $\mathbb{Z}^\infty$ is the set of all sequences $\mathbf{s} = (s_j)_{j=1}^\infty$ with $s_j \in \mathbb{Z}$. Furthermore, $\mathbb{Z}^\infty_+ := \{\mathbf{s} \in \mathbb{Z}^\infty : s_j \ge 0,\ j = 1,2,...\}$, y_j is the jth coordinate of $\mathbf{y} \in \mathbb{R}^\infty$. Moreover, $\mathbb{F}$ is a subset of $\mathbb{Z}^\infty_+$ of all $\mathbf{s}$ such that $\mathrm{supp}(\mathbf{s})$ is finite, where $\mathrm{supp}(\mathbf{s})$ is the support of $\mathbf{s}$, that is the set of all $j \in \mathbb{N}$ such that $s_j \neq 0$. If $\mathbf{s} \in \mathbb{F}$, we define

$$\mathbf{s}! := \prod_{j=1}^\infty s_j!, \quad |\mathbf{s}|_1 := \sum_{j=1}^\infty s_j, \quad \text{and} \quad \mathbf{b}^{\mathbf{s}} := \prod_{j=1}^\infty b_j^{s_j}$$

for a sequence $\mathbf{b} = (b_j)_{j\in\mathbb{N}}$ of positive numbers. We use the convention: $\frac{0}{0} := 0$.

2. Parametric operator equations. Let us briefly recall the setting of [11]. Denote by X a real separable Banach space over the field $\mathbb{R}$ and by X' its topological dual, i.e., the space of bounded linear functionals. We consider a map

$$\mathcal{G} : (\mathbb{I}^\infty, \|\cdot\|_\infty) \to \mathcal{L}_I(X, X'), \quad \mathbf{y} \mapsto \mathcal{G}(\mathbf{y}) = G_{\mathbf{y}},$$

where $\|\mathbf{y}\|_\infty := \sup_{j\in\mathbb{N}} |y_j|$ and $\mathcal{L}_I(X, X')$ denotes the space of boundedly invertible linear operators $X \to X'$. By $G_{\mathbf{y}}^{-1} \in \mathcal{L}_I(X', X)$, we denote the element such that $G_{\mathbf{y}} \circ G_{\mathbf{y}}^{-1} = \mathrm{Id}_{X'}$ and $G_{\mathbf{y}}^{-1} \circ G_{\mathbf{y}} = \mathrm{Id}_X$. We define

$$\mathcal{G}^{-1} : (\mathbb{I}^\infty, \|\cdot\|_\infty) \to \mathcal{L}_I(X', X), \quad \mathbf{y} \mapsto \mathcal{G}^{-1}(\mathbf{y}) = G_{\mathbf{y}}^{-1}.$$

We assume that $\mathcal{G}^{-1}$ is bounded by $C(\mathcal{G})$ i.e., that

$$\sup_{\substack{\mathbf{y}\in\mathbb{I}^\infty \\ \|\mathbf{y}\|_\infty \le 1}} \left\|\mathcal{G}^{-1}(\mathbf{y})\right\|_{\mathcal{L}(X',X)} = \sup_{\mathbf{y}\in\mathbb{I}^\infty} \left\|G_{\mathbf{y}}^{-1}\right\|_{\mathcal{L}(X',X)} \le C(\mathcal{G}). \tag{2.1}$$

Moreover, we assume that $\mathcal{G}$ is analytic with respect to every y_j with $j \in \mathbb{N}$ and that there is a sequence $\mathbf{d} : \mathbb{N} \to \mathbb{R}$ with $\mathbf{d} \in \ell^p(\mathbb{N})$ for a fixed $0 < p \le 1$ such that for all $\mathbf{s} \in \mathbb{F} \setminus \{\mathbf{0}\}$

$$\sup_{\mathbf{y}\in\mathbb{I}^\infty} \left\|\mathcal{G}^{-1}(\mathbf{0})\partial_{\mathbf{y}}^{\mathbf{s}}\mathcal{G}(\mathbf{y})\right\|_{\mathcal{L}(X,X)} \le C(\mathcal{G})\mathbf{d}^{\mathbf{s}}. \tag{2.2}$$

Furthermore, we observe that we can write the solution $u \in X$ of the operator equation $G_{\mathbf{y}}u(\mathbf{y}) = f$ for given $f \in X'$ in terms of the solution operator

$$\mathcal{S} : \mathbb{I}^\infty \times X' \to X, \quad (\mathbf{y}, f) \mapsto \mathcal{S}(\mathbf{y}, f) := \mathcal{G}^{-1}(\mathbf{y})f = G_{\mathbf{y}}^{-1}f,$$

and [11, Thm. 4] provides the bound

$$\sup_{\mathbf{y}\in\mathbb{I}^\infty} \sup_{\substack{f\in X'\\ \|f\|_{X'}=1}} \left\|\partial_{\mathbf{y}}^{\mathbf{s}}\mathcal{S}(\mathbf{y},f)\right\|_X \le \frac{C(\mathcal{G})}{\ln(2)}\,|\mathbf{s}|!\mathbf{d}^{\mathbf{s}} \tag{2.3}$$

for all $\mathbf{s}\in\mathbb{F}\setminus\{\mathbf{0}\}$. This implies a (generalized) Taylor's series representation of

$$u(\mathbf{y}) = u_f(\mathbf{y}) = \sum_{s\in\mathbb{F}} \frac{1}{\mathbf{s}!}\partial_{\mathbf{y}}^{\mathbf{s}} u_f(\mathbf{y})\mathbf{y}^{\mathbf{s}} = \sum_{s\in\mathbb{F}} \left(\frac{1}{\mathbf{s}!}\,\partial_{\mathbf{y}}^{\mathbf{s}} u_f(\mathbf{y})\big|_{\mathbf{y}=0}\right)\mathbf{y}^{\mathbf{s}} = \sum_{s\in\mathbb{F}} \left(\frac{1}{\mathbf{s}!}\,\partial_{\mathbf{y}}^{\mathbf{s}}\mathcal{S}(\mathbf{y},f)\big|_{\mathbf{y}=0}\right)\mathbf{y}^{\mathbf{s}}. \tag{2.4}$$

Hence, the coefficients are bounded by

$$\left\|\frac{1}{\mathbf{s}!}\,\partial_{\mathbf{y}}^{\mathbf{s}} u_f(\mathbf{y})\big|_{\mathbf{y}=0}\right\|_X \;\le\; \frac{C(\mathcal{G})}{\ln(2)}\frac{|\mathbf{s}|!}{\mathbf{s}!}\mathbf{d}^{\mathbf{s}}\,\|f\|_{X'}\,, \tag{2.5}$$

which fits exactly into our framework, i.e., the upper bound has the structure of $\rho_{a,\mathbf{b}}^{-1}(\mathbf{k},\mathbf{s})$ with $a=0$ from (1.2). We will, however, study a more specific example in more detail, since we also need spatial regularity results, which allows also for $a>0$. For the elliptic PDEs (2.6) formulated in the next section, some particular estimates for the coefficients in the Taylor and Legendre expansions which are similar to (2.3) and (2.5) were established in earlier papers [1, 4, 5].

2.1. Parametric elliptic PDEs. Here, we consider a more specific problem which fits into the framework outlined above. We chose $X = H_0^1(\Omega)$ and hence $X' = H^{-1}(\Omega)$ where $\Omega := [0,1]^m$. The operator is

$$\mathcal{G}_a : \mathbb{I}^\infty \to \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega)), \quad \mathbf{y}\mapsto \left(H_0^1(\Omega)\ni u \mapsto -\operatorname{div}(a(\mathbf{y})\nabla_{\mathbf{x}} u)\in H^{-1}(\Omega)\right),$$

where $a:\Omega\times\mathbb{I}^\infty\to\mathbb{R}_+$ is a function satisfying

$$0 \;<\; r \;<\; a(\mathbf{x},\mathbf{y}) \le \;R\; < \;\infty, \quad \mathbf{x}\in\Omega,\ \mathbf{y}\in\mathbb{I}^\infty.$$

In order to derive spatial regularity, we will restrict ourselves to $f\in L_2(\Omega)\subset H^{-1}(\Omega)$. Moreover, we restrict ourselves to periodic problems, that is $a(\mathbf{y})(\mathbf{x}) := a(\mathbf{x},\mathbf{y})$ is a function of $\mathbf{x}=(x_1,...,x_m)\in\Omega$ and of parameters $\mathbf{y}=(y_1,y_2,...)\in\mathbb{I}^\infty$ on $\Omega\times\mathbb{I}^\infty$, and the function $f(\mathbf{x})$ is a function of $\mathbf{x}=(x_1,...,x_m)\in\Omega$. Hence, we consider the parametric elliptic problem

$$-\operatorname{div}(a(\mathbf{y})\nabla_{\mathbf{x}} u(\mathbf{y})) \;=\; f \quad\text{in}\quad \Omega, \quad u|_{\partial\Omega} \;=\; 0, \quad \mathbf{y}\in\mathbb{I}^\infty. \tag{2.6}$$

Throughout the present paper, we assume that $a(\mathbf{y})$ and f as functions on $\mathbf{x}$ can be extended to 1-periodic functions in each variable x_j on the whole $\mathbb{R}^m$, and hence $a(\mathbf{y})$ and f can be considered as functions defined on $\mathbb{T}^m$. We also preliminarily assume that $f\in L_2(\mathbb{T}^m)$ and the diffusions a satisfy the *uniform ellipticity assumption* which ensures condition (2.1)

$$0 \;<\; r \;<\; a(\mathbf{y})(\mathbf{x}) = a(\mathbf{x},\mathbf{y}) \;\le\; R \;<\; \infty, \quad \mathbf{x}\in\mathbb{T}^m,\ \mathbf{y}\in\mathbb{I}^\infty.$$

Let $V := H_0^1(\mathbb{T}^m)$ and denote by W the subspace of V equipped with the semi-norm and norm

$$|v|_W := \;\|\Delta v\|_{L_2(\mathbb{T}^m)}, \quad \|v\|_W := \;\left(\|v\|_V^2 + |v|_W^2\right)^{1/2}.$$

Note that if $v\in L_2(\mathbb{T}^m)\cap W$ and

$$v \;=\; \sum_{\mathbf{k}\in\mathbb{Z}_*^m} v_{\mathbf{k}} e_{\mathbf{k}},$$

where $e_{\mathbf{k}}(\mathbf{x}) := e^{i2\pi\mathbf{k}\mathbf{x}}$, i.e., $\{e_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^m}$ is the usual orthonormal basis of $L_2(\mathbb{T}^m)$ (the $e_{\mathbf{0}}$ part is taken care of by the boundary condition of $V := H_0^1(\mathbb{T}^m)$), then from the definition and Parseval's identity we have

$$(2\pi)^2 \sum_{\mathbf{k}\in\mathbb{Z}_*^m} |\mathbf{k}|_\infty^2 |v_{\mathbf{k}}|^2 \;\le\; \|v\|_V^2 = \sum_{i=1}^m \|\partial_i v\|_{L^2(\mathbb{T}^m)}^2 = (2\pi)^2 \sum_{\mathbf{k}\in\mathbb{Z}_*^m} |\mathbf{k}|_2^2 |v_{\mathbf{k}}|^2 \;\le\; (2\pi)^2 m \sum_{\mathbf{k}\in\mathbb{Z}_*^m} |\mathbf{k}|_\infty^2 |v_{\mathbf{k}}|^2, \tag{2.7}$$

and

$$(2.8)\qquad (2\pi)^4 \sum_{\mathbf{k}\in\mathbb{Z}^m_*} |\mathbf{k}|_\infty^4 |v_\mathbf{k}|^2 \;\le\; |v|_W^2 \;\le\; (2\pi)^4 m^2 \sum_{\mathbf{k}\in\mathbb{Z}^m_*} |\mathbf{k}|_\infty^4 |v_\mathbf{k}|^2,$$

where we used the norm equivalence $|\mathbf{k}|_\infty \le |\mathbf{k}|_2 \le \sqrt{m}|\mathbf{k}|_\infty$ for all $\mathbf{k}\in\mathbb{Z}^m$.

