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Low-Rank Approximation to Heterogeneous Elliptic Problems

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Abstract

In this work, we investigate the low-rank approximation of elliptic problems in heterogeneous media by means of Kolmogrov n-width and asymptotic expansion. This class of problems arises in practical applications involving high-contrast media, and their efficient approximation often relies crucially on certain low-rank structure of the solutions. We provide conditions on the permeability coefficient κ that ensure a favorable low-rank approximation. These conditions are expressed in terms of the distribution of the inclusions in the coefficient κ, e.g., the values, locations and diameters of the heterogeneous regions. Further, we provide a new asymptotic analysis for high-contrast elliptic problems based on the perfect conductivity problems and layer potential techniques. These results provide theoretical underpinnings for several multiscale model reduction algorithms.
Keywords: low-rank approximation, heterogeneous elliptic problems, eigenvalue decays, asymptotic expansion, layer potential techniques

1 Introduction

Elliptic problems with heterogeneous coefficients, where the value of the coefficient can vary over several order of magnitudes, arise in many practical applications, e.g., reservoir simulation, subsurface flow, battery modeling, and material sciences [13, 14]. This class of problems is computationally very challenging due to the disparity of scales, which often renders the classical numerical treatment inefficient or even infeasible. In recent years, a number of multiscale model reduction techniques, e.g., Multiscale Finite Element Methods (MsFEM) and Generalized Multiscale Finite Element Methods (GMsFEM), have been proposed in the literature [22, 12], and they have achieved great success in the efficient and accurate simulation of heterogeneous problems. Conceptually, all these techniques rely crucially on a certain low-rank structure of the solution manifold of the heterogeneous problem, in the sense that the solution can be effectively approximated by a few specialized basis functions. Nonetheless, despite the extensive numerical evidences, the existence of such low-rank structure has rarely been theoretically established, and the excellent empirical efficiency remains rather mysterious. In this paper, we investigate conditions on the coefficient that ensure a favorable low-rank approximation, thereby providing theoretical underpinnings for related algorithms.

Now we mathematically formulate the problem precisely. Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain with a boundary ∂D. Then we seek a function $u \in V := H_0^1(D)$ such that

$$\begin{aligned} \mathcal{L}u := -\nabla \cdot (\kappa \nabla u) = f \qquad & \text{in } D, \\ u = 0 \qquad & \text{on } \partial D, \end{aligned} \tag{1.1}$$

where the force term $f \in L^2(D)$. The permeability coefficient κ is assume to be in $L^\infty(D)$ with $\alpha \leq \kappa(x) \leq \beta$ almost everywhere in the domain D for some lower bound $\alpha > 0$ and upper bound $\beta > \alpha$. We denote by $\Lambda := \frac{\beta}{\alpha}$ the ratio of these bounds, which reflects the contrast of the coefficient κ. Throughout, let the space $V := H_0^1(D)$ be equipped with the (weighted) inner product $\langle v_1, v_2 \rangle_D = \int_D \kappa \nabla v_1 \cdot \nabla v_2 \mathrm{d}x$ and the associated energy norm $\|v\|^2_{H^1_\kappa(D)} := \langle v, v \rangle_D$, and denote by $W = L^2(D)$, equipped with the usual norm $\|\cdot\|_{L^2(D)}$ and inner product $(\cdot,\cdot)_D$.

The weak formulation for problem (1.1) is to find $u \in V$ such that

$$\langle u, v \rangle_D = (f, v)_D \quad \text{for all } v \in V. \tag{1.2}$$

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The Lax-Milgram theorem implies the well-posedness of problem (1.2). We denote by $\mathcal{S} = \mathcal{L}^{-1} : W \to V$ the solution operator. By the compactness of the Sobolev embedding $V \hookrightarrow W$ [1], the solution operator $\mathcal{S}$ is compact on W. Further, we denote by $\mathcal{U}$ the image of the unit ball in W under the mapping $\mathcal{S}$, i.e.,

$$\mathcal{U} := \big\{\mathcal{S}(f) : f \in W \text{ with } \|f\|_{L^2(D)} \leq 1\big\}.$$

Then the (low-rank) approximation property of the set $\mathcal{U}$ can be formulated as follows. Given a tolerance $\delta > 0$, we aim at finding a linear subspace $X_N \subset V$ of dimension N, dependent of δ, satisfying

$$\sup_{u\in\mathcal{U}} \inf_{v\in X_N} \|u - v\|_{H^1_\kappa(D)} \leq C\delta, \tag{1.3}$$

where C denotes a constant independent of N. The (low-rank) approximation in (1.3) underpins the efficiency of numerical techniques for multiscale problems: for a given tolerance δ, the smaller the dimension N of the approximating subspace X_N is, the cheaper the effective problem complexity becomes. Thus property (1.3) provides a theoretical lower bound on any numerical treatment, and it is of central importance for the theoretical justifications of multiscale model reduction algorithms.

Generally, the existence of a low-rank approximation is not *a priori* ensured. To see this, consider the following example. Let $\kappa = \kappa(\frac{x}{\epsilon})$ for some $0 < \epsilon \ll 1$, i.e., problem (1.1) corresponds to a periodic and rapidly oscillating elliptic operator. It is well known that the eigenvalues of the solution operator $\mathcal{S}$ decay as $\mathcal{O}(n^{-\frac{2}{d}})$ [26, 24]. In particular, this and the discussions in Section 2 below (cf. (2.3)) imply that the problem actually does not admit a low-rank approximation when the dimension d is large. Thus, a low-rank approximation is not always feasible for every problem.

In this paper, we investigate the situation when a low-rank approximation to (1.1) is favorable, especially for high-contrast problems where the contrast $\Lambda \to \infty$ in some regions [6, 23]. It is well-known that when the source term f has high regularity or a special structure, e.g., low-rank expression, there will be a fast decay in the Kolmogorov n-width [25, 11, 20]. In a slightly different context of stochastic homogenization, the recent work [15, Corollary 4] provides a low-rank approximation of a κ-harmonic function that grows at most polynomially at the infinity. This assertion is proved under the assumption that the scalar and vector potentials of the harmonic coordinates in (1.1) grow sublinearly, which holds if the coefficient κ is stationary and qualitatively ergodic. In this paper, we will not make use of special assumptions on the source term f. The focus of this work is on *structural conditions of the permeability field* κ that provide a favorable low-rank structure in the sense of (1.3), in terms of spectral gap in the Kolmogorov n-width.

The contributions of this work are three-folded. First, we formulate the main goal (1.3) into the eigenvalue decay estimate of the solution operator $\mathcal{S}$, and provide one sufficient condition that ensures a favorable low-rank approximation to the corresponding elliptic equations (cf. Proposition 3.1). Second, we give a detailed study on the eigenvalue estimate of the operator $\mathcal{S}$ in the context of heterogeneous media (with piecewise constant high-contrast coefficient). This is achieved by a precise characterization of the dominant eigenmodes in Theorem 4.1 and a novel orthogonal decomposition of the space in Theorem 4.3. To the best of our knowledge, there is no known analogous result on the eigenvalue estimate in the literature. Third and last, based on the aforementioned decay estimate, layer potential techniques and the perfect conductivity problem (i.e., the weak H^1 limit of the solution when the contrast $\Lambda \to \infty$), we derive an accurate asymptotic expansion for the high-contrast case in Theorem 5.1, which improves several known results [6, 7]. In particular, it provides a rather explicit low-rank approximation.

We conclude this section by discussing related results in the literature. So far there are only a few results in the literature. In [3, Lemma 2.6], a rank N of order $\log(\frac{1}{\delta})$ was given, which estimates locally in L^2 norm for any arbitrary L^∞-coefficient and any given prescribed error δ. In the work [19], a local (generalized) finite element basis (i.e., AL basis) was constructed. With H being the mesh width of the finite element mesh, it consists of $\mathcal{O}((\log \frac{1}{H})^{d+1})$ basis functions per nodal point, and preserves the convergence rate of the classical finite element method for Poisson-type problems. Nonetheless, these results [3, 19] remain κ dependent and make no specific assumptions on the permeability coefficient κ which are critical for an efficient low-rank approximation. In contrast, in this work, we shall exploit certain structures on the permeability coefficient κ in order to obtain a favorable low-rank approximation.