2.2. Spatial regularity. By the well-known Lax-Milgram lemma, for every $\mathbf{y}\in\mathbb{I}^\infty$ there exists a unique (weak) solution $u\in V$ to equation (2.6) which satisfies the variational equation

$$\int_{\mathbb{T}^m} a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\mathbf{y})\cdot\nabla v(\mathbf{x})\,\mathrm{d}\mathbf{x} \;=\; \int_{\mathbb{T}^m} f(\mathbf{x})\,v(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \forall v\in V.$$

We skip the explicit dependence on the parameter $\mathbf{y}$ in this section. Moreover, this solution satisfies the inequality

$$\|u\|_V \;\le\; \frac{\|f\|_{V^*}}{r},$$

where $V^* = H^{-1}(\mathbb{T}^m)$ denotes the dual of V. Observe that there holds the embedding $L_2(\mathbb{T}^m)\hookrightarrow V^*$ and the inequality

$$\|f\|_{V^*} \;\le\; \|f\|_{L_2(\mathbb{T}^m)}.$$

Now, denote by $W^1_\infty(\mathbb{T}^m)$ the space of functions v on $\mathbb{T}^m$, equipped with the semi-norm and the norm

$$|v|_{W^1_\infty(\mathbb{T}^m)} := \max_{1\le i\le m} \|\partial_{x_i} v\|_{L_\infty(\mathbb{T}^m)}, \quad \|v\|_{W^1_\infty(\mathbb{T}^m)} := \|v\|_{L_\infty(\mathbb{T}^m)} + |v|_{W^1_\infty(\mathbb{T}^m)}$$

respectively. If we assume that $a\in W^1_\infty(\mathbb{T}^m)$, then the solution u of (2.6) is in W. Moreover, u satisfies the estimates

$$|u|_W \;\le\; \frac{1}{r}\left(1+\frac{|a|_{W^1_\infty(\mathbb{T}^m)}}{r}\right)\|f\|_{L_2(\mathbb{T}^m)},$$

and

$$\|u\|_W \;\le\; \frac{1}{r}\left[1+\left(1+\frac{|a|_{W^1_\infty(\mathbb{T}^m)}}{r}\right)\right]\|f\|_{L_2(\mathbb{T}^m)}.$$

This spatial regularity implies certain approximation rate if we use trigonometric polynomials in a Galerkin approach. For a real positive number $T\ge 1$ we define the index set

$$G_{\mathbb{Z}^m}(T) := \{\mathbf{k}\in\mathbb{Z}^m :\; |\mathbf{k}|_\infty \le T\}.$$

Denote by $\mathcal{T}_n$ with $n = (2\lfloor T\rfloor)^m = |G_{\mathbb{Z}^m}(T)|$ the space of trigonometric polynomials

$$\mathcal{T}_n := \left\{ v : \sum_{\mathbf{k}\in G_{\mathbb{Z}^m}(T)} v_\mathbf{k} e_\mathbf{k} \right\}$$

of dimension n. Let P_n be the projection from $L_2(\mathbb{T}^m)$ onto $\mathcal{T}_n$. Then, using $2^{-1}n^{1/m} = \lfloor T\rfloor \le T \le 2^{-1}n^{1/m}+1$ and $T\ge 1$, we get that

$$(2.9)\qquad \begin{aligned} \|u-P_n(u)\|_V &\le 2\pi\left(m\sum_{\mathbf{k}\in\mathbb{Z}^m\setminus G_{\mathbb{Z}^m}(T)} |\mathbf{k}|_\infty^2 |u_\mathbf{k}|^2\right)^{\frac12} \le 2\pi\sqrt{m}\left(\sum_{\mathbf{k}\in\mathbb{Z}^m\setminus G_{\mathbb{Z}^m}(T)} T^{-2}|\mathbf{k}|_\infty^4 |u_\mathbf{k}|^2\right)^{\frac12} \\ &\le 2\pi\sqrt{m}T^{-1}|u|_W \le 4\pi\sqrt{m}n^{-1/m}|u|_W \end{aligned}$$

holds for all $u \in W$. Furthermore, we obtain $n_\varepsilon(\mathbf{W}, V) \lesssim |G_{\mathbb{Z}^m}(\varepsilon^{-1})|$ for $0 < \varepsilon \leq 1$, where $\mathbf{W} := \{v \in W : |v|_W \leq 1\}$. Let u_n be the Galerkin approximation, i.e., the unique solution of the problem

$$\int_{\mathbb{T}^m} a(\mathbf{x}, \mathbf{y}) \nabla u_n(\mathbf{x}, \mathbf{y}) \cdot \nabla v(\mathbf{x}) \, \mathrm{d}\mathbf{x} \;=\; \int_{\mathbb{T}^m} f(\mathbf{x}) \, v(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad \forall v \in \mathcal{T}_n.$$

Then, we get with Céa's lemma and (2.9) that

$$\|u - u_n\|_V \;\leq\; \sqrt{\frac{R}{r}} \inf_{v \in \mathcal{T}_n} \|u - v\|_V \;=\; \sqrt{\frac{R}{r}} \, \|u - P_n(u)\|_V \;\leq\; \sqrt{\frac{R}{r}} \, 4\pi \sqrt{m} n^{-1/m} |u|_W \;\leq\; C \, n^{-1/m},$$

where we can explicitly compute the constant to be

$$C := \; 4\pi \sqrt{\frac{mR}{r}} \, \frac{1}{r} \left(1 + \frac{|a|_{W^1_\infty(\mathbb{T}^m)}}{r} \right) \|f\|_{L_2(\mathbb{T}^m)}.$$

2.3. Parametric regularity. A probability measure on $\mathbb{I}^\infty$ is the infinite tensor product measure μ of the univariate uniform probability measures on the one-dimensional $\mathbb{I}$, i.e.

$$\mathrm{d}\mu(\mathbf{y}) \;=\; \bigotimes_{j \in \mathbb{N}} \frac{1}{2} \mathrm{d}y_j.$$

Here, the sigma algebra Σ for μ is generated by the finite rectangles $\prod_{j \in \mathbb{N}} I_j$, where only a finite number of the I_j are different from $\mathbb{I}$ and those that are different are intervals contained in $\mathbb{I}$. Then, $(\mathbb{I}^\infty, \Sigma, \mu)$ is a probability space.

Now, let $L_2(\mathbb{I}^\infty, \mu)$ denote the Hilbert space of functions on $\mathbb{I}^\infty$ equipped with the inner product

$$\langle v, w \rangle := \int_{\mathbb{I}^\infty} v(\mathbf{y}) \overline{w(\mathbf{y})} \, \mathrm{d}\mu(\mathbf{y}).$$

The norm in $L_2(\mathbb{I}^\infty, \mu)$ is defined as $\|v\| := \langle v, v \rangle^{1/2}$. In what follows, μ is fixed, and, for convention, we write $L_2(\mathbb{I}^\infty, \mu) := L_2(\mathbb{I}^\infty)$. Furthermore, let $L_2(\mathbb{T}^m)$ be the usual Hilbert space of Lebesgue square-integrable functions on $\mathbb{T}^m$ based on the univariate normed Lebesgue measure. Then, we define

$$L_2(\mathbb{T}^m \times \mathbb{I}^\infty) := \; L_2(\mathbb{T}^m) \otimes L_2(\mathbb{I}^\infty).$$

For $s \in \mathbb{N}$ the space $L_2(\mathbb{T}^m \times \mathbb{I}^s) = L_2(\mathbb{T}^m) \otimes L_2(\mathbb{I}^s)$ can be considered as a subspace of $L_2(\mathbb{T}^m \times \mathbb{I}^\infty)$.

Let us reformulate the parametric equation (2.6) in the variational form. For every $\mathbf{y} \in \mathbb{I}^\infty$, by the well-known Lax-Milgram lemma, there exists a unique solution $u(\mathbf{y}) \in V$ in weak form which satisfies the variational equation

$$\int_{\mathbb{T}^m} a(\mathbf{x}, \mathbf{y}) \nabla u(\mathbf{y})(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \, \mathrm{d}\mathbf{x} \;=\; \int_{\mathbb{T}^m} f(\mathbf{x}) \, v(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad \forall v \in V.$$

Moreover, $u(\mathbf{y})$ satisfies the estimate

$$\|u(\mathbf{y})\|_V \;\leq\; \frac{\|f\|_{V^*}}{r}, \; \forall \mathbf{y} \in \mathbb{I}^\infty. \tag{2.10}$$

Denote by $L_2(\mathbb{I}^\infty, V, \mu)$ the Bochner space of all mappings v from $\mathbb{I}^\infty$ to V such that the norm

$$\|v\|_{L_2(\mathbb{I}^\infty, V, \mu)} := \left(\int_{\mathbb{I}^\infty} \|v(\mathbf{y})\|_V^2 \, \mathrm{d}\mu(\mathbf{y}) \right)^{1/2}$$

is finite. Furthermore, denote by $L_\infty(\mathbb{I}^\infty, V)$ the space of all mappings v from $\mathbb{I}^\infty$ to V such that v is defined everywhere in $\mathbb{I}^\infty$ and uniformly bounded in V, and that the norm

$$\|v\|_{L_\infty(\mathbb{I}^\infty, V)} := \sup_{\mathbf{y} \in \mathbb{I}^\infty} \|v(\mathbf{y})\|_V$$

is finite. Since μ is a probability measure, we have that $\|v\|_{L_2(\mathbb{I}^\infty,V,\mu)} \le \|v\|_{L_\infty(\mathbb{I}^\infty,V)}$ for every $v \in L_\infty(\mathbb{I}^\infty,V)$. Hence, the space $L_\infty(\mathbb{I}^\infty,V)$ is continuously embedded into $L_2(\mathbb{I}^\infty,V,\mu)$ and we can write $L_\infty(\mathbb{I}^\infty,V) \subset L_2(\mathbb{I}^\infty,V,\mu)$. On the other hand, by (2.10) we have that $u \in L_\infty(\mathbb{I}^\infty,V)$. From the inclusions $u \in L_\infty(\mathbb{I}^\infty,V) \subset L_2(\mathbb{I}^\infty,V,\mu)$ it follows that u admits the unique expansions

$$u = \sum_{\mathbf{s}\in\mathbb{F}} u_{\mathbf{s}}\, L_{\mathbf{s}}, \tag{2.11}$$

where $\{L_s\}_{s=0}^\infty$ is the family of univariate orthonormal Legendre polynomials in $L_2(\mathbb{I})$ and

$$L_{\mathbf{s}}(\mathbf{y}) := \prod_{j\in\mathrm{supp}(\mathbf{s})} L_{s_j}(y_j).$$

The expansion (2.11) for u converges in $L_2(\mathbb{I}^\infty,V,\mu)$, where the Legendre coefficients $u_{\mathbf{s}} \in V$ are defined by

$$u_{\mathbf{s}} := \langle u, L_{\mathbf{s}}\rangle := \int_{\mathbb{I}^\infty} u(\mathbf{y}) L_{\mathbf{s}}(\mathbf{y})\,\mathrm{d}\mu(\mathbf{y}) \qquad \mathbf{s}\in\mathbb{F}.$$

From [1, Theorem 2.1] (or from the more general bound (2.3) for the parametric elliptic PDEs (2.6)) and the formulas for the Legendre coefficients

$$u_{\mathbf{s}} \;=\; \frac{1}{\mathbf{s}!}\prod_{j:\ s_j\neq 0}\frac{\sqrt{2s_j+1}}{2^{s_j}}\int_{\mathbb{I}^\infty}\partial_{\mathbf{y}}^{\mathbf{s}}u(\mathbf{y})\prod_{j:\ s_j\neq 0}(1-y_j^2)^{s_j}\,\mathrm{d}\mu(\mathbf{y})$$

we derive the following result.

Lemma 2.1. *Assume that the diffusions a are infinitely times differentiable with respect to $\mathbf{y}$ and that there exists a positive sequence $\mathbf{a} = (a_j)_{j\in\mathbb{N}}$ such that*

$$\|\partial_{\mathbf{y}}^{\mathbf{s}} a(\mathbf{y})\|_V \;\le\; \mathbf{a}^{\mathbf{s}}, \quad \mathbf{y}\in\mathbb{I}^\infty, \quad \mathbf{s}\in\mathbb{F}.$$

Then we have

$$\|u_{\mathbf{s}}\|_V \;\le\; K\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\,\mathbf{d}^{\mathbf{s}}, \quad \mathbf{s}\in\mathbb{F},$$

where $K := \frac{\|f\|_{V'}}{r}$ and $\mathbf{d} := \frac{\mathbf{a}}{\ln 2}$.

For the proof of the following lemma see [6, Lemma 5.5].

Lemma 2.2. *Assume that $f \in L_2(\mathbb{T}^m)$, and assume that the diffusions $a \in L_\infty(\mathbb{I}^\infty, W_\infty^1(\mathbb{T}^m))$ and they are affinely dependent with respect to $\mathbf{y}$ as*

$$a(\mathbf{y})(\mathbf{x}) \;=\; \overline{a}(\mathbf{x}) \;+\; \sum_{j=1}^\infty y_j\,\psi_j(\mathbf{x}), \quad \mathbf{x}\in\mathbb{T}^m,\ \mathbf{y}\in\mathbb{I}^\infty, \quad \overline{a},\psi_j \in W_\infty^1(\mathbb{T}^m). \tag{2.12}$$

Then we have that

$$\|u_{\mathbf{s}}\|_W \;\le\; K\frac{|\mathbf{s}|_1}{!}\mathbf{s}!\,\mathbf{d}^{\mathbf{s}}, \quad \mathbf{s}\in\mathbb{F},$$

where

$$K := \frac{1}{r}\left(1+\left(1+\frac{|a|_{L_\infty(\mathbb{I}^\infty,W_\infty^1(\mathbb{T}^m))}}{r}\right)\right)\|f\|_{L_2(\mathbb{T}^m)}$$

and $\mathbf{d} = (d_j)_{j\in\mathbb{N}}$ with

$$d_j := \frac{1}{r\sqrt{3}}\left(\left(\frac{|a|_{L_\infty(\mathbb{I}^\infty,W_\infty^1(\mathbb{T}^m))}}{r}+2\right)\|\psi_j\|_{L_\infty(\mathbb{T}^m)} + |\psi_j|_{W_\infty^1(\mathbb{T}^m)}\right).$$

The affine structure in (2.12) makes it easy to check the condition (2.2). Furthermore, see [12, Section 2.3] for more details where the setting of general operator equations includes parametric elliptic PDEs as special case.

We will see in Section 4 that the spatial and parametric regularities of the solution u to (2.6) induce a joint regularity in infinite tensor product Hilbert spaces which is appropriate to hyperbolic cross approximation in infinite dimension.

3. Approximation in infinite tensor product Hilbert spaces of joint regularity. In this section, we recall some results on approximation in infinite tensor product Hilbert spaces of joint regularity which were proven in [7, Subsection 3.1]. We first introduce the notion of the infinite tensor product of separable Hilbert spaces. Let H_j, $j = 1, ..., m$, be separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_j$. First, we define the finite-dimensional tensor product of H_j, $j = 1, ..., m$, as the tensor vector space $H_1 \otimes H_2 \otimes \cdots \otimes H_m$ equipped with the inner product

$$\langle \otimes_{j=1}^m \phi_j, \otimes_{j=1}^m \psi_j \rangle := \prod_{j=1}^m \langle \phi_j, \psi_j \rangle_j \quad \text{for all } \phi_j, \psi_j \in H_j. \tag{3.1}$$

By taking the completion under this inner product, the resulting Hilbert space is defined as the tensor product space $H_1 \otimes H_2 \otimes \cdots \otimes H_m$ of H_j, $j = 1, ..., m$. Next, we consider the infinite-dimensional case. If $H_j, j \in \mathbb{N}$, is a collection of separable Hilbert spaces and $\xi_j, j \in \mathbb{N}$, is a collection of unit vectors in these Hilbert spaces then the infinite tensor product $\otimes_{j\in\mathbb{N}} H_j$ is the completion of the set of all finite linear combinations of simple tensor vectors $\otimes_{j\in\mathbb{N}} \phi_j$ where all but finitely many of the ϕ_j's are equal to the corresponding ξ_j. The inner product of $\otimes_{j\in\mathbb{N}} \phi_j$ and $\otimes_{j\in\mathbb{N}} \psi_j$ is defined as in (3.1) with $m = \infty$. For details on infinite tensor product of Hilbert spaces, see [2].

Now, we will need a tensor product of Hilbert spaces of a special structure. Let H_1 and H_2 be two given infinite-dimensional separable Hilbert spaces. Consider the infinite tensor product Hilbert space

$$\mathcal{L} := H_1^m \otimes H_2^\infty \qquad \text{where} \qquad H_1^m := \otimes_{j=1}^m H_1, \quad H_2^\infty := \otimes_{j=1}^\infty H_2.$$

Let $\{\phi_{1,k}\}_{k\in\mathbb{Z}}$ and $\{\phi_{2,s}\}_{s\in\mathbb{N}}$ be given orthonormal bases of H_1 and H_2, respectively. Then, $\{\phi_{1,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^m}$ and $\{\phi_{2,\mathbf{s}}\}_{\mathbf{s}\in\mathbb{F}}$ are orthonormal bases of H_1^m and H_2^∞, respectively, where

$$\phi_{1,\mathbf{k}} := \otimes_{j=1}^m \phi_{1,k_j}, \ \mathbf{k} \in \mathbb{Z}^m, \quad \phi_{2,\mathbf{s}} := \otimes_{j=1}^\infty \phi_{2,s_j}, \ \mathbf{s} \in \mathbb{F}.$$

Moreover, the set $\{\phi_{\mathbf{k},\mathbf{s}}\}_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}}$ is an orthonormal basis of $\mathcal{L}$, where

$$\phi_{\mathbf{k},\mathbf{s}} := \phi_{1,\mathbf{k}} \otimes \phi_{2,\mathbf{s}}.$$

Thus, every $v \in \mathcal{L}$ can be represented by the series

$$v = \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} v_{\mathbf{k},\mathbf{s}} \, \phi_{\mathbf{k},\mathbf{s}},$$

where for $v = v_1 \otimes v_2$,

$$v_{\mathbf{k},\mathbf{s}} := \langle v, \phi_{\mathbf{k},\mathbf{s}} \rangle_{\mathcal{L}} = (v_1, \phi_{1,\mathbf{k}})_{H_1^m} \, (v_2, \phi_{2,\mathbf{s}})_{H_2^\infty}$$

is the Fourier coefficient with index $(\mathbf{k}, \mathbf{s})$ of v with respect to the orthonormal basis $\{\phi_{\mathbf{k},\mathbf{s}}\}_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}}$. Furthermore, there holds Parseval's identity

$$\|v\|_{\mathcal{L}}^2 = \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} |v_{\mathbf{k},\mathbf{s}}|^2.$$

Now let us assume that a general sequence of scalars $\lambda := \{\lambda(\mathbf{k}, \mathbf{s})\}_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}}$ is given. Then, we define the associated space

$$\mathcal{L}^\lambda := \left\{ v \in \mathcal{L} \ : \ \tilde{v} := \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} \lambda(\mathbf{k}, \mathbf{s}) \, v_{\mathbf{k},\mathbf{s}} \phi_{\mathbf{k},\mathbf{s}} \in \mathcal{L} \right\}. \tag{3.2}$$

For $v \in \mathcal{L}^\lambda$, we define

$$\|v\|_{\mathcal{L}^\lambda}^2 := \|\tilde{v}\|_{\mathcal{L}}^2 = \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} |\lambda(\mathbf{k}, \mathbf{s})|^2 \, |v_{\mathbf{k},\mathbf{s}}|^2, \tag{3.3}$$

where the last equality stems from Parseval's identity.

Define $\mathbb{F}_s := \{\, \mathbf{s} \in \mathbb{F} : \ \mathrm{supp}(\mathbf{s}) \subset \{1, \cdots, s\} \,\}$. We consider

$$\mathcal{L}_s := \left\{ v = \sum_{(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^m \times \mathbb{F}_s} v_{\mathbf{k},\mathbf{s}} \, \phi_{\mathbf{k},\mathbf{s}} \right\} \quad \text{and} \quad \mathcal{L}_s^\lambda := \ \mathcal{L}^\lambda \cap \mathcal{L}_s. \tag{3.4}$$

Moreover, let the non-vanishing sequences of scalars $\lambda := \{\lambda(\mathbf{k},\mathbf{s})\}_{(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^m \times \mathbb{F}}$ and $\nu := \{\nu(\mathbf{k},\mathbf{s})\}_{(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^m \times \mathbb{F}}$ be given with associated spaces $\mathcal{L}^\lambda$ and $\mathcal{L}^\nu$ with corresponding norms and subspaces $\mathcal{L}_s^\lambda$ and $\mathcal{L}_{\mathbf{s}}^\nu$, c.f. (3.4). As in Section 2.2, we define for $T \geq 1$ the index-set

$$G_{\mathbb{Z}^m \times \mathbb{F}}(T) := \ \big\{ (\mathbf{k},\mathbf{s}) \in \mathbb{Z}^m \times \mathbb{F} : \, 0 < |\lambda(\mathbf{k},\mathbf{s})/\nu(\mathbf{k},\mathbf{s})| \leq T \big\},$$

which induces a subspace

$$\mathcal{P}(T) := \left\{ g \in \mathcal{L} \ : \ v \ = \sum_{(\mathbf{k},\mathbf{s}) \in G_{\mathbb{Z}^m \times \mathbb{F}}(T)} v_{\mathbf{k},\mathbf{s}} \, \phi_{\mathbf{k},\mathbf{s}} \right\} \subset \mathcal{L}.$$

We are interested in the $\mathcal{L}^\nu$-norm approximation of elements from $\mathcal{L}^\lambda$ by elements from $\mathcal{P}(T)$. To this end, for $v \in \mathcal{L}$ and $T \geq 1$, we define the operator $\mathcal{S}_T$ as

$$\mathcal{S}_T(v) := \sum_{(\mathbf{k},\mathbf{s}) \in G_{\mathbb{Z}^m \times \mathbb{F}}(T)} v_{\mathbf{k},\mathbf{s}} \, \phi_{\mathbf{k},\mathbf{s}}.$$

We make the assumption throughout this section that $G_{\mathbb{Z}^m \times \mathbb{F}}(T)$ is a finite set for every $T \geq 1$. Obviously, $\mathcal{S}_T$ is the orthogonal projection onto $\mathcal{P}(T)$. Furthermore, we define the set $G_{\mathbb{Z}^m \times \mathbb{F}_s}(T)$, the subspace $\mathcal{P}_s(T)$ and the operator $\mathcal{S}_{s,T}(v)$ in the same way by replacing $\mathbb{F}$ with $\mathbb{F}_s$.