The remainder of this paper is organized as follows. In Section 2, we provide an approximation to Kolmogorov n-width $d_n(\mathcal{S}(W); W)$ and $d_n(\mathcal{S}(W); V)$ through the eigenvalues of the solution operator $\mathcal{S}$. This highlights the central role of eigenvalue decay estimate in the analysis. Then in Section 3 we present one sufficient condition for the low-rank approximations to the solutions of some elliptic equations. In Section 4, we identify the characteristic of the dominant eigenmodes of the operator $\mathcal{S}$, and thereupon, we derive estimates on leading eigenvalues. In Section 5, we derive a new asymptotic expansion for high-contrast problems with the weak limit as the zeroth order approximant, and as a by product, also an estimate on the decay of Kolmogorov n-width. Finally, a conclusion is given in Section 6.

2 Low-rank approximation and eigenvalues

In this section, we establish the estimate (1.3) via the definition of Kolmogorov n-width, and discuss its relation with the eigenvalues of the solution map $\mathcal{S}$ (with the help of approximation number). We shall derive two estimates on Kolmogorov n-width in terms of the eigenvalues of $\mathcal{S}$.

First, let us recall the definitions of Kolmogrov n-width and approximation numbers. The Kolmogrov n-width for the solution operator $\mathcal{S} : W \to W$ [28, pp. 29] is defined by

$$d_n(\mathcal{S}(W); W) = \inf_{X_n} \sup_{y\in\mathcal{U}} \inf_{x\in X_n} \|x - y\|_{L^2(D)}, \tag{2.1}$$

with the infimum taken over all n-dimensional subspaces $X_n \subset W$. The n-dimensional subspace X_n that attains $d_n(\mathcal{S}(W); W)$ is called the optimal space. The compactness of $\mathcal{S}$ on W immediately indicates that $d_n(\mathcal{S}(W); W) \to 0$ as $n \to \infty$. Since $\mathcal{S} : W \to V$ is a bounded linear operator, we can have an analogous definition

$$d_n(\mathcal{S}(W); V) = \inf_{X_n} \sup_{y\in\mathcal{U}} \inf_{x\in X_n} \|x - y\|_{H^1_\kappa(D)}, \tag{2.2}$$

where the infimum is taken over all n-dimensional subspaces $X_n \subset V$. However, generally there is no guarantee that $d_n(\mathcal{S}(W); V) \to 0$ as $n \to \infty$.

The Kolmogorov n-width $d_n(\mathcal{S}(W); W)$ can be characterized precisely by the spectrum of the operator $\mathcal{S}$. Since the operator $\mathcal{S} : W \to W$ is nonnegative, compact and self-adjoint, by the standard spectral theory [32], it has at most countably many discrete eigenvalues, with zero being the only accumulation point, and each nonzero eigenvalue has only finite multiplicity. Let $\{(\lambda_j, v_j)\}_{j=1}^{\infty}$ be the eigenvalues and corresponding $L^2(D)$ normalized eigenfunctions of $\mathcal{S}$ listed according to their algebraic multiplicities and the eigenvalues ordered nonincreasingly. Then, the eigenfunctions $\{v_j\}_{j=1}^{\infty}$ form an orthonormal basis in $L^2(D)$, and $\{\sqrt{\lambda_j} v_j\}_{j=1}^{\infty}$ form an orthonormal basis in V. Then an application of Theorem 2.2 of [28, Chapter IV] yields immediately

$$d_n(\mathcal{S}(W); W) = \lambda_{n+1} \tag{2.3}$$

with the subspace $V_n := \mathrm{span}\{v_1, \cdots, v_n\}$ being an optimal space for $n = 1, 2, \cdots$.

Next we estimate the Kolmogorov n-width $d_n(\mathcal{S}(W); V)$. To this end, we first recall the definition of the approximation number for a bounded linear operator in W. The $(n+1)^{\text{th}}$ approximation number (cf. [27, Section 2.3.1]), denoted by $a_{n+1}(\mathcal{S})$, of an operator $\mathcal{S} \in \mathcal{B}(W, W)$ is defined by

$$a_{n+1}(\mathcal{S}) := \inf\{\|\mathcal{S} - L\|_{W\to W} : L \in \mathfrak{F}(W, W), \mathrm{rank}(L) \le n\}, \tag{2.4}$$

where the notation $\mathfrak{F}(W, W)$ represents the set of all finite-rank operators in W and $\|\cdot\|_{W\to W}$ denotes the operator norm on the space W. The finite rank operator attains the infimum is called the optimal operator. The approximation number $a_n(\mathcal{S})$ provides a lower bound of the worst-case convergence rate for any finite-rank approximation to $\mathcal{S}$ (in particular, any numerical treatment). The definition of s-numbers implies that $d_n(\mathcal{S}(W); W)$ and $a_n(\mathcal{S})$ are both s-numbers for the compact operator $\mathcal{S}$. By the uniqueness of s-numbers of any operator between Hilbert spaces [27, Section 2.11.9], we deduce

$$a_{n+1}(\mathcal{S}) = d_n(\mathcal{S}(W); W) = \lambda_{n+1}. \tag{2.5}$$

Remark 2.1. *The choice of the finite-rank operator in the definition* (2.4) *is fairly flexible. In particular, assume that D is a bounded, convex polygon and the coefficient $\kappa \in \mathcal{C}^2$. Let L be a finite-rank operator constructed from the conforming P_1 finite element discretization of $\mathcal{S}$. Then the standard FEM a priori estimate [21, Chapter 4] and* (2.4) *imply*

$$a_{n+1}(\mathcal{S}) \le C\Lambda n^{-\frac{2}{d}},$$

where C denotes a positive constant independent of α, β and n.

Our next endeavor is to estimate the Kolmogorov n-width $d_n(\mathcal{S}(W); V)$ in terms of the eigenvalues λ_n. This is achieved by constructing a finite-rank operator to approximate $\mathcal{S}$ directly, then invoking (2.2) to obtain the desired estimate. The finite-rank operator is constructed below. Given $n \in \mathbb{N}_+$, we define an orthogonal projection operator $\Pi_n : V \to V_n := \mathrm{span}(\{v_i\}_{i=1}^n)$ by

$$\langle v - \Pi_n v, \varphi\rangle_D = 0 \quad \text{for all } \varphi \in V_n. \tag{2.6}$$

Let $\mathfrak{F}(W, V) \ni \mathcal{S}_n := \Pi_n \mathcal{S}$ be a rank $\leq n$ operator. A simple calculation yields

$$\|\mathcal{S} - \mathcal{S}_n\|_{W \to W} = \lambda_{n+1}.$$

Now we can state an *a priori* estimate on the projection operator Π_n.

Lemma 2.1. *Let u be the solution to* (1.1). *For the projection operator Π_n, there holds*

$$\|u - \Pi_n u\|_{H^1_\kappa(D)} \leq C_{poin}(D)\sqrt{\lambda_{n+1}}\|f\|_{L^2(D)}, \tag{2.7}$$

where $C_{poin}(D)$ denotes the Poincaré constant for the domain D.

Proof. Since $\{v_j\}_{j=1}^\infty$ and $\{\sqrt{\lambda_j}v_j\}_{j=1}^\infty$ form an orthonormal basis in $L^2(D)$ and V, respectively, for any $u \in V \subset L^2(D)$, there exists a sequence $\{c_j\}_{j=1}^\infty \in \ell^2$ such that $u = \sum_{j=1}^\infty c_j v_j$ and $\Pi_n u = \sum_{j=1}^n c_j v_j$ by the definition (2.6), which gives directly $c_j = \lambda_j \langle u, v_j\rangle_D$. Further, we have

$$\begin{aligned}\|u - \Pi_n u\|^2_{L^2(D)} &= \sum_{j=n+1}^\infty c_j^2 = \sum_{j=n+1}^\infty \frac{\lambda_j}{\lambda_j} c_j^2 \\ &\leq \lambda_{n+1} \sum_{j=n+1}^\infty \frac{1}{\lambda_j} c_j^2 = \lambda_{n+1}\|u - \Pi_n u\|^2_{H^1_\kappa(D)}.\end{aligned} \tag{2.8}$$

By taking $v = (u - \Pi_n u)$ as the test function in (1.2) and applying (2.6), we obtain

$$\|u - \Pi_n u\|^2_{H^1_\kappa(D)} = (f, u - \Pi_n u)_D.$$

Now the desired assertion follows from (2.8), Cauchy-Schwarz inequality and Poincaré inequality. □

Remark 2.2. *The condition $f \in L^2(D)$ is essential for obtaining the convergence rate in Lemma 2.1. If $f \in H^{-1}(D)$ only, the convergence estimate is generally not true.*

Now an upper bound for the Kolmogorov n-width $d_n(\mathcal{S}(W); V)$ follows directly from Lemma 2.1 and the definition (2.2).