The following lemma gives an upper bound for the error of the orthogonal projection $\mathcal{S}_T$ with respect to the parameter T.

Lemma 3.1. *For arbitrary $T \geq 1$, we have that*

$$\|v - \mathcal{S}_T(v)\|_{\mathcal{L}^\nu} \ \leq \ T^{-1} \|v\|_{\mathcal{L}^\lambda}, \qquad \forall v \in \mathcal{L}^\lambda \cap \mathcal{L}^\nu.$$

Recall that $\mathcal{U}^\lambda$ is the unit ball in $\mathcal{L}^\lambda$, i.e., $\mathcal{U}^\lambda := \ \{v \in \mathcal{L}^\lambda : \|v\|_{\mathcal{L}^\lambda} \leq 1\}$, and denote by $\mathcal{U}_s^\lambda$ the unit ball in $\mathcal{L}_s^\lambda$, i.e., $\mathcal{U}_s^\lambda := \ \{v \in \mathcal{L}_s^\lambda : \|v\|_{\mathcal{L}_s^\lambda} \leq 1\}$. We then have the following corollary.

Corollary 3.2. *For arbitrary $T \geq 1$,*

$$\sup_{v \in \mathcal{U}^\lambda} \inf_{w \in \mathcal{P}(T)} \|v - w\|_{\mathcal{L}^\nu} \ = \ \sup_{v \in \mathcal{U}^\lambda} \|v - \mathcal{S}_T(v)\|_{\mathcal{L}^\nu} \ \leq \ T^{-1}.$$

Now we are in the position to give lower and upper bounds on the ε-dimension $n_\varepsilon(\mathcal{U}^\lambda, \mathcal{L}^\nu)$.

Lemma 3.3. *Let $\varepsilon \in (0,1]$. Then, we have*

$$|G_{\mathbb{Z}^m \times \mathbb{F}}(1/\varepsilon)| - 1 \ \leq \ n_\varepsilon(\mathcal{U}^\lambda, \mathcal{L}^\nu) \ \leq \ |G_{\mathbb{Z}^m \times \mathbb{F}}(1/\varepsilon)|.$$

In a similar way, by using the set $G_{\mathbb{Z}^m \times \mathbb{F}_s}(T)$, the subspace $\mathcal{P}_s(T)$ and the operator $\mathcal{S}_{s,T}(f)$, we can prove the following lemma for $n_\varepsilon(\mathcal{U}_s^\lambda, \mathcal{L}_{\mathbf{s}}^\nu)$.

Lemma 3.4. *Let $\varepsilon \in (0,1]$. Then we have*

$$|G_{\mathbb{Z}^m \times \mathbb{F}_s}(1/\varepsilon)| - 1 \ \leq \ n_\varepsilon(\mathcal{U}_s^\lambda, \mathcal{L}_{\mathbf{s}}^\nu) \ \leq \ |G_{\mathbb{Z}^m \times \mathbb{F}_s}(1/\varepsilon)|.$$

These lemmas show that we need to estimate the cardinality of the index sets $|G_{\mathbb{Z}^m \times \mathbb{F}}(1/\varepsilon)|$ and $|G_{\mathbb{Z}^m \times \mathbb{F}_s}(1/\varepsilon)|$. We will treat this problem in Section 5 for infinite tensor product Hilbert spaces of joint regularity which are related to the solution of parametric PDEs.

4. Joint regularity of the solution of parametric elliptic PDEs. In order to apply our results on approximation in Section 3 to the parametric elliptic model problem (2.6) we show that the solution to this problem belongs to certain infinite tensor product Hilbert spaces of joint regularity. To this end, we combine the results from Subsections 2.2 and 2.3 to derive explicit formulas for the sequences λ and ν for these spaces.

We focus on functions defined in $L_2(\mathbb{T}^m) \otimes L_2(\mathbb{I}^\infty)$. Let $e_k(x) := e^{i2\pi kx}$. Then $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $L_2(\mathbb{T})$. Let $\{L_s\}_{s=0}^\infty$ be the family of univariate orthonormal Legendre polynomials in $L_2(\mathbb{I})$. For $(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^m \times \mathbb{F}$, we define

$$L_{(\mathbf{k},\mathbf{s})}(\mathbf{x},\mathbf{y}) := \; e_{\mathbf{k}}(\mathbf{x}) L_{\mathbf{s}}(\mathbf{y}), \quad e_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^m e_{k_j}(x_j), \; L_{\mathbf{s}}(\mathbf{y}) := \prod_{j\in \mathrm{supp}(\mathbf{s})} L_{s_j}(y_j).$$

Note that $\{L_{(\mathbf{k},\mathbf{s})}\}_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}}$ is an orthonormal basis of $L_2(\mathbb{T}^m \times \mathbb{I}^\infty)$. Moreover, we have the following expansion for every $v \in L_2(\mathbb{T}^m \times \mathbb{I}^\infty)$,

$$v = \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} v_{\mathbf{k},\mathbf{s}} L_{(\mathbf{k},\mathbf{s})},$$

where for $(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^m \times \mathbb{F}$, $v_{\mathbf{k},\mathbf{s}} := \langle v, L_{(\mathbf{k},\mathbf{s})}\rangle$ denotes the Fourier coefficient with index $(\mathbf{k},\mathbf{s})$ of v with respect to the orthonormal basis $\{L_{(\mathbf{k},\mathbf{s})}\}_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}}$.

We present two specific examples for sequences λ and their associated function spaces $\mathcal{L}^\lambda$ which naturally arise in the regularity theory of parametric elliptic partial differential equations, in particular, of problem (2.6). Let the pair $\alpha, \mathbf{b}$ be given by

$$\alpha > 0; \quad \mathbf{b} = (b_j)_{j\in\mathbb{N}}, \; b_j > 0, \; j \in \mathbb{N}.$$

For each $(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^m \times \mathbb{F}$, we define the scalar $\rho(\mathbf{k},\mathbf{s})$ by

$$\rho(\mathbf{k},\mathbf{s}) \;=\; \rho_{\alpha,\mathbf{b}}(\mathbf{k},\mathbf{s}) := \; |\mathbf{k}|_\infty^\alpha \, \frac{\mathbf{s}!}{|\mathbf{s}|_1!} \mathbf{b}^{-\mathbf{s}}. \tag{4.1}$$

Then, we define the associated space

$$\begin{aligned} \mathcal{L}^\rho &=: A^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty) \\ &= \left\{ v \in L_2(\mathbb{T}^m \times \mathbb{I}^\infty) \; : \; \tilde{v} := \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} \rho_{\alpha,\mathbf{b}}(\mathbf{k},\mathbf{s})\, v_{\mathbf{k},\mathbf{s}}\, L_{(\mathbf{k},\mathbf{s})} \in L_2(\mathbb{T}^m \times \mathbb{I}^\infty) \right\}. \end{aligned}$$

For $v \in A^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$, we define

$$\|v\|^2_{A^{\alpha,\mathbf{b}}(\mathbb{T}^m\times\mathbb{I}^\infty)} \; := \|\tilde{v}\|^2_{L_2(\mathbb{T}^m\times\mathbb{I}^\infty)} = \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} \rho^2_{\alpha,\mathbf{b}}(\mathbf{k},\mathbf{s}) \, |v_{\mathbf{k},\mathbf{s}}|^2 .$$

Next, for $\beta \geq 0$, we set

$$\theta(\mathbf{k},\mathbf{s}) = \theta_\beta(\mathbf{k},\mathbf{s}) := |\mathbf{k}|_\infty^\beta$$

where β determines the spatial regularity and use this to define the Sobolev-type space

$$\begin{aligned} \mathcal{L}^\theta &= K^\beta(\mathbb{T}^m \times \mathbb{I}^\infty) \\ &:= \left\{ v \in L_2(\mathbb{T}^m \times \mathbb{I}^\infty) \; : \; \tilde{v} := \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} \theta_\beta(\mathbf{k},\mathbf{s})\, v_{\mathbf{k},\mathbf{s}}\, L_{(\mathbf{k},\mathbf{s})} \in L_2(\mathbb{T}^m \times \mathbb{I}^\infty) \right\}. \end{aligned}$$

Again, for $v \in K^\beta(\mathbb{T}^m \times \mathbb{I}^\infty)$, we define

$$\|v\|^2_{K^\beta(\mathbb{T}^m\times\mathbb{I}^\infty)} \; := \|\tilde{v}\|^2_{L_2(\mathbb{T}^m\times\mathbb{I}^\infty)} = \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} |v_{\mathbf{k},\mathbf{s}}|^2 \, \theta^2_\beta(\mathbf{k},\mathbf{s}).$$

Lemma 4.1. *We have*

$$\|v\|_{L_2(\mathbb{I}^\infty,V,\mu)} \;\le\; 2\pi\sqrt{m}\|v\|_{K^1(\mathbb{T}^m\times\mathbb{I}^\infty)}, \quad v\in K^1(\mathbb{T}^m\times\mathbb{I}^\infty).$$

and

$$\|v\|_{L_2(\mathbb{I}^\infty,W,\mu)} \;\le\; \sqrt{2}(2\pi)^2 m\|v\|_{K^2(\mathbb{T}^m\times\mathbb{I}^\infty)}, \quad v\in K^2(\mathbb{T}^m\times\mathbb{I}^\infty).$$

Proof. For a function $v\in K^1(\mathbb{T}^m\times\mathbb{I}^\infty)$ of the form

$$v \;=\; \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} v_{\mathbf{k},\mathbf{s}} L_{(\mathbf{k},\mathbf{s})} \;=\; \sum_{\mathbf{s}\in\mathbb{F}} v_{\mathbf{s}}\, L_{\mathbf{s}}, \quad v_{\mathbf{s}} := \sum_{\mathbf{k}\in\mathbb{Z}^m} v_{\mathbf{k},\mathbf{s}} e_{\mathbf{k}}, \tag{4.2}$$

we have by (2.7)

$$\|v\|^2_{L_2(\mathbb{I}^\infty,V,\mu)} \;=\; \sum_{\mathbf{s}\in\mathbb{F}}\|v_{\mathbf{s}}\|^2_V \;\le (2\pi)^2 m \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} |\mathbf{k}|^2_\infty |v_{\mathbf{k},\mathbf{s}}|^2 \;=\; (2\pi)^2 m\|v\|^2_{K^1(\mathbb{T}^m\times\mathbb{I}^\infty)}.$$

Similarly, if $v\in K^2(\mathbb{T}^m\times\mathbb{I}^\infty)$ and is of the form (4.2), we obtain with (2.7) and (2.8)

$$\begin{aligned}
\|v\|^2_{L_2(\mathbb{I}^\infty,W,\mu)} \;&=\; \sum_{\mathbf{s}\in\mathbb{F}}\|v_{\mathbf{s}}\|^2_W \;=\; \sum_{\mathbf{s}\in\mathbb{F}}\big(\|v_{\mathbf{s}}\|^2_V + |v_{\mathbf{s}}|^2_W\big)\\
&\le \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} \big((2\pi)^2 m|\mathbf{k}|^2_\infty + (2\pi)^4 m^2|\mathbf{k}|^4_\infty\big)|v_{\mathbf{k},\mathbf{s}}|^2\\
&\le 2(2\pi)^4 m^2 \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} \theta_2^2(\mathbf{k},\mathbf{s})|v_{\mathbf{k},\mathbf{s}}|^2 \;=\; 2(2\pi)^4 m^2\|v\|^2_{K^2(\mathbb{T}^m\times\mathbb{I}^\infty)}.
\end{aligned} \tag{4.3}$$

□

Lemma 4.2. *Let $0<p\le\infty$ and $\mathbf{b}=(b_j)_{j\in\mathbb{N}}$ be a positive sequence. Then the sequence $\big(\mathbf{b}^{\mathbf{s}}\big)_{\mathbf{s}\in\mathbb{F}}$ belongs to $\ell_p(\mathbb{F})$ if and only if $\|\mathbf{b}\|_{\ell_\infty(\mathbb{N})}<1$ and $\mathbf{b}\in\ell_p(\mathbb{N})$.*

Proof. The proof of this lemma is the same as that of Lemma 7.1 in [4]. □

Lemma 4.3. *Let the assumptions and notation of Lemma 2.1 hold. Let furthermore $\mathbf{c}=(c_j)_{j\in\mathbb{N}}$ be any positive sequence such that $c_j>1$ and such that the sequence $\mathbf{c}^{-1}=(c_j^{-1})_{j\in\mathbb{N}}$ belongs to $\ell_2(\mathbb{N})$. Then, for the sequence*

$$\mathbf{b} := (b_j)_{j\in\mathbb{N}}, \quad b_j := c_j d_j,$$

the solution u to (2.6) belongs to $A^{1,\mathbf{b}} := A^{1,\mathbf{b}}(\mathbb{T}^m\times\mathbb{I}^\infty)$ and

$$\|u\|_{A^{1,\mathbf{b}}} \;\le\; K\,\|(\mathbf{c}^{-\mathbf{s}})\|_{\ell_2(\mathbb{F})}.$$

Proof. Notice that by Lemma 4.2 the sequence $\big(\mathbf{c}^{-\mathbf{s}}\big)_{\mathbf{s}\in\mathbb{F}}$ belongs to $\ell_2(\mathbb{F})$. Hence, by (2.7) and Lemma 2.1 we have that

$$\begin{aligned}
\|u\|^2_{A^{1,\mathbf{b}}} \;&=\; \sum_{(\mathbf{k},\mathbf{s})\in\mathbb{Z}^m\times\mathbb{F}} |\mathbf{k}|^2_\infty \left(\frac{\mathbf{s}!}{|\mathbf{s}|_1!}\mathbf{b}^{-\mathbf{s}}\right)^2 |u_{\mathbf{k},\mathbf{s}}|^2 \;\le\; \sum_{\mathbf{s}\in\mathbb{F}}\left(\frac{\mathbf{s}!}{|\mathbf{s}|_1!}\mathbf{b}^{-\mathbf{s}}\right)^2\|u_{\mathbf{s}}\|^2_V\\
&\le\; K^2\sum_{\mathbf{s}\in\mathbb{F}}\mathbf{c}^{-2\mathbf{s}} \;=\; K^2\,\|(\mathbf{c}^{-\mathbf{s}})\|^2_{\ell_2(\mathbb{F})} \;<\; \infty.
\end{aligned}$$

□

In the same way, from Eq. (2.8), Lemma 4.2 and Lemma 2.2 we deduce the following result.

LEMMA 4.4. *Let the assumptions and notation of Lemma 2.2 hold. Let furthermore* $\mathbf{c} = (c_j)_{j\in\mathbb{N}}$ *be any positive sequence such that* $c_j > 1$ *and such that the sequence* $\mathbf{c}^{-1} = (c_j^{-1})_{j\in\mathbb{N}}$ *belongs to* $\ell_2(\mathbb{N})$. *For the sequence*

$$\mathbf{b} := (b_j)_{j\in\mathbb{N}}, \quad b_j := c_j d_j,$$

the solution u *to* (2.6) *then belongs to* $A^{2,\mathbf{b}} := A^{2,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$ *and*

$$\|u\|_{A^{2,\mathbf{b}}} \;\le\; K\,\|(\mathbf{c}^{-\mathbf{s}})\|_{\ell_2(\mathbb{F})}. \tag{4.4}$$

5. The cardinality of infinite-dimensional hyperbolic crosses. For $T > 0$, consider the hyperbolic cross

$$E_{a,\mathbf{b}}(T) := \big\{(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^m \times \mathbb{F} : 0 < \rho_{a,\mathbf{b}}(\mathbf{k},\mathbf{s}) \le T\big\},$$

in the infinite-dimensional case, where we recall

$$\rho_{a,\mathbf{b}}(\mathbf{k},\mathbf{s}) := \; |\mathbf{k}|_\infty^a \, \frac{\mathbf{s}!}{|\mathbf{s}|_1!}\mathbf{b}^{-\mathbf{s}}.$$

In order to obtain estimates on the ε-dimension in the norm of $K^\beta(\mathbb{T}^m \times \mathbb{I}^\infty)$ of the unit ball in $A^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$, we want to employ Lemma 3.3 or Lemma 3.4 respectively. This, however, needs an estimate on $n = |E_{a,\mathbf{b}}(T)|$ with $a = \alpha - \beta$. In this section, we establish such an estimate for the cardinality of $E_{a,\mathbf{b}}(T)$.

As a preparatory step, we first have to study sharp conditions for the inclusion $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)_{\mathbf{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$ with $0 < p < \infty$. The main difference to the existing literature is, that we explicitly allow for $p > 1$. This result, though it is of its own interest, will be used in defining the constant in (5.12) for the cost estimate.

5.1. A condition for summability of sequences. In this subsection, given a sequence $\mathbf{b} = (b_j)_{j=1}^\infty$, we are interested in a necessary and sufficient condition for the inclusion $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)_{\mathbf{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$ with $0 < p < \infty$. We first recall a previous result for the case $0 < p \le 1$ which has been proven in [4].

THEOREM 5.1. *Let* $0 < p \le 1$ *and* $\mathbf{b} = (b_j)_{j=1}^\infty$ *be a positive sequence. Then the sequence* $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)_{\mathbf{s}\in\mathbb{F}}$ *belongs to* $\ell_p(\mathbb{F})$ *if and only if* $\|\mathbf{b}\|_{\ell_1(\mathbb{N})} < 1$ *and* $\mathbf{b} \in \ell_p(\mathbb{N})$.

As shown in [3, 4, 5], the $\ell_p(\mathbb{F})$-summability with some $0 < p < 1$ of the sequence of the energy norm of the coefficients in chaos polynomial Taylor and Legendre expansions, together with Stechkin's lemma plays a basic role in construction of nonlinear n-term approximation methods for the solution of parametric and stochastic elliptic PDEs. The proof of this $\ell_p(\mathbb{F})$-summability relies upon Theorem 5.1.

In the present paper, we need a necessary and sufficient condition on the sequence $\mathbf{b} = (b_j)_{j=1}^\infty$ for the $\ell_p(\mathbb{F})$-summability of the sequence $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)_{\mathbf{s}\in\mathbb{F}}$ in the case $0 < p < \infty$ which is a basic condition for construction of a linear approximation by orthogonal projection in the space $K^\beta := K^\beta(\mathbb{T}^m \times \mathbb{T}^\infty)$ for functions from $A^{\alpha,\mathbf{b}} := A^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{T}^\infty)$ and hence, collective Galerkin approximation in the Bochner space $L_2(\mathbb{I}^\infty, V, \mu)$ of the solution u of the parametric elliptic problem (2.6). This necessary and sufficient condition of the $\ell_p(\mathbb{F})$-summability in the case $1 < p < \infty$ as well as its proof are different from those in the case $0 < p \le 1$. In the proof, we use in particular, the following well known inequality between the arithmetic and geometric means, see, e.g., [9, 2.5, pp. 17-18]. For nonnegative numbers $a_1, ..., a_n$ and positive numbers $p_1, ..., p_n$, there holds true the inequality

$$a_1^{p_1} \cdots a_n^{p_n} \;<\; \left(\frac{a_1 p_1 + \cdots + a_n p_n}{p_1 + \cdots + p_n}\right)^{p_1+\cdots+p_n} \tag{5.1}$$

unless all the $a_1, ..., a_n$ are equal.

THEOREM 5.2. *Let* $1 < p < \infty$ *and* $\mathbf{b} = (b_j)_{j=1}^\infty$ *be a nonnegative sequence with infinitely many positive* b_j. *Then, the sequence* $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)_{\mathbf{s}\in\mathbb{F}}$ *belongs to* $\ell_p(\mathbb{F})$ *if and only if* $\|\mathbf{b}\|_{\ell_1(\mathbb{N})} \le 1$.

Proof.
Necessity. Assume that the sequence $\mathbf{b}$ is given and $\|\mathbf{b}\|_{\ell_1(\mathbb{N})} > 1$. Then we fix a $J \in \mathbb{N}$ large enough so that

$$B := \ b_1 + \cdots + b_J \ > \ 1.$$

For each $s \in \mathbb{N}$, we define $\mathbf{s}^* \in \mathbb{F}$ by

$$s_j^* = \left\lfloor s\frac{b_j}{B} \right\rfloor + 1 \ \text{ if } \ 1 \le j \le J, \ \text{ and } \ s_j^* = 0 \ \text{ if } \ j > J.$$

So $s_j^* \ge s\frac{b_j}{B}$ for every $1 \le j \le J$, and then

$$\frac{|\mathbf{s}^*|_1}{s_j^*} \ge \frac{s}{s_j^*} \ge \frac{s}{s\frac{b_j}{B}+1} = \frac{B}{b_j}\left(\frac{1}{1+\frac{B}{sb_j}}\right) \ge \frac{B}{b_j}\lambda_s, \quad \text{for all } 1 \le j \le J,$$

where

$$\Lambda_s = \min\left\{\left(1+\frac{B}{sb_j}\right)^{-1} \ : \ 1 \le j \le J\right\}.$$

Hence, we have with

$$C(\mathbf{s}^*) := \frac{|\mathbf{s}^*|_1!}{\sqrt{2\pi|\mathbf{s}^*|_1}(|\mathbf{s}^*|_1/e)^{|\mathbf{s}^*|_1}}$$

that

$$\begin{aligned}\frac{|\mathbf{s}^*|_1!}{\mathbf{s}^*!}\mathbf{b}^{\mathbf{s}^*} &= C(\mathbf{s}^\star)\left(\prod_{j=1}^J \frac{\sqrt{2\pi s_j^*}(s_j^*/e)^{s_j^*}}{s_j^*!}\right)\left(\frac{\sqrt{2\pi|\mathbf{s}^*|_1}}{\prod_{j=1}^J\sqrt{2\pi s_j^*}}\right)\left(\prod_{j=1}^J\left(\frac{|\mathbf{s}^*|_1}{s_j^*}b_j\right)^{s_j^*}\right)\\ &\ge (2\pi)^{(1-J)/2}C(\mathbf{s}^\star)\left(\prod_{j=1}^J \frac{\sqrt{2\pi s_j^*}(s_j^*/e)^{s_j^*}}{s_j^*!}\right)\left(\frac{|\mathbf{s}^*|_1}{\prod_{j=1}^J s_j^*}\right)^{1/2}(\Lambda_s B)^{|\mathbf{s}^*|_1}.\end{aligned}$$

Observe that there are a number $\sigma > 1$ and a number $\bar{s} := \bar{s}(J) \in \mathbb{N}$ large enough such that

$$\Lambda_s B \ \ge \ \sigma, \ \forall s \ge \bar{s}.$$

From the estimate

$$\frac{|\mathbf{s}^*|_1}{\prod_{j=1}^J s_j^*} \ \ge \ J^J|\mathbf{s}^*|_1^{1-J} \ \ge \ J^J(s+J)^{1-J},$$

which stems from an application of (5.1) and the observation that $|\mathbf{s}^*|_1 \le s+J$, and the Stirling formula

$$\lim_{k\to\infty}\frac{k!}{\sqrt{2\pi k}\left(\frac{k}{e}\right)^k} = 1,$$

we obtain

$$\frac{|\mathbf{s}^*|_1!}{\mathbf{s}^*!}\mathbf{b}^{\mathbf{s}^*} \ge C_J(\lambda_s B)^{|\mathbf{s}^*|_1}(s+J)^{(1-J)/2} \ge C_J\sigma^s(s+J)^{(1-J)/2}, \ \forall s \ge \bar{s},$$

where C_J is a positive constant depending on J only. Therefore, for arbitrary $s \ge \bar{s}$

$$\sum_{\mathbf{s}\in\mathbb{F}}\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)^p \ge \left(\frac{|\mathbf{s}^*|_1!}{\mathbf{s}^*!}\mathbf{b}^{\mathbf{s}^*}\right)^p \ge C_J^p\left(\sigma^s(s+J)^{(1-J)/2}\right)^p \ \to \ \infty, \ s\to\infty.$$

The necessity is proven.