Proposition 2.1. *The rank $\leq n$ operator $\mathcal{S}_n := \Pi_n \mathcal{S}$ is an optimal operator to the solution operator $\mathcal{S}$ for $n \in \mathbb{N}_+$. There holds*

$$d_n(\mathcal{S}(W); V) \leq C_{poin}(D)\sqrt{\lambda_{n+1}}.$$

Lemma 2.1 (as well as Proposition 2.1) implies that V_n is the optimal space for approximating solutions to problem (1.1) and the convergence rate in V_n is essentially determined by either the eigenvalue decay rate of the solution operator $\mathcal{S}$ or the existence of a spectral gap. Here a spectral gap means that there is an integer $L \in \mathbb{N}_+$ and $0 < \epsilon \ll 1$ such that

$$d_1(\mathcal{S}(W); V) \geq d_2(\mathcal{S}(W); V) \geq \cdots \geq d_L(\mathcal{S}(W); V) \gg \epsilon \geq d_{L+1}(\mathcal{S}(W); V) \geq \cdots. \tag{2.9}$$

The identity (2.3) and Proposition 2.1 both highlight the central role of the eigenvalue decay/spectral gap in the study of the low-rank approximation of heterogeneous elliptic problems: a fast eigenvalue decay or spectral gap implies that the solution operator can be effectively approximated by a set of basis functions of low-dimensionality. We shall analyze the spectral gap for elliptic problems in high-contrast media in Sections 4 and 5. Before that, we first provide one sufficient condition that ensures the low-rank structure.

3 One sufficient condition for low-rank approximation

In this part, we provide one sufficient condition for the low-rank approximation to problem (1.1) via its error equation, for the case of a bounded contrast Λ.

To motivate the construction, we begin with a simple situation. Given a prescribed tolerance $\epsilon > 0$, let κ_0 be an approximation to the permeability coefficient κ (e.g., on a coarse mesh) and u_0 be the solution to problem (1.1) with κ_0 in place of κ (assuming also $\alpha \leq \kappa_0 \leq \beta$). Then the following implication holds

$$\text{If } \|\kappa - \kappa_0\|_{L^\infty(D)} \leq \epsilon, \text{ then } |u - u_0|_{H^1(D)} \leq \frac{\epsilon}{\alpha^2} C_{\text{poin}}(D) \|f\|_{L^2(D)}, \tag{3.1}$$

with $C_{\text{poin}}(D)$ being the Poincaré constant for the domain D and $|\cdot|_{H^1(\omega)}$ denoting the $H^1(\omega)$-semi norm on $\omega \subset D$. This assertion can be verified directly by a perturbation argument and the *a priori* estimate for elliptic problems with rough coefficient as follows. The equation for the difference $u - u_0 \in V$ is given by

$$-\nabla \cdot (\kappa \nabla (u - u_0)) = \nabla \cdot ((\kappa - \kappa_0) \nabla u_0) \quad \text{in } D.$$

This equation together with the coercivity of the elliptic problem yields

$$\begin{aligned}
\alpha |u - u_0|^2_{H^1(D)} &\leq \langle u - u_0, u - u_0 \rangle_D = -\int_D (\kappa - \kappa_0) \nabla u_0 \cdot \nabla (u - u_0) \mathrm{d}x \\
&\leq \|\kappa - \kappa_0\|_{L^\infty(D)} |u_0|_{H^1(D)} |u - u_0|_{H^1(D)} \leq C_{\text{poin}}(D) \frac{\epsilon}{\alpha} \|f\|_{L^2(D)} |u - u_0|_{H^1(D)},
\end{aligned}$$

and the assertion (3.1) follows directly by dividing $\alpha |u - u_0|_{H^1(D)}$ from both sides. In the last line we have employed the Hölder's inequality and the following *a priori* error estimate

$$\alpha |u_0|^2_{H^1(D)} \leq \|f\|_{L^2(D)} \|u_0\|_{L^2(D)} \leq \|f\|_{L^2(D)} C_{\text{poin}}(D) |u_0|_{H^1(D)}.$$

Our focus in the remainder of this section is to relax the condition in (3.1). Then in addition to the term u_0, extra basis functions are needed in order to get a good approximation. To this end, we analyze one specific situation, which generalizes assertion (3.1). Let

$$\kappa_0 = \fint_D \kappa(x) \mathrm{d}x := \frac{1}{|D|} \int_D \kappa(x) \mathrm{d}x \tag{3.2}$$

be a zeroth-order approximation to the permeability field κ. Accordingly, we define $u_0 \in V$ to be the corresponding solution to the problem

$$-\nabla \cdot (\kappa_0 \nabla u_0) = f \quad \text{in } D. \tag{3.3}$$

For any given $\delta > 0$, let $D_\delta = \{x \in D : \text{dist}(x, \partial D) \leq \delta\}$. Further, let η be a cutoff function on the domain D satisfying $\eta = 1$ in $D \backslash D_\delta$, $\eta = 0$ on ∂D, $0 \leq \eta \leq 1$ and $\|\nabla \eta\|_{L^\infty(D)} \leq \frac{1}{\delta}$. Now we can give a sufficient condition for the existence of a low-rank approximation. The construction is based on certain harmonic functions in the interior of the domain D.

Proposition 3.1. *Let $d \leq 3$, $f \in L^2(D)$, $0 < \epsilon$ be a given tolerance, and κ_0 and u_0 be defined in* (3.2) *and* (3.3)*, respectively. Further, assume that there are harmonic functions $\{\phi_i\}_{i=1}^n$, for some $n \in \mathbb{N}_+$, such that*

$$|u_0 + \sum_{i=1}^n \phi_i|_{H^1(D \backslash D_\delta)} \leq \epsilon^{\frac{1}{3}}, \quad \|\nabla \phi_i\|_{L^\infty(D)} \leq 1 \quad \textit{and} \quad \|\phi_i\|_{L^2(D)} \leq \epsilon. \tag{3.4}$$

Then there holds

$$|u - \Big(u_0 + \eta \sum_{i=1}^n \phi_i\Big)|_{H^1(D)} \leq C(D, n) \epsilon^{\frac{1}{3}} \Big(\frac{\beta}{\alpha^2} \|f\|_{L^2(D)} + \frac{\beta}{\alpha}\Big),$$

where $C(D, n)$ is a constant depending on the domain D and n.

Proof. Let $v = u - (u_0 + \eta \sum_{i=1}^n \phi_i)$. Clearly $v = 0$ on ∂D. Using the governing equations (1.1) and (3.3), and noting that ϕ_is are harmonic, we deduce that the difference v satisfies

$$\begin{aligned}\tilde{f} := -\nabla \cdot (\kappa \nabla v) &= f + \nabla \cdot (\kappa \nabla u_0) + \sum_{i=1}^n \nabla \cdot (\kappa \nabla (\eta \phi_i)) \\ &= f + \nabla \cdot ((\kappa - \kappa_0 + \kappa_0)\nabla u_0) + \sum_{i=1}^n \nabla \cdot ((\kappa - \kappa_0)\nabla(\eta\phi_i)) - \sum_{i=1}^n \nabla \cdot (\kappa_0 \nabla((1-\eta)\phi_i)) \\ &= \nabla \cdot ((\kappa - \kappa_0)\nabla(u_0 + \eta \sum_{i=1}^n \phi_i)) - \sum_{i=1}^n \nabla \cdot (\kappa_0 \nabla((1-\eta)\phi_i)).\end{aligned}$$

Next we estimate the residual $\tilde{f}$. By Hölder's inequality, we obtain

$$\begin{aligned}|\int_D \tilde{f} v \mathrm{d}x| &\leq \int_D |(\kappa - \kappa_0)\nabla\Big(u_0 + \eta \sum_{i=1}^n \phi_i\Big) \cdot \nabla v| \mathrm{d}x + \sum_{i=1}^n \int_D |\kappa_0 \nabla((1-\eta)\phi_i) \cdot \nabla v| \mathrm{d}x \\ &\leq \beta\Big(|u_0 + \eta \sum_{i=1}^n \phi_i|_{H^1(D)} + \sum_{i=1}^n |(1-\eta)\phi_i|_{H^1(D)}\Big)|v|_{H^1(D)}.\end{aligned}$$

It remains to bound the two terms in the parenthesis. For the first term, we appeal to the splitting

$$|u_0 + \eta \sum_{i=1}^n \phi_i|^2_{H^1(D)} = |u_0 + \sum_{i=1}^n \phi_i|^2_{H^1(D \setminus D_\delta)} + |u_0 + \eta \sum_{i=1}^n \phi_i|^2_{H^1(D_\delta)} := \mathrm{I} + \mathrm{II},$$

where the first term I is bounded by $\epsilon^{\frac{2}{3}}$, by Assumption (3.4). To bound the second term II, we apply Young's inequality

$$\mathrm{II} \leq 3 \int_{D_\delta} |\nabla u_0|^2 \mathrm{d}x + 3 \int_{D_\delta} |\eta \sum_{i=1}^n \nabla \phi_i|^2 \mathrm{d}x + 3 \int_{D_\delta} |\nabla \eta \sum_{i=1}^n \phi_i|^2 \mathrm{d}x = 3 \sum_{j=1}^3 \mathrm{II}_j.$$

To bound the term II_1, we employ a corollary of the following *a priori* estimate on u_0 [21, Theorem 3.1.2.1] and the Sobolev embedding that $H^2(D) \hookrightarrow L^\infty(D)$ when $d \leq 3$ that

$$\|\nabla u_0\|_{L^\infty(D)} \leq \frac{C(D)}{\alpha} \|f\|_{L^2(D)}$$

for some constant $C(D)$ depending on the domain D. Upon noting $|D_\delta| \leq C(D)|\delta|$, we have

$$\mathrm{II}_1 \leq \|\nabla u_0\|^2_{L^\infty(D)} |D_\delta| \leq \frac{C(D)}{\alpha^2} \|f\|^2_{L^2(D)} \delta.$$

Next by the property of the cutoff function η and the bounds $\|\nabla \phi_i\|_{L^\infty(D)} \leq 1$, cf. Assumption (3.4), we have

$$\mathrm{II}_2 \leq \Big(\|\eta\|_{L^\infty(D)} \sum_{i=1}^n \|\nabla \phi_i\|_{L^\infty(D)}\Big)^2 |D_\delta| \leq n^2 C(D) \delta\cdot$$

For the third term II_3, we appeal to the property of the cutoff function again

$$\mathrm{II}_3 \leq \Big(\|\nabla \eta\|_{L^\infty(D)} \sum_{i=1}^n \|\phi_i\|_{L^2(D)}\Big)^2 \leq n^2 \frac{\epsilon^2}{\delta^2}.$$

Combining the preceding three estimates yields

$$\mathrm{II} \leq C(D,n)\big(\delta(\tfrac{1}{\alpha^2}\|f\|^2_{L^2(D)} + 1) + \tfrac{\epsilon^2}{\delta^2}\big),$$

for some constant $C(D,n)$ depending on D and n only. Similarly, from Assumption 3.4, we derive

$$\begin{aligned}|(1-\eta)\phi_i|^2_{H^1(D)} = |(1-\eta)\phi_i|^2_{H^1(D_\delta)} &\leq 2\Big(\int_{D_\delta} |(1-\eta)\nabla \phi_i|^2 \mathrm{d}x + \int_{D_\delta} |(\nabla \eta)\phi_i|^2 \mathrm{d}x\Big) \\ &\leq 2\Big(C(D)\delta + \frac{\epsilon^2}{\delta^2}\Big).\end{aligned}$$

Taking $\delta = \epsilon^{\frac{2}{3}}$ yields

$$\int_D \kappa |\nabla v|^2 \mathrm{d}x = |\int_D \tilde{f} v \mathrm{d}x| \leq C(D,n)\epsilon^{\frac{1}{3}}\Big(\frac{\beta}{\alpha}\|f\|_{L^2(D)} + \beta\Big)|v|_{H^1(D)},$$

which implies directly the desired result, since κ is bounded from below by α. □

Proposition 3.1 gives one sufficient condition (3.4) for problem (1.1) to admit a low-rank approximation. Under condition (3.4), the triangle inequality gives

$$|u|_{H^1(D\backslash D_\delta)} \leq C(D,n)\epsilon^{\frac{1}{3}}\Big(\frac{\beta}{\alpha^2}\|f\|_{L^2(D)} + \frac{\beta}{\alpha}\Big).$$

The condition (3.4) actually imposes certain (implicit) structural assumptions on the permeability field κ. Though Proposition 3.1 gives one sufficient condition, it is unfortunately not constructive in nature, and the precise assumption on the permeability field κ is not transparent. Nonetheless, it motivates further analysis by constructing specialized harmonic functions within the domain. In the remainder of this paper, we focus on the elliptic operator with high-contrast piecewise constant coefficients κ, for which the dominant eigenmodes can be identified and eigenvalue estimates in the spirit of Proposition 3.1 can be derived. Specifically, we make the following structural assumptions on the domain D and the coefficient κ.

Assumption 3.1. *(Structure of D and κ_η.) Let D be a domain with a $C^{2,a}$ $(0 < a < 1)$ boundary ∂D, and $\{D_i \subset D\}_{i=1}^m$ be m pairwise disjoint strictly convex open subsets, each with a $C^{2,a}$ boundary $\Gamma_i := \partial D_i$, and denote $D_0 = D\backslash\overline{\cup_{i=1}^m D_i}$. Further, there exists an open set $\omega \subset D$, such that $\cup_{i=1}^m D_i \subset \omega$ and dist$(\partial\omega, \partial D) \geq \tau$, for some $\tau > 0$. Let the permeability coefficient κ_η be piecewise constant defined by*

$$\kappa_\eta = \begin{cases} \eta_i & \text{in } D_i, \\ 1 & \text{in } D_0. \end{cases} \tag{3.5}$$

Let $\eta_{min} := \min_i\{\eta_i\} \geq 1$.

Throughout, we always take 1 and ϵ_i as the diameters of D and D_i, respectively. Let $\eta = (\eta_1, \cdots, \eta_m)$ and $\epsilon = (\epsilon_1, \cdots, \epsilon_m)$. Denote $\tau_i := \mathrm{dist}(D_i, \partial D)$, $\delta_{ij} := \mathrm{dist}(D_i, D_j)$ and $\delta_j := \min_{i\neq j}\{\delta_{ij}\}$. We assume that $\tau_j \geq \delta_j$, for $j = 1, 2, \cdots, m$. Without loss of generality, we may relabel the indices for the inclusions D_j such that $|D_1| \geq |D_2| \geq \cdots \geq |D_m|$. Further, we use the notation $A \lesssim B$ if $A \leq CB$ for some constant C independent of ϵ_i, η_i, δ_i and τ_i. The notation $C_{\text{poin}}(\omega)$ denotes the Poincaré constant in the subdomain $\omega \subset D$ for all functions in $H_0^1(\omega)$, i.e., $C_{\text{poin}}(\omega) = \sup_{v\in H_0^1(\omega)} \int_D v^2 \mathrm{d}x / \int_D |\nabla v|^2 \mathrm{d}x$. A scaling argument shows that $C_{\text{poin}}(\omega) \lesssim \mathrm{diam}(\omega)^2$.

Below, we denote by $n_i(x)$ the unit outward normal (relatively to D_i) to the interface Γ_i at the point $x \in \Gamma_i$. For a function w defined on $\mathbb{R}^2\backslash\Gamma_i$ for $i = 1, 2, \cdots, m$, we define for $x \in \Gamma_i$,

$$w(x)|_\pm := \lim_{t\to 0^+} w(x \pm t n_i(x)) \quad \text{and} \quad \frac{\partial}{\partial n_i^\pm} w(x) := \lim_{t\to 0^+} (\nabla w(x \pm t n_i(x)) n_i(x))$$

if the limit on the right hand side exists. We denote by $[w]$ the jump across the interface Γ_i defined by

$$[w(x)] := \lim_{t\to 0^+} (w(x + t n_i(x)) - w(x - t n_i(x))) \text{ and } [\kappa_\eta \frac{\partial w}{\partial n_i}] := \frac{\partial w}{\partial n_i^+} - \eta_i \frac{\partial w}{\partial n_i^-}.$$

4 Eigenvalue decay rate

In this section, we establish the eigenvalue estimates for the operator $\mathcal{S}$ through the maxmin principle and a novel orthogonal decomposition of the space V. Specifically, we seek $\{(v_n, \lambda_n)\} \in V \times \mathbb{R}$ such that

$$\begin{cases} \mathcal{S} v_n = \lambda_n v_n & \text{in } D, \\ \quad v_n = 0 & \text{on } \partial D. \end{cases} \tag{4.1}$$

The weak formulation for the eigenvalue problem is to seek $v_n \in V$ and $\lambda_n \in \mathbb{R}$ satisfying

$$(v_n, \phi)_D = \lambda_n \langle v_n, \phi\rangle_D \quad \text{for all } \phi \in V.$$

One approach to characterize the sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ is through the Rayleigh quotient

$$R(v) = \frac{(v,v)_D}{\langle v,v\rangle_D} := \frac{\int_D v^2 \mathrm{d}x}{\int_D \kappa|\nabla v|^2 \mathrm{d}x}. \tag{4.2}$$

As a corollary of the maxmin principle, there holds

$$\lambda_n = \max_{\substack{V_n\subset V\\ \dim(V_n)\leq n}} \min_{v\in V_n} R(v) = R(v_n). \tag{4.3}$$

First, we show that piecewise harmonic functions v with high oscillations on the interface Γ_i for $i=1,2,\cdots,m$ generate unimportant eigenmodes, i.e., the value of the Rayleigh quotient $R(v)$ is small. For simplicity, let $D_i := B(O_i,\epsilon_i)$ be balls centering at O_i with radius ϵ_i. Then the set of functions

$$\{\cos k\theta, \sin(k+1)\theta,\ k=0,1,\cdots\}$$

forms an orthogonal basis set of $H^{\frac{1}{2}}(\Gamma_i)$, where the angle θ is with respect to the center O_i.