Sufficiency. Assume that the sequence $\mathbf{b}$ is given and $\|\mathbf{b}\|_{\ell_1(\mathbb{N})} \le 1$. We fix an integer m satisfying the inequality $m(p-1) > 2$. Since the sequence $\mathbf{b}$ has infinitely many positive terms b_j, without loss of generality we may assume that $b_j > 0$ for all $j = 1, ..., m+1$. Put $\mathbf{s} = (\mathbf{s}', \mathbf{s}^{''})$ with $\mathbf{s}' = (s_1, ..., s_m)$ and $\mathbf{s}^{''} = (s_{m+1}, s_{m+2}, ...)$ for $\mathbf{s} \in \mathbb{F}$. We have

$$\begin{aligned}\sum_{\mathbf{s}\in\mathbb{F}} \left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)^p &= \sum_{M=0}^{\infty} \sum_{\mathbf{s}\in\mathbb{F},|\mathbf{s}|_1=M} \left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)^p \\ &= \sum_{M=0}^{\infty} \sum_{\mathbf{s}':|\mathbf{s}'|_1\le M} \sum_{\mathbf{s}''\in\mathbb{F}:|\mathbf{s}''|_1=M-|\mathbf{s}'|_1} \left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)^p \\ &\le \sum_{M=0}^{\infty} \sum_{\mathbf{s}':|\mathbf{s}'|_1\le M} \left(\sum_{\mathbf{s}''\in\mathbb{F},|\mathbf{s}''|_1=M-|\mathbf{s}'|_1} \frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)^p.\end{aligned}$$

From the well-known multinomial theorem we derive that

$$\sum_{\mathbf{s}''\in\mathbb{F},|\mathbf{s}''|_1=M-|\mathbf{s}'|_1} \frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}} = \frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}},$$

where for the sake of readability, we have made the definitions

$$\mathbf{k} = (k_1, k_2, \dots, k_m, k_{m+1}) = (s_1, s_2, \dots, s_m, M - |\mathbf{s}'|_1),$$

$$\mathbf{a} = (a_1, a_2, \dots, a_m, a_{m+1}) = (b_1, b_2, \dots, b_m, b_{m+1} + b_{m+2} + \dots).$$

Hence,

$$\sum_{\mathbf{s}\in\mathbb{F}} \left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)^p \le \sum_{M=0}^{\infty} \sum_{\mathbf{k}\in\mathbb{Z}_+^{m+1}:|\mathbf{k}|_1=M} \left(\frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}}\right)^p.$$

Putting

$$J_{1,M} := \{\mathbf{k} \in \mathbb{N}^{m+1} : |\mathbf{k}|_1 = M\}, \quad J_{2,M} := \{\mathbf{k} \in \mathbb{Z}_+^{m+1} : |\mathbf{k}|_1 = M, \prod_{j=1}^{m+1} k_j = 0\},$$

we obtain

$$\sum_{\mathbf{s}\in\mathbb{F}} \left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)^p \le \sum_{M=0}^{\infty} \sum_{\mathbf{k}\in J_{1,M}} \left(\frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}}\right)^p + \sum_{M=0}^{\infty} \sum_{\mathbf{k}\in J_{2,M}} \left(\frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}}\right)^p =: I_1 + I_2.$$

We have with the notation

$$\hat{\mathbf{k}}_j := (k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_{m+1})$$

that

$$I_2 \le \sum_{M=0}^{\infty} \sum_{j=1}^{m+1} \sum_{\mathbf{k}\in\mathbb{Z}_+^{m+1}:|\hat{\mathbf{k}}_j|_1=M} \left(\frac{M!}{\hat{\mathbf{k}}_j!} a_1^{k_1} \dots a_{j-1}^{k_{j-1}} a_{j+1}^{k_{j+1}} \dots a_{m+1}^{k_{m+1}}\right)^p.$$

On the other hand,

$$\sum_{j=1}^{m+1}\sum_{\mathbf{k}:|\hat{\mathbf{k}}_j|_1=M}\left(\frac{M!}{\hat{\mathbf{k}}_j!}a_1^{k_1}\dots a_{j-1}^{k_{j-1}}a_{j+1}^{k_{j+1}}\dots a_{m+1}^{k_{m+1}}\right)^p$$
$$\leq \sum_{j=1}^{m+1}\left(\sum_{\mathbf{k}:|\hat{\mathbf{k}}_j|_1=M}\frac{M!}{\hat{\mathbf{k}}_j!}a_1^{k_1}\dots a_{j-1}^{k_{j-1}}a_{j+1}^{k_{j+1}}\dots a_{m+1}^{k_{m+1}}\right)^p$$
$$= \sum_{j=1}^{m+1}\Big(a_1+\dots a_{j-1}+a_{j+1}+\dots+a_{m+1}\Big)^{Mp} =: \sum_{j=1}^{m+1}A_j^{Mp}.$$

Since $\mathbf{b}$ is a nonnegative sequence with infinitely many positive terms b_j, and $\|\mathbf{b}\|_{\ell_1(\mathbb{N})}\leq 1$, we deduce that $\mathbf{a}$ is a positive vector in $\mathbb{R}^{m+1}$ with $a_1+\dots+a_{m+1}\leq 1$, and consequently, $A_j<1$ for $1\leq j\leq m+1$. Hence,

$$I_2 \leq \sum_{j=1}^{m+1}\sum_{M=0}^{\infty}A_j^{Mp}<\infty.$$

Let us estimate I_1. Putting

$$J_{3,M} := \{\mathbf{k}\in\mathbb{N}^{m+1} : |\mathbf{k}|_1 = M, \frac{a_jk_i}{a_ik_j}\in[1/2,2] \ \text{ for all } \ i,j=1,...,m+1\},$$

$$J_{4,M} := \{\mathbf{k}\in\mathbb{N}^{m+1} : |\mathbf{k}|_1 = M, \frac{a_jk_i}{a_ik_j}\notin[1/2,2] \ \text{ for some } \ i,j=1,...,m+1\},$$

we split I_1 into two sums I_3 and I_4 as

$$I_1 \ = \ \sum_{M=0}^{\infty}\sum_{\mathbf{k}\in J_{3,M}}\left(\frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}}\right)^p + \sum_{M=0}^{\infty}\sum_{\mathbf{k}\in J_{4,M}}\left(\frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}}\right)^p =: I_3+I_4.$$

By Stirling's approximation,

$$\frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}} \ \leq \ C\,(2\pi)^{-m/2}\prod_{j=1}^{m+1}(Ma_j/k_j)^{k_j}\left(M\prod_{j=1}^{m+1}k_j^{-1}\right)^{1/2},$$

where C is an absolute constant.

We estimate I_3. For all $\mathbf{k}\in J_{3,M}$, we have by definition

$$a_j/k_j \leq 2(a_1+...+a_{m+1})/(k_1+...+k_{m+1}) \leq 2/M,$$

and therefore,

$$k_j^{-1}\leq 2/(a_{\min}M),$$

where

$$a_{\min} = \min\{a_1,\cdots,a_{m+1}\}>0.$$

Also, as mentioned above, we have $a_1+\dots+a_{m+1}\leq 1$. All these together with the inequality (5.1) give

$$\prod_{j=1}^{m+1}(Ma_j/k_j)^{k_j}\left(M\prod_{j=1}^{m+1}k_j^{-1}\right)^{1/2} \leq \left(\sum_{j=1}^{m+1}a_j\right)^M 2^{(m+1)/2}\left(a_{\min}^{-m-1}M^{-m}\right)^{1/2} \leq 2^{(m+1)/2}a_{\min}^{-(m+1)/2}M^{-m/2}.$$

Therefore, again, by using the well-known multinomial theorem we have that

$$\begin{aligned}\sum_{\mathbf{k}\in J_{3,M}}\left(\frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}}\right)^p &\le \left(2^{(m+1)/2}a_{\min}^{-(m+1)/2}M^{-m/2}\right)^{p-1}\sum_{\mathbf{k}\in J_{3,M}}\frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}}\\ &\le C_3M^{-m(p-1)/2}\sum_{\mathbf{k}\in J_{1,M}}\frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}}\\ &\le C_3M^{-m(p-1)/2}\left(\sum_{j=1}^{m+1}a_j\right)^M \le C_3M^{-m(p-1)/2}.\end{aligned} \tag{5.2}$$

This and the inequality $m(p-1)/2>1$ imply that

$$I_3\le C_3\sum_{M=0}^{\infty}M^{-m(p-1)/2} \ < \ \infty.$$

Now, we estimate I_4. Take any $\mathbf{k}\in J_{4,M}$, and rearrange $(1,2,\dots,m+1)$ to $(i_1,i_2,\dots,i_{m+1})$ so that

$$\frac{a_{i_1}}{k_{i_1}}\ge\frac{a_{i_2}}{k_{i_2}}\ge\cdots\ge\frac{a_{i_{m+1}}}{k_{i_{m+1}}}. \tag{5.3}$$

Then denoting $\alpha_{i_j}:=\frac{a_{i_j}}{k_{i_j}}$, by definition we have

$$\frac{\alpha_{i_1}}{\alpha_{i_{m+1}}}>2,$$

Therefore, since

$$\frac{\alpha_{i_1}}{\alpha_{i_2}}\cdot\frac{\alpha_{i_2}}{\alpha_{i_3}}\cdots\frac{\alpha_{i_m}}{\alpha_{i_{m+1}}}=\frac{\alpha_{i_1}}{\alpha_{i_{m+1}}}>2,$$

there exists $\nu\in\mathbb{N}, 1\le\nu\le m+1$ such that

$$\frac{\alpha_{i_\nu}}{\alpha_{i_{\nu+1}}}\ge\ \sqrt[m]{2}. \tag{5.4}$$

From (5.3) we have

$$\frac{a_{i_1}+a_{i_2}+\cdots+a_{i_\nu}}{k_{i_1}+k_{i_2}+\cdots+k_{i_\nu}}\ge\frac{a_{i_\nu}}{k_{i_\nu}}=\alpha_{i_\nu}$$

and

$$\frac{a_{i_{\nu+1}}+a_{i_{\nu+2}}+\cdots+a_{i_{m+1}}}{k_{i_{\nu+1}}+k_{i_{\nu+2}}+\cdots+k_{i_{m+1}}}\le\frac{a_{i_{\nu+1}}}{k_{i_{\nu+1}}}=\alpha_{i_{\nu+1}}.$$

We define the two nonempty sets: $e=\{i_1,i_2,\dots i_\nu\}\subset\{1,2,..,m\}$ and $e'=\{1,2,..,m\}\setminus e$. From (5.4) we obtain

$$\frac{(\sum_{j\in e}a_j)/(\sum_{j\in e}k_j)}{(\sum_{j\in e'}a_j)/(\sum_{j\in e'}k_j)}\ \ge\ \sqrt[m]{2}. \tag{5.5}$$