Theorem 4.1. *Let $D_i := B(O_i,\epsilon_i)$ and $v\in V$ satisfying*

$$-\Delta v = 0 \quad \textit{in } D\backslash \cup_{i=1}^m \Gamma_i.$$

If $v=\sin k_i\theta$ on the interface Γ_i, where $k_i\in\mathbb{N}_+$ and $i=1,\cdots,m$, then there holds

$$R(v) \leq \frac{1}{\pi\eta_{min}\sum_{i=1}^m k_i}.$$

Proof. It can be verified directly that $v(x) = (\frac{|x-O_i|}{\epsilon_i})^{k_i}\sin k_i\theta$ in D_i, for $i=1,2,\cdots,m$. Hence, a direct calculation together with the Dirichlet's principle [10] and the maximum principle yields

$$\forall i=1,\cdots,m : \pi k_i = |v|^2_{H^1(D_i)} \text{ and } (v,v)_D \leq |D| \leq 1.$$

Thus we obtain

$$R(v) = \frac{(v,v)_D}{\langle v,v\rangle_D} \leq \frac{1}{\sum_{i=1}^m \pi k_i\eta_i} \leq \frac{1}{\pi\eta_{\min}\sum_{i=1}^m k_i},$$

and the desired estimate follows. □

Theorem 4.1 indicates that, in the high-contrast limit $\eta\to\infty$, the dominant piecewise harmonic eigenfunctions in (4.1) must have low oscillations on the interfaces $\{\Gamma_i\}_{i=1}^m$. Naturally, this observation suggests itself a constructive approach to retrieve the dominant eigenfunctions of $\mathcal{S}$. Specifically, we define axillary functions on the domain D that are piecewise constant on $\cup_{j=1}^m D_j$: $\{w_i\}_{i=1}^m \subset H_0^1(D)$ satisfying

$$\begin{cases} -\Delta w_i = 0 & \text{in } D\setminus \cup_i\Gamma_i,\\ \quad w_i = \delta_{ik} & \text{on } \Gamma_k,\ k=1,2,\cdots,m,\\ \quad w_i = 0 & \text{on } \partial D, \end{cases} \tag{4.4}$$

where δ_{ik} is the Kronecker symbol. The well-posedness of problem (4.4) can be established by a variational method [2]. Below, we provide some *a priori* estimates, which are useful for deriving the lower bound of the Rayleigh quotient $R(w_i)$.

Lemma 4.1. *Assume that $\epsilon_i \leq \frac{1}{2}\delta_i$. For $i=1,2,\cdots,m$, there holds*

$$\int_{D_0} |\nabla w_i|^2 \mathrm{d}x \leq \begin{cases} \pi(1+4\frac{\epsilon_i}{\delta_i}) & \textit{if } d=2,\\ \frac{4}{3}\pi(\frac{1}{2}\delta_i + 3\epsilon_i + 6\frac{\epsilon_i^2}{\delta_i}) & \textit{if } d=3. \end{cases} \tag{4.5}$$

Proof. We denote by O_i the center of D_i and $B(O_i, \frac{1}{2}\delta_i + \epsilon_i)$ a ball centering at O_i with radius $(\frac{1}{2}\delta_i + \epsilon_i)$. Then $D_i \subset B(O_i, \frac{1}{2}\delta_i + \epsilon_i)$ and $D_j \cap B(O_i, \frac{1}{2}\delta_i + \epsilon_i) = \emptyset$ for $j \neq i$. Further, we define a cutoff function $\rho_i \in \mathcal{C}^2(D)$ by

$$\rho_i(x) = \begin{cases} 1 & x \in B(O_i, \epsilon_i), \\ 0 & x \in D \backslash B(O_i, \frac{1}{2}\delta_i + \epsilon_i), \\ \text{affine} & \text{otherwise.} \end{cases}$$

By construction, $0 \leq \rho_i \leq 1$, $\|\nabla \rho_i\|_{L^\infty(D)} \leq \frac{2}{\delta_i}$ and $\rho_i = w_i$ on ∂D_0. The Dirichlet's principle [10] implies

$$\int_{D_0} |\nabla w_i|^2 \mathrm{d}x \leq \int_{D_0} |\nabla \rho_i|^2 \mathrm{d}x.$$

Together with the identity

$$|B(O_i, \frac{1}{2}\delta_i + \epsilon_i) \backslash B(O_i, \epsilon_i)| = \begin{cases} \pi(\frac{1}{4}\delta_i^2 + \epsilon_i\delta_i) & \text{if } d = 2, \\ \frac{4}{3}\pi(\frac{1}{8}\delta_i^3 + \frac{3}{4}\epsilon_i\delta_i^2 + \frac{3}{2}\epsilon_i^2\delta_i) & \text{if } d = 3, \end{cases}$$

we immediately obtain

$$\int_{D_0} |\nabla w_i|^2 \mathrm{d}x \leq \int_{D_0} |\nabla \rho_i|^2 \mathrm{d}x \leq \|\nabla \rho_i\|^2_{L^\infty(D)} |B(O_i, (\delta_i + \epsilon_i)) \backslash B(O_i, \epsilon_i)|.$$

Combining the preceding two estimates shows the desired result. □

Now we can derive a lower bound on the Rayleigh quotient $R(w_i)$ for $i = 1, 2, \cdots, m$.

Theorem 4.2. *Assume that $\epsilon_i \leq \frac{1}{2}\delta_i$. For $i = 1, 2, \cdots, m$, there holds*

$$R(w_i) \geq \begin{cases} [\pi(1 + 4\frac{\epsilon_i}{\delta_i})]^{-1}|D_i| & \text{if } d = 2, \\ [\frac{4}{3}\pi(\frac{1}{2}\delta_i + 3\epsilon_i + 6\frac{\epsilon_i^2}{\delta_i})]^{-1}|D_i| & \text{if } d = 3. \end{cases} \tag{4.6}$$

Proof. By definition, we have

$$R(w_i) := \frac{\int_D w_i^2 \mathrm{d}x}{\int_D \kappa |\nabla w_i|^2 \mathrm{d}x} \geq \frac{|D_i|}{\int_{D_0} |\nabla w_i|^2 \mathrm{d}x},$$

where the inequality follows, since $w_i \equiv 1$ in D_i. Then the desired result follows from Lemma 4.1. □

Remark 4.1. *The spatial dimensionality d impacts directly the lower bound on $R(w_i)$: in 3d, the factor δ_i^{-1} enters the estimate, whereas in 2d, it is a constant factor 1 if $\epsilon_i \ll \delta_i$.*

To estimate the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ by the maxmin principle, we also need an upper bound on the Rayleigh quotient $R(v)$. To this end, we appeal to a novel orthogonal decomposition of the full space $(V; \langle\cdot,\cdot\rangle_D)$. This decomposition is motivated by the dominant modes of the perfect conductivity problem (5.1) in Section 5 below, which represents the limit problem when $\eta \to \infty$.

Theorem 4.3. *The following orthogonal decomposition of the space $(V; \langle\cdot,\cdot\rangle_D)$ holds.*

$$V := V_m \oplus V^h \oplus V^b \oplus V_0^b, \tag{4.7}$$

where the subspaces V_m, V^h, V^b and V_0^b are respectively defined by

$$\begin{aligned}
V_m &= \mathrm{span}\{w_1, w_2, \cdots, w_m\}, \\
V^h &= \{v \in V : -\Delta v = 0 \textit{ in } D \backslash \cup_{j=1}^m \Gamma_j, \quad \int_{\Gamma_i} \frac{\partial v}{\partial n_i^+} \mathrm{d}s(x) = 0 \quad \textit{for } i = 1, 2, \cdots, m\}, \\
V^b &= \{v \in V : v = 0 \textit{ in } \bar{D}_0\}, \\
V_0^b &= \{v \in V : v = 0 \textit{ in } \cup_{i=1}^m \bar{D}_i\}.
\end{aligned}$$

Prior to proceeding to its proof, we first note a few useful facts. The orthogonality of the spaces V_m, V^b and V_0^b can be shown directly. Indeed, firstly, the orthogonality of V^b and V_0^b is trivial since their supports are disjoint. Secondly, since the functions in V^b are supported in $\cup_{i=1}^m D_i$, where V_m is piecewise constant, therefore, V^b is orthogonal to V_m in $(V; \langle\cdot,\cdot\rangle_D)$. Thirdly, take $v \in V_0^b$, the divergence theorem yields

$$\langle v, w_i\rangle_D = \int_{D_0} \nabla v \cdot \nabla w_i \mathrm{d}x = -\sum_{j=1}^m \int_{\Gamma_j} \frac{\partial w_i}{\partial n_j^+} v \mathrm{d}s(x) = 0.$$

Upon letting $\widetilde{V} := V_m \oplus V^b \oplus V_0^b$, then the preceding discussions indicate that (4.7) is equivalent to

$$V^h = \widetilde{V}^\perp := \{v \in V : \langle v, w\rangle_D = 0 \text{ for all } w \in \widetilde{V}\}. \tag{4.8}$$

Proof. We only need to show (4.8). The proof consists of two steps.

Step 1. We show that $V^h \subset \widetilde{V}^\perp$. For any $v \in V^h$, by definition, $v \in H_A(D_j)$ for $j = 0, 1, \cdots, m$, where $H_A(D_j) := \{v \in V : -\Delta v = 0 \text{ in } D_j\}$. Therefore, $v \in {V^b}^\perp$ and $v \in {V_0^b}^\perp$. We only need to prove

$$\langle v, w\rangle_D = 0 \qquad \forall w \in V_m.$$

Actually, since w is constant in each inclusion D_i for all $i = 1, 2, \cdots, m$ and $v \in H_A(D_0)$, the divergence theorem leads directly to

$$\langle v, w\rangle_D = \int_{D_0} \nabla v \cdot \nabla w \mathrm{d}x = \sum_{i=1}^m w|_{\Gamma_i} \int_{\Gamma_i} \frac{\partial}{\partial n_i^+} v \mathrm{d}s(x) = 0,$$

where the last identity follows from the definition of the space V^h.

Step 2. We show that $V^h \supset \widetilde{V}^\perp$. For any $v \in \widetilde{V}^\perp$, we have $v \in {V^b}^\perp$ and $v \in {V_0^b}^\perp$. This indicates $v \in H_A(D_j)$ for $j = 0, 1, \cdots, m$. Then $v \in V_m^\perp$ yields $\int_{\Gamma_j} \frac{\partial v}{\partial n_j^+} \mathrm{d}s(x) = 0$ and this completes the proof. $\square$

By Theorem 4.2, the functions in the m-dimensional subspace V_m constitute the dominant eigenmodes. Further, in Section 5 (cf. Proposition 5.3), we will show

$$R(v) \lesssim \eta_{\min}^{-1} \quad \text{ for all } v \in V^h \text{ when } \eta \to \infty. \tag{4.9}$$

Thus it suffices to estimate the Rayleigh quotient $R(v)$ for $v \in V^b \oplus V_0^b$ to obtain the eigenvalue estimate, which will be discussed next separately.

For $v \in V^b$, an application of the Poincaré inequality in each inclusion D_i yields

$$\int_{D_i} |v|^2 \mathrm{d}x \le C_{\text{poin}}(D_i) \int_{D_i} |\nabla v|^2 \mathrm{d}x \quad \text{ for } v \in V^b.$$

This together with the characterization of the space V^b implies

$$R(v) \le \max_i \{\eta_i^{-1} C_{\text{poin}}(D_i)\} \quad \text{ for } v \in V^b.$$

That is, in the high-contrast limit, the contribution of the space V^b to the Rayleigh quotient $R(v)$ is negligible, and will not contribute much to the dominant eigenmodes.

It remains to estimate the contribution of V_0^b to the Rayleigh quotient $R(v)$. Note that the space V_0^b represents the solution space of the degenerate elliptic problem with holes in the domain and a homogeneous Dirichlet boundary condition [30]. To the best of our knowledge, in this case, the Rayleigh quotient $R(v)$ exhibits fairly complex behavior and is still not fully understood, except in the following two scenarios. The first result [8] we are aware of is in the case that every compact set $K \subset D$ belongs to D_0 if $\epsilon := (\epsilon_1, \ldots, \epsilon_m)$ is small enough, for which, there holds the estimate $\max_{v \in V_0^b} R(v) \le C_{\text{poin}}(D)$, with $C_{\text{poin}}(D)$ being the inverse of the smallest eigenvalue for the Laplacian in the unperturbed domain D. This indicates that there exist infinitely many important modes in the space V_0^b, since the eigenvalues of the inverse of Laplacian in D decay as $\mathcal{O}(n^{-\frac{2}{d}})$, and thus the problem does not admit a low-rank structure. The second result asserts that $R(v) \to 0$ for all $v \in V_0^b$ if the characteristic function of the set of holes weakly $\star$ converges to a strictly positive function in $L^\infty(D)$ as $\epsilon \to 0$ [30, Chapter 15]. As a consequence, the functions in V_0^b contribute negligibly to the Rayleigh quotient $R(v)$. In this paper, we are mainly interested in the spectral gap, which implies a low rank structure in V_0^b. Thus, we make the following assumption on the Poincaré constant $C_{\text{poin}}(D_0)$ on the domain D_0:

Assumption 4.1 (Poincaré constant in the perforated domain D_0)**.**

$$C_{poin}(D_0) \ll \min_{i=1,\cdots,m}\{R(w_i)\}.$$

Now we can state an upper bound on the $(m+1)$-th eigenvalue λ_{m+1}.

Theorem 4.4. *The following statements hold.*

(a) Assume that $\epsilon_i \leq \frac{1}{2}\delta_i$, $\epsilon := (\epsilon_1, \ldots, \epsilon_m) \to 0$, $\eta \to \infty$ and that $\cup_{i=1}^m D_i$ are periodically embedded into the global domain D. Then there holds

$$\lambda_{m+1} \lesssim \min_i\{\epsilon_i^2\}.$$

(b) Fix ϵ. Let $\epsilon_i \leq \frac{1}{2}\delta_i$ and $\eta \to \infty$, and Assumption 4.1 hold. Then there holds

$$\lambda_{m+1} \ll \lambda_m.$$

Proof. In either case, the dominant modes lie in the spaces $V_m \oplus V_0^b$. In the periodic setting (a), due to [29, Lemma 1, Appendix], there holds

$$R(v) \leq C(D_0)\epsilon_i^2.$$

This and the maxmin principle (4.3) yields the desired assertion. Case (b) is direct from Assumption 4.1. $\square$

Theorem 4.4 provides a highly desirable spectral gap, under the designate conditions on the inclusions, i.e., the coefficient is periodic with $\epsilon_i \to 0$ or the perforated domain D_0 satisfies suitable Poincaré constant in the space V_0^b. As a byproduct, Theorem 4.4 and the discussions in Section 2 yield also a gap in Kolmogorov n-width. Despite the compelling evidences, Assumption 4.1 remains largely unexplored, and it is of much interest to further analyze the problem, which we leave to future work. In the next section, we will present an asymptotic expansion for high-contrast coefficient based on the decomposition (4.7), which verifies the assertion (4.9) and yields a low-rank approximation to (1.1) under Assumption 4.1.

5 Asymptotic expansion for high-contrast coefficient case

In this section, we establish the low-rank approximation to (1.1) for high-contrast coefficients, i.e., $\eta \to \infty$, by means of layer potential techniques and asymptotic expansion. The spectral gap problem has been considered in various settings, e.g., an efficient preconditioner for high-contrast problems, effective conductivity and multiscale basis functions construction [5, 18, 4, 16, 17]. We focus our discussions on the 2-d case, and the argument is similar for 3-d case.