Therefore, by the inequality (5.1),

$$
\begin{aligned}
&\prod_{j=1}^{m+1}(Ma_j/k_j)^{k_j}\left(M\prod_{j=1}^{m+1}k_j^{-1}\right)^{1/2}\\
&\le M^{1/2}\prod_{j=1}^{m+1}(Ma_j/k_j)^{k_j}\\
&\le M^{1/2}\left(\frac{M\sum_{j\in e}a_j}{\sum_{j\in e}k_j}\right)^{\sum_{j\in e}k_j}\left(\frac{M\sum_{j\in e'}a_j}{\sum_{j\in e'}k_j}\right)^{\sum_{j\in e'}k_j}\\
&=: M^{1/2}\left(\frac{Mc_1}{r_1}\right)^{r_1}\left(\frac{Mc_2}{r_2}\right)^{r_2},
\end{aligned}
\tag{5.6}
$$

with $r_1+r_2=M$ and $c_1+c_2=\|\mathbf{b}\|_{\ell_1(\mathbb{N})}\le 1$, where

$$
c_1=\sum_{j\in e}a_j, c_2=\sum_{j\in e'}a_j, r_1=\sum_{j\in e}k_j, r_2=\sum_{j\in e'}a_j.
$$

Consider the function

$$
h(x)=\left(\frac{x}{Mc_1}\right)^{x}\left(\frac{M-x}{Mc_2}\right)^{M-x},\quad x\in(0,M).
$$

Notice that the function h has an absolute minimum in the interval $(0,M)$ at the point $x_{\min}=\frac{Mc_1}{c_1+c_2}$, and is decreasing in the interval $(0,x_{\min})$ and increasing in the interval $(x_{\min},M)$. By (5.5) we have

$$
\frac{c_1/r_1}{c_2/(M-r_1)}\ge\sqrt[m]{2}
$$

which implies that

$$
0\ <\ r_1\ \le\ \frac{Mc_1}{c_1+c_2\sqrt[m]{2}}\ <\ \frac{Mc_1}{c_1+c_2}\ =\ x_{\min},
$$

and therefore,

$$
(Mc_1/r_1)^{r_1}(Mc_2/(M-r_1))^{(M-r_1)}=1/h(r_1)\le 1/h(Mc_1/(c_1+c_2\sqrt[m]{2}))\ =\ \delta^M,
$$

where

$$
\delta:=\ \left(c_1+c_2\sqrt[m]{2}\right)^{c_1/(c_1+c_2\sqrt[m]{2})}\left(\frac{c_1+c_2\sqrt[m]{2}}{\sqrt[m]{2}}\right)^{(c_2\sqrt[m]{2})/(c_1+c_2\sqrt[m]{2})}.
\tag{5.7}
$$

Combining this with (5.6) we obtain

$$
\prod_{j=1}^{m+1}(Ma_j/k_j)^{k_j}\left(M\prod_{j=1}^{m+1}k_j^{-1}\right)^{1/2}\ \le\ \delta^M M^{1/2}.
$$

Hence, similarly to (5.2) we derive that

$$
\sum_{\mathbf{k}\in J_{4,M}}\left(\frac{M!}{\mathbf{k}!}\mathbf{a}^{\mathbf{k}}\right)^{p}\ \le\ C_4\delta^{(p-1)M}M^{(p-1)/2}.
\tag{5.8}
$$

Observe that the positive numbers c_1, c_2 and therefore, the positive number δ as defined in (5.7) depend only on the nonempty set $e \subset \{1, ..., m\}$, i.e., $c_1 = c_1(e)$, $c_2 = c_2(e)$ and $\delta = \delta(e)$. Consider the production in the right hand of (5.7). Since

$$c_1(e) + c_2(e)\sqrt[m]{2} \;>\; \frac{c_1(e) + c_2(e)\sqrt[m]{2}}{\sqrt[m]{2}},$$

applying the inequality (5.1) to this production with $c_1(e_{\max}), c_2(e_{\max})$, gives for all the nonempty sets $e \subset \{1, ..., m\}$,

$$0 \;<\; \delta(e) \;\le\; \delta_{\max} := \;\delta(e_{\max}) \;<\; c_1(e_{\max}) + c_2(e_{\max}) \;\le\; 1,$$

where $e_{\max} \subset \{1, ..., m\}$ is a set such that

$$\delta(e_{\max}) \;=\; \max_{e\subset\{1,...,m\},\; e\neq\varnothing} \delta(e).$$

Thus, provided with (5.8) and $\delta \le \delta_{\max} < 1$, we arrive at

$$I_4 \le C_4 \sum_{M=0}^{\infty} \delta_{\max}^{(p-1)M} M^{(p-1)/2} \;<\; \infty.$$

The proof of sufficiency is complete. □

In Theorem 5.2, the assumption that the nonnegative sequence $\mathbf{b} = (b_j)_{j=1}^{\infty}$ has infinitely many positive b_j, is essential. Indeed, if $\mathbf{b} = (b_1, b_2, 0, 0, \dots)$ with $b_1 = b_2 = 1/2$, then a computation shows that $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^s\right)_{s\in\mathbb{F}} \notin \ell_p(\mathbb{F})$ for all $p \le 2$. However, one can prove that for $3 < p < \infty$ and any non-negative sequence $\mathbf{b} = (b_j)_{j=1}^{\infty}$, the sequence $\left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^s\right)_{s\in\mathbb{F}}$ belongs to $\ell_p(\mathbb{F})$ if and only if $\|\mathbf{b}\|_{\ell_1(\mathbb{N})} \le 1$. For application we will consider only positive sequences $\mathbf{b} = (b_j)_{j=1}^{\infty}$ when this assumption always holds.

5.2. Estimates of the cardinality of infinite-dimensional hyperbolic crosses. We are now in the position to derive an estimate for the cardinality of $E_{a,\mathbf{b}}(T)$. We will do this together with estimating the cardinality of finite-dimensional hyperbolic crosses. For arbitrary $s \in \mathbb{N}$, consider the hyperbolic cross

$$E_{a,\mathbf{b},s}(T) := \;\big\{(\mathbf{k}, \mathbf{s}) \in \mathbb{Z}^m \times \mathbb{F}_s : 0 < \rho_{a,\mathbf{b}}(\mathbf{k}, \mathbf{s}) \le T\big\}$$

with the notation from (3.4).

THEOREM 5.3. *Let $a > 0$, $\mathbf{b} = (b_j)_{j\in\mathbb{N}}$ be a positive sequence. Then we have that*

$$|E_{a,\mathbf{b}}(T)| < \infty, \;\; \forall T \ge 1 \tag{5.9}$$

if only if

$$\begin{cases} \|\mathbf{b}\|_{\ell_1(\mathbb{N})} \;<\; 1, \quad \mathbf{b} \in \ell_{m/a}(\mathbb{N}), & m/a \le 1, \\ \|\mathbf{b}\|_{\ell_1(\mathbb{N})} \;\le\; 1, & m/a > 1. \end{cases} \tag{5.10}$$

If (5.9) holds, i.e., if $|E_{a,\mathbf{b}}(T)| < \infty$, then we have for every $T \ge 1$ and every $s \ge 0$,

$$2^m \left(\lfloor T^{1/a}\rfloor - 1\right)^m \;\le\; |E_{a,\mathbf{b},s}(T)| \;\le\; |E_{a,\mathbf{b}}(T)| \;\le\; C\,T^{m/a}, \tag{5.11}$$

where

$$C := \;3^m \sum_{\mathbf{s}\in\mathbb{F}} \left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!}\mathbf{b}^{\mathbf{s}}\right)^{m/a}. \tag{5.12}$$

Proof. Let $T \geq 1$ be given. For $\mathbf{s} \in \mathbb{F}$, we put

$$T_{\mathbf{s}} \;:=\; T^{1/a} \left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!} \mathbf{b}^{\mathbf{s}} \right)^{1/a}.$$

By definition we have

$$\begin{aligned} |E_{a,\mathbf{b}}(T)| \;&=\; \sum_{\mathbf{s}\in\mathbb{F}} \sum_{\mathbf{k}\in\ \mathbb{Z}^m:\ 0<|\mathbf{k}|_\infty\leq T_{\mathbf{s}}} 1 \;\leq\; \sum_{\mathbf{s}\in\mathbb{F}} (2T_{\mathbf{s}}+1)^m \;\leq\; \sum_{\mathbf{s}\in\mathbb{F}} (3T_{\mathbf{s}})^m \\ &\leq\; 3^m\, T^{m/a} \sum_{\mathbf{s}\in\mathbb{F}} \left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!} \mathbf{b}^{\mathbf{s}} \right)^{m/a} \end{aligned} \tag{5.13}$$

With the definition (5.12), this already shows the upper bound of (5.11). On the other hand, we have

$$|E_{a,\mathbf{b}}(T)| \;\geq\; \sum_{\mathbf{s}\in\mathbb{F}} T_{\mathbf{s}}^m \;=\; T^{m/a} \sum_{\mathbf{s}\in\mathbb{F}} \left(\frac{|\mathbf{s}|_1!}{\mathbf{s}!} \mathbf{b}^{\mathbf{s}} \right)^{m/a}.$$

Hence, Theorems 5.1 and 5.2 establish the equivalence between (5.9) and (5.10).

The remaining lower bound in (5.11) can be proven in the same way as that for [7, Theorem 2.13]. To be precise, we consider $(\mathbf{k}, \mathbf{0}) \in \mathbb{Z}^m \times \mathbb{F}_s \subset \mathbb{Z}^m \times \mathbb{F}$ with $k_j \leq T^{1/a} - 1$. Then a direct counting argument yields the assertion. ∎

6. Final approximation rates.

6.1. ε-dimension and n-widths. For a finite subset G in $\mathbb{Z}^m \times \mathbb{F}$, denote by $\mathcal{V}(G)$ the subspace in $L_2(\mathbb{T}^m \times \mathbb{I}^\infty)$ of all functions f of the form

$$v \;=\; \sum_{(\mathbf{k},\mathbf{s})\in G} v_{\mathbf{k},\mathbf{s}} L_{(\mathbf{k},\mathbf{s})}$$

and define the linear operator $\mathcal{S}_G :\, L_2(\mathbb{T}^m \times \mathbb{I}^\infty) \to \mathcal{V}(G)$ by

$$\mathcal{S}_G v := \sum_{(\mathbf{k},\mathbf{s})\in G} v_{\mathbf{k},\mathbf{s}} L_{(\mathbf{k},\mathbf{s})}.$$

Moreover, let $\mathcal{S}_{s,G}$ be the restriction of the operator $\mathcal{S}_G$ on $L_2(\mathbb{T}^m \times \mathbb{I}^s)$.

Then, for $s \in \mathbb{N}$, we define the spaces $A_s^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$, $K_s^\beta(\mathbb{T}^m \times \mathbb{I}^\infty)$ and $\mathcal{V}_s(G)$ as the intersections of $A^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$, $K^\beta(\mathbb{T}^m \times \mathbb{I}^\infty)$ and $\mathcal{V}(G)$ with $L_2(\mathbb{T}^m \times \mathbb{I}^s)$. Furthermore, let $U^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$ and $U_s^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$ be the unit ball in $A^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$ and $A_s^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$, respectively. In the following theorems, we drop for convenience $(\mathbb{T}^m \times \mathbb{I}^\infty)$ from the relevant notations. For example, we write $U^{\alpha,\mathbf{b}}$ instead $U^{\alpha,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$.

From the results on the cardinality of infinite-dimensional hyperbolic crosses in Section 5 and the results on approximation in infinite tensor product Hilbert spaces in Section 3 we can now deduce results on approximation in the norm of K^β of functions from $U^{\alpha,\mathbf{b}}$ and in the norm of K_s^β of functions from $U_s^{\alpha,\mathbf{b}}$ in terms of ε-dimension and n-widths as follows.