5.1 The perfect conductivity problem

The starting point of our analysis is the perfect conductivity problem, whose solution naturally serves as the zeroth order approximation. Specifically, we analyze the solution u_η (where the subscript η is to emphasize its dependence on the contrast η) to problem (1.1) with a source term $f \in L^2(D)$ and the coefficient $\kappa := \kappa_\eta$. Upon passing to a subsequence, we have $u_\eta \rightharpoonup u_\infty$ in $H^1(D)$ as $\eta \to \infty$, where u_∞ is the solution to the perfect conductivity problem:

$$\begin{cases} -\Delta u_\infty = f & \text{in } D_0, \\ u_\infty(x)|_+ = u_\infty(x)|_- & \text{on } \Gamma_i, i = 1, 2, \cdots, m, \\ \nabla u_\infty \equiv 0 & \text{in } D_i, i = 1, 2, \cdots, m, \\ \int_{\Gamma_i} \frac{\partial u_\infty}{\partial n_i^+}\mathrm{d}s(x) = -\int_{D_i} f \mathrm{d}x & i = 1, 2, \cdots, m, \\ u_\infty = 0 & \text{on } \partial D. \end{cases} \tag{5.1}$$

Problem (5.1) can be derived by a variational method along the line of [2, Appendix]. Further, we can obtain the following *a priori* estimate

$$|u_\infty|_{H^1(D_0)} \leq C_{\text{poin}}(D)\|f\|_{L^2(D)}. \tag{5.2}$$

Actually, multiplying u_∞ on both sides of the governing equation in (5.1), integration by parts, and appealing to the interface condition in (5.1) and the fact that u_∞ is piecewise constant on the inclusions $\cup_{i=1}^m \overline{D}_i$ lead directly to

$$
\begin{aligned}
|u_\infty|^2_{H^1(D_0)} &= -\sum_{i=1}^m \int_{\Gamma_i} \frac{\partial u_\infty}{\partial n_i^+} u_\infty \mathrm{d}s(x) + \int_{D_0} u_\infty f \mathrm{d}x \\
&= -\sum_{i=1}^m u_\infty \int_{\Gamma_i} \frac{\partial u_\infty}{\partial n_i^+} \mathrm{d}s(x) + \int_{D_0} u_\infty f \mathrm{d}x \\
&= \sum_{i=1}^m \int_{D_i} u_\infty f \mathrm{d}x + \int_{D_0} u_\infty f \mathrm{d}x = \int_D u_\infty f \mathrm{d}x.
\end{aligned}
$$

Then the combination of the Hölder's inequality and Poincaré inequality yields the desired *a priori* estimate.

It can be verified directly that the solution u_∞ to problem (5.1) can be decomposed into

$$
u_\infty = w_0 + \sum_{i=1}^m c_i w_i,
$$

where c_i are constants that can be uniquely determined through (5.1), the functions $\{w_i\}_{i=1}^m$ are defined in (4.4) and w_0 satisfies

$$
\begin{cases}
-\Delta w_0 = f & \text{in } D_0, \\
w_0 = 0 & \text{on } \cup_{k=1}^m \Gamma_k, \\
w_0 = 0 & \text{on } \partial D.
\end{cases}
$$

This last problem is commonly known as the perforated problem with a homogeneous Dirichlet boundary condition in the literature. The Hölder's inequality and Poincaré inequality imply

$$
|w_0|_{H^1(D_0)} \le C_{\text{poin}}(D_0) \|f\|_{L^2(D_0)},
$$

with $C_{\text{poin}}(D_0)$ being the Poincaré constant for D_0.

First, we give a useful orthogonality relation between the difference $u_\eta - u_\infty$ and the space V_m spanned by $\{w_j\}$, defined in (4.4). This result will be used to analyze the leading term approximation below.

Lemma 5.1. *For the functions w_j, $j = 1, \cdots, m$, defined in* (4.4)*, there holds*

$$
\int_D \kappa_\eta \nabla(u_\eta - u_\infty) \cdot \nabla w_j \mathrm{d}x = 0.
$$

Proof. Since w_j is piecewise constant on the domain $D \backslash D_0$, by the divergence theorem, we obtain

$$
\begin{aligned}
\int_D \kappa_\eta \nabla(u_\eta - u_\infty) \cdot \nabla w_j \mathrm{d}x &= \int_{D_0} \kappa_\eta \nabla(u_\eta - u_\infty) \cdot \nabla w_j \mathrm{d}x \\
&= -\int_{\Gamma_j} \kappa_\eta \frac{\partial}{\partial n_j^+}(u_\eta - u_\infty) w_j \mathrm{d}s(x) - \int_{D_0} \nabla \cdot (\kappa_\eta \nabla(u_\eta - u_\infty)) w_j \mathrm{d}x.
\end{aligned}
$$

By virtue of the governing equations for u_η and u_∞, the second term on the right hand side vanishes. For the first term, since $w_j = 1$ on Γ_j and $\kappa_\eta = 1$ in D_0, we have

$$
\int_D \kappa_\eta \nabla(u_\eta - u_\infty) \cdot \nabla w_j \mathrm{d}x = -\int_{\Gamma_j} \frac{\partial}{\partial n_j^+}(u_\eta - u_\infty) \mathrm{d}s(x).
$$

The continuity of the flux for u_η on the interface Γ_j and the interface condition for u_∞ imply

$$
\begin{aligned}
\int_{\Gamma_j} \frac{\partial}{\partial n_j^+}(u_\eta - u_\infty) \mathrm{d}s(x) &= \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} u_\eta \mathrm{d}s(x) - \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} u_\infty \mathrm{d}s(x) \\
&= \int_{\Gamma_j} \kappa_\eta \frac{\partial}{\partial n_j^-} u_\eta \mathrm{d}s(x) + \int_{D_j} f \mathrm{d}x \\
&= \int_{D_j} \big(\nabla \cdot (\kappa_\eta \nabla u_\eta) + f\big) \mathrm{d}x = 0,
\end{aligned}
$$

and this yields the desired result. $\square$

Let us examine the energy error committed when approximating the solution u_η by the leading term u_∞. The following energy error follows by a straightforward application of the divergence theorem

$$
\begin{aligned}
\|u_\eta - u_\infty\|^2_{H^1_\kappa(D)} &= \langle u_\eta - u_\infty, u_\eta - u_\infty\rangle_D \\
&= \sum_{j=1}^m \int_{\Gamma_j} \left[\kappa_\eta \frac{\partial u_\infty}{\partial n_j}\right](u_\eta - u_\infty)\mathrm{d}s(x) + \sum_{j=1}^m \int_{D_j} f(u_\eta - u_\infty)\mathrm{d}x.
\end{aligned}
\tag{5.3}
$$

This estimate indicates that there are two sources of the energy error: (i) the nonzero source term f on each inclusion D_j and (ii) the mismatch of the interface flux, namely,

$$
\left[\kappa_\eta \frac{\partial u_\infty}{\partial n_j}\right] = \frac{\partial u_\infty}{\partial n_j^+} \neq 0 \quad \text{on } \Gamma_j \text{ for } j = 1, 2, \cdots, m. \tag{5.4}
$$

In order to obtain a good approximation, one has to decrease these two sources of errors, which will be carried out below by means of layer potential techniques and asymptotic expansion.

5.2 Asymptotic expansion

Now we derive a novel asymptotic expansion, by carefully analyzing (5.4) using layer potential techniques and asymptotic expansion. This expansion lends itself to a useful low-rank approximation. First, we build auxiliary basis functions to decrease the mismatch on the interfaces. To this end, we denote by $z_j \in L^2_0(\Gamma_j) := \{v \in L^2(\Gamma_j) \text{ with } \int_{\Gamma_j} v\mathrm{d}s(x) = 0\}$, the unknown layer potential density for obtaining the auxiliary function in order to decrease the flux mismatch on the interface Γ_j for $j = 1, 2, \cdots, m$, cf. (5.4). Let

$$
z(x) = \sum_{j=1}^m z_j \delta_{\Gamma_j},
$$

and define the operator $\hat{\mathcal{R}} : L^2(D) \to H^1_0(D)$ by

$$
\Delta \hat{\mathcal{R}}(z) = z \quad \text{in } D, \qquad \text{with } \hat{\mathcal{R}}(z) = 0 \quad \text{on } \partial D. \tag{5.5}
$$