THEOREM 6.1. *Let $\alpha > \beta \geq 0$ and $\mathbf{b} = (b_j)_{j\in\mathbb{N}}$ be a positive sequence. Assume that*

$$\begin{cases} \|\mathbf{b}\|_{\ell_1(\mathbb{N})} \;<\; 1, \quad \mathbf{b} \in \ell_{m/(\alpha-\beta)}(\mathbb{N}), & m/(\alpha-\beta) \leq 1, \\ \|\mathbf{b}\|_{\ell_1(\mathbb{N})} \;\leq\; 1, & m/(\alpha-\beta) > 1. \end{cases}$$

Then we have for every $s \in \mathbb{N}$ and every $\varepsilon \in (0,1]$,

$$2^m \left(\lfloor \varepsilon^{-1/(\alpha-\beta)} \rfloor - 1 \right)^m \;\leq\; n_\varepsilon(U_s^{\alpha,\mathbf{b}}, K_s^\beta) \;\leq\; n_\varepsilon(U^{\alpha,\mathbf{b}}, K^\beta) \;\leq\; C\, \varepsilon^{-m/(\alpha-\beta)}, \tag{6.1}$$

where C is the constant defined in (5.12).

Proof. By putting $H_1 = L_2(\mathbb{T})$ and $H_2 = L_2(\mathbb{I})$; $\phi_{1,k} := e_k$ and $\phi_{2,s} := L_s$; $\lambda(\mathbf{k},\mathbf{s}) := \rho_{\alpha,\mathbf{b}}(\mathbf{k},\mathbf{s})$; $\nu(\mathbf{k},\mathbf{s}) := |\mathbf{k}|_\infty^\beta$, we have $\mathcal{L} = L_2(\mathbb{T}^m \times \mathbb{I}^\infty)$; $\mathcal{U}^\lambda = U^{\alpha,\mathbf{b}}$; $\mathcal{L}^\nu = K^\beta$; $G_{\mathbb{Z}^m\times\mathbb{F}}(1/\varepsilon) = E_{\alpha-\beta,\mathbf{b}}(1/\varepsilon)$; $G_{\mathbb{Z}^m\times\mathbb{F}_s}(1/\varepsilon) = E_{\alpha-\beta,\mathbf{b},s}(1/\varepsilon)$. The conditions on the sequence $\mathbf{b}$ allow to use (5.11) from Theorem 5.3, i.e.,

$$2^m \left(\lfloor T^{1/a}\rfloor - 1\right)^m \;\le\; |E_{a,\mathbf{b},s}(T)| \;\le\; |E_{a,\mathbf{b}}(T)| \;\le\; C\,T^{m/a}$$

with $T = \varepsilon^{-1}$. The fact that we can can replace the cardinalities of the index set by the ε-dimensions follows from Lemmas 3.3 and 3.4. ∎

Similarly, from Corollary 3.2 and Theorem 5.3 we obtain

THEOREM 6.2. *Under the assumptions of Theorem 6.1, and with $E(T) := E_{\alpha-\beta,\mathbf{b}}(T)$ and $n := |E(T)|$, we have for every $s \in \mathbb{N}$,*

$$\begin{aligned} d_n(U_s^{\alpha,\mathbf{b}}, K_s^\beta) \;\le\; d_n(U^{\alpha,\mathbf{b}}, K^\beta) \;&\le\; \sup_{v\in U^{\alpha,\mathbf{b}}} \inf_{g\in\mathcal{V}(E(T))} \|v-g\|_{K^\beta} \\ &=\; \sup_{v\in U^{\alpha,\mathbf{b}}} \|v - \mathcal{S}_{E(T)}(v)\|_{K^\beta} \;\le\; C^{(\alpha-\beta)/m}\, n^{-(\alpha-\beta)/m}. \end{aligned}$$

Notice that from Theorem 6.1 one can also derive the lower bound

$$d_n(U^{\alpha,\mathbf{b}}, K^\beta) \;\ge\; d_n(U_s^{\alpha,\mathbf{b}}, K_s^\beta) \;\ge\; C'\, n^{-(\alpha-\beta)/m},$$

where C' is a positive constant depending on α, β, m only.

6.2. Application to Galerkin approximation of parametric elliptic PDEs. We now apply our results on the ε-dimension and n-widths of Subsection 6.1 to the Galerkin approximation of parametric elliptic PDEs (2.6).

Since $u \in L_2(\mathbb{I}^\infty, V, \mu)$, it can be defined as the unique solution of the variational problem: Find $u \in L_2(\mathbb{I}^\infty, V, \mu)$ such that

$$B(u,v) \;=\; F(v) \quad \forall v \in L_2(\mathbb{I}^\infty, V, \mu),$$

where

$$\begin{aligned} B(u,v) &:= \int_{\mathbb{I}^\infty}\int_{\mathbb{T}^m} a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\mathbf{y})\cdot\nabla v(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu(\mathbf{y}), \\ F(v) &:= \int_{\mathbb{I}^\infty}\int_{\mathbb{T}^m} f(\mathbf{x})\,v(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu(\mathbf{y}). \end{aligned}$$

We define the *Galerkin approximation* u_G to u as the unique solution to the problem: Find $u_G \in \mathcal{V}(G)$ such that

$$B(u_G, v) \;=\; F(v) \quad \forall v \in \mathcal{V}(G).$$

By Céa's lemma we have the estimate

$$\|u - u_G\|_{L_2(\mathbb{I}^\infty,V,\mu)} \;\le\; \sqrt{\frac{R}{r}} \inf_{v\in\mathcal{V}(G)} \|u-v\|_{L_2(\mathbb{I}^\infty,V,\mu)},$$

and consequently,

$$\|u - u_G\|_{L_2(\mathbb{I}^\infty,V,\mu)} \;\le\; \sqrt{\frac{R}{r}}\, \|u - \mathcal{S}_G u\|_{L_2(\mathbb{I}^\infty,V,\mu)}. \tag{6.2}$$

THEOREM 6.3. *Let the assumptions and the notation of Lemma 2.2 hold. Let furthermore* $\mathbf{c} = (c_j)_{j\in\mathbb{N}}$ *be any positive sequence such that* $c_j > 1$*, such that the sequence* $\mathbf{c}^{-1} = (c_j^{-1})_{j\in\mathbb{N}}$ *belongs to* $\ell_2(\mathbb{N})$ *and such that for the sequence*

$$\mathbf{b} := (b_j)_{j\in\mathbb{N}}, \quad b_j := c_j d_j, \tag{6.3}$$

there holds the condition

$$\begin{cases} \|\mathbf{b}\|_{\ell_1(\mathbb{N})} \;<\; 1, & m = 1, \\ \|\mathbf{b}\|_{\ell_1(\mathbb{N})} \;\le\; 1, & m > 1. \end{cases}$$

For any $T \ge 1$*, put* $n := |E_{1,\mathbf{b}}(T)|$*;* $\mathcal{V}_n := \mathcal{V}\big(E_{1,\mathbf{b}}(T)\big)$*;* $\mathcal{P}_n := \mathcal{S}_{E_{1,\mathbf{b}}(T)}$*. Then* $\mathcal{P}_n$ *is the orthogonal projector from* $L_2(\mathbb{I}^\infty, V, \mu)$ *onto the space* $\mathcal{V}_n$ *of dimension* n*, and*

$$\|u - u_{E_{1,\mathbf{b}}(T)}\|_{L_2(\mathbb{I}^\infty,V,\mu)} \;\le\; \sqrt{\frac{R}{r}}\,\big\|u - \mathcal{P}_n u\big\|_{L_2(\mathbb{I}^\infty,V,\mu)} \;\le\; B\,n^{-1/m},$$

where

$$B := \; 4\pi\, C^{1/m} \sqrt{\frac{mR}{r}}\, K\, \|(\mathbf{c}^{-\mathbf{s}})\|_{\ell_2(\mathbb{F})},$$

C *is the constant defined in* (5.12) *for* $a = 1$ *and* $\mathbf{b}$ *as in* (6.3).

Proof. By Lemma 4.4 the solution u belongs to $A^{2,\mathbf{b}} := A^{2,\mathbf{b}}(\mathbb{T}^m \times \mathbb{I}^\infty)$. Hence, by (6.2), Lemma 4.1, Theorem 6.2 and (4.4) we have

$$\begin{aligned} \|u - u_{E_{1,\mathbf{b}}(T)}\|_{L_2(\mathbb{I}^\infty,V,\mu)} \;&\le\; \sqrt{\frac{R}{r}}\,\|u - \mathcal{P}_n(u)\|_{L_2(\mathbb{I}^\infty,V,\mu)} \;\le\; \sqrt{\frac{R}{r}}\,2\pi\sqrt{m}\|u - \mathcal{P}_n(u)\|_{K^1(\mathbb{T}^m\times\mathbb{I}^\infty)} \\ &\le\; \sqrt{\frac{R}{r}}\,4\pi\sqrt{m}\,C^{1/m}\|u\|_{A^{2,\mathbf{b}}}\,n^{-1/m} \;\le\; \sqrt{\frac{R}{r}}\,4\pi\sqrt{m}\,C^{1/m}\,K\,\|\mathbf{c}^{-1}\|_{\ell_2(\mathbb{F})}\,n^{-1/m} \\ &=\; B\,n^{-1/m}. \end{aligned}$$

□

The following theorem can be proven in a similar way.

THEOREM 6.4. *Let the assumptions and the notation of Lemma 2.1 hold. Let* $\mathbf{c} = (c_j)_{j\in\mathbb{N}}$ *be any positive sequence such that* $c_j > 1$*, such that the sequence* $\mathbf{c}^{-1} = (c_j^{-1})_{j\in\mathbb{N}}$ *belongs to* $\ell_2(\mathbb{N})$ *and such that for the sequence*

$$\mathbf{b} := (b_j)_{j\in\mathbb{N}}, \quad b_j := c_j d_j, \tag{6.4}$$

there holds the condition

$$\begin{cases} \|\mathbf{b}\|_{\ell_1(\mathbb{N})} \;<\; 1, & m = 1, \\ \|\mathbf{b}\|_{\ell_1(\mathbb{N})} \;\le\; 1, & m > 1. \end{cases}$$

For any $T \ge 1$*, put* $n := |E_{1,\mathbf{b}}(T)|$*;* $\mathcal{V}_n := \mathcal{V}\big(E_{1,\mathbf{b}}(T)\big)$*;* $\mathcal{P}_n := \mathcal{S}_{E_{1,\mathbf{b}}(T)}$*. Then* $\mathcal{P}_n$ *is the orthogonal projector from* $L_2(\mathbb{T}^m \times \mathbb{I}^\infty)$ *onto the space* $\mathcal{V}_n$ *of dimension* n*, and*

$$\big\|u - \mathcal{P}_n u\big\|_{L_2(\mathbb{T}^m\times\mathbb{I}^\infty)} \;\le\; B\,n^{-1/m},$$

where

$$B := \; 4\pi\, C^{1/m}\, K\, \|(\mathbf{c}^{-\mathbf{s}})\|_{\ell_2(\mathbb{F})}$$

and C *is the constant defined in* (5.12) *for* $a = 1$ *and* $\mathbf{b}$ *as in* (6.4).

7. Concluding remarks. We derived upper and lower bounds of the ε-dimension and Komogorov n-widths of Sobolev-analytic-type function spaces with certain product and order depended weights. The methodology of the paper follows a strict guideline: First, we fix the error and construct an index set that can realize this error. Then, we compute a bound for the cardinality of this index set. The index set, we study here, might also arise in different applications and hence are of its own interest. Thus, our approach is quite general and can also be applied in other situations.

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