Further, we define

$$
\mathcal{R}(z, f) := \hat{\mathcal{R}}(z) + \hat{u},
$$

where $\hat{u} \in H^1_0(D)$ satisfies

$$
-\nabla \cdot (\kappa_\eta \nabla \hat{u}) = f \qquad \text{in } D_j \qquad \text{with } \hat{u} = 0 \quad \text{on } \Gamma_j, \tag{5.6}
$$

and a zero extension on D_0. Multiplying both sides of the governing equation by $\hat{u}$, integrating over the domain D, an application of the Hölder's inequality and the Poincaré inequality result in

$$
\sum_{j=1}^m |\hat{u}|^2_{H^1(D_j)} = \sum_{j=1}^m \eta_j^{-1} \int_{D_j} f\hat{u}\mathrm{d}x \leq \sum_{j=1}^m \eta_j^{-1} C_{\text{poin}}(D_j)\|f\|_{L^2(D_j)}|\hat{u}|_{H^1(D_j)}.
$$

Then an application of the Young's inequality yields

$$
|\hat{u}|_{H^1(D)} \leq \max_{j=1,2,\cdots,m} \{C_{\text{poin}}(D_j)\eta_j^{-1}\}\|f\|_{L^2(D)}. \tag{5.7}
$$

Therefore, the solution $\|\hat{u}\|_{H^1(D_j)}$ and $\|\frac{\partial \hat{u}}{\partial n_j^-}\|_{H^{-\frac{1}{2}}(\Gamma_j)}$ are both of order $\mathcal{O}(\eta_j^{-1})$, and $\frac{\partial \hat{u}}{\partial n_j^+} = 0$. The solution $\hat{u}$ will be used to correct the force term in the inclusions $\{D_i\}_{i=1}^m$, cf. (5.3).

This section is dedicated to seeking functions $\{z_j\}_{j=1}^m$ such that

$$
u_\eta = u_\infty + \mathcal{R}(z, f). \tag{5.8}
$$

By the continuity of the flux $\kappa \frac{\partial u_\eta}{\partial n_j}$ for u_η across the interface Γ_j and in view of the relation (5.4), this is equivalent to

$$
\left[\kappa_\eta \frac{\partial}{\partial n_j}\mathcal{R}(z, f)\right] = -\frac{\partial u_\infty}{\partial n_j^+} \quad \text{on } \Gamma_j \text{ for } j = 1, 2, \cdots, m. \tag{5.9}
$$

The definition (5.5) indicates that $\hat{\mathcal{R}}$ is harmonic in $D \backslash \cup_{j=1}^m \Gamma_j$. Moreover, the next result gives an important characterization of $\hat{\mathcal{R}}(z)$, i.e., $\hat{\mathcal{R}}(z) \in V^h$.

Lemma 5.2. *For $j = 1, 2, \ldots, m$, there holds*

$$\int_{\Gamma_j} \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z) \mathrm{d}s(x) = 0.$$

Proof. First, the defining identity (5.5) and the divergence theorem imply

$$\int_{\Gamma_j} \kappa_\eta \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z) \mathrm{d}s(x) = 0.$$

Meanwhile the identity (5.9) and the fact $\frac{\partial \hat{u}}{\partial n_j^+} = 0$ imply

$$[\kappa_\eta \frac{\partial}{\partial n_j} \hat{\mathcal{R}}(z)] = -\frac{\partial u_\infty}{\partial n_j^+} - [\kappa_\eta \frac{\partial \hat{u}}{\partial n_j}] = -\frac{\partial u_\infty}{\partial n_j^+} + \kappa_\eta \frac{\partial \hat{u}}{\partial n_j^-}.$$

By integrating over Γ_j, and applying the divergence theorem, the governing equation (5.6), and the interface condition (5.1), we obtain

$$\int_{\Gamma_j} -\frac{\partial u_\infty}{\partial n_j^+} + \kappa_\eta \frac{\partial \hat{u}}{\partial n_j^-} \mathrm{d}s(x) = \int_{D_j} f \mathrm{d}x + \int_{D_j} \nabla \cdot (\kappa_\eta \nabla \hat{u}) \mathrm{d}x = 0,$$

from which the desired assertion follows directly. □

Our main tool to identify the unknown $\{z_j\}_{j=1}^m$ is the layer potential techniques. First, we recall a few preliminary results. We denote by $\Phi(x, y) = (2\pi)^{-1} \log |x - y|$ the fundamental solution of the Laplacian in $\mathbb{R}^2$. Then the Green's function $G(x, y)$ for the unperturbed domain D is given by

$$G(x, y) = \Phi(x, y) - H(x, y),$$

where $H(x, y)$ represents its regular part satisfying

$$\begin{cases} \Delta_x H(x, y) = 0 & x, y \in D, \\ \quad H(x, y) = (2\pi)^{-1} \log |x - y| & x \in \partial D, y \in D. \end{cases}$$

Thus, using Green's function $G(x, y)$, the function $\hat{\mathcal{R}}(z)$ admits a (formal) expression

$$\hat{\mathcal{R}}(z) = \int_D G(x, y) z(y) \mathrm{d}y = \sum_{j=1}^m \Big(\int_{\Gamma_j} \Phi(x, y) z_j(y) \mathrm{d}s(y) - \int_{\Gamma_j} H(x, y) z_j(y) \mathrm{d}s(y) \Big). \tag{5.10}$$

The single layer potential $\mathcal{S}_{D_j} z_j(x)$ of the density function z_j on the interface Γ_j is defined by

$$\mathcal{S}_{D_j} z_j(x) = \int_{\Gamma_j} \Phi(x, y) z_j(y) \mathrm{d}s(y),$$

and there hold the well-known jump formula [31]:

$$\frac{\partial}{\partial n_j^\pm} \mathcal{S}_{D_j} z_j(x) = (\pm \tfrac{1}{2} + \mathcal{K}_{D_j}^*) z_j(x), \quad x \in \Gamma_j \quad \text{for } j = 1, 2, \cdots, m, \tag{5.11}$$

where $\mathcal{K}_{D_j}^*$ is the $L^2(\Gamma_j)$-adjoint of the operator $\mathcal{K}_{D_j}$, defined by

$$\begin{aligned} \mathcal{K}_{D_j} z_j(x) &= \frac{1}{2\pi} \text{p.v.} \int_{\Gamma_j} \frac{(y - x, n_j(y))}{|x - y|^2} z_j(y) \mathrm{d}s(y) \\ &:= \frac{1}{2\pi} \lim_{t \to 0^+} \int_{\Gamma_j \cap |x-y| > t} \frac{(y - x, n_j(y))}{|x - y|^2} z_j(y) \mathrm{d}s(y). \end{aligned}$$

Here, p.v. denotes taking the Cauchy principal value. It is well known that if the interface Γ_j is Lipschitz, then the singular integral operator $\mathcal{K}_{D_j}$ is bounded on the space $L^2(\Gamma_j)$ [9]. Further, the identities (5.10) and (5.11) together with the regularity of $H(x,y)$ yield

$$\frac{\partial \hat{\mathcal{R}}(z)}{\partial n_j^+} - \frac{\partial \hat{\mathcal{R}}(z)}{\partial n_j^-} = z_j \quad \text{on } \Gamma_j. \tag{5.12}$$

Next, we choose $\{z_j\}_{j=1}^m$ to satisfy the flux condition (5.9). By the definitions of $\mathcal{R}(z,f)$ and $\hat{u}$, the flux condition (5.9) is equivalent to

$$\Big[\kappa_\eta \frac{\partial \hat{\mathcal{R}}(z)}{\partial n_j}\Big] := -\frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z)\eta_j + \frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z) = -\frac{\partial u_\infty}{\partial n_j^+} + \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \quad \text{on } \Gamma_j. \tag{5.13}$$

This relation forms the basis of the asymptotic expansion below. The expression of $\hat{\mathcal{R}}(z)$ in (5.10) and the jump formula (5.11) imply

$$\begin{aligned}\Big(\frac{1}{2}z_j - \sum_{i=1}^m \text{p.v.}\int_{\Gamma_i} \frac{\partial G(x,y)}{\partial n_i(x)} z_i(y)\mathrm{d}s(y)\Big)\eta_j + \frac{1}{2}z_j + \sum_{i=1}^m \text{p.v.}\int_{\Gamma_i} \frac{\partial G(x,y)}{\partial n_i(x)} z_i(y)\mathrm{d}s(y) \\ = -\frac{\partial u_\infty}{\partial n_j^+} + \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \quad \text{on } \Gamma_j.\end{aligned}$$

Now we are ready to determine the leading terms of the asymptotic expansion for each $\{z_j\}_{j=1}^m$. This can be achieved as follows. First, assume that they admit the formal expansion

$$z_j(x) = \sum_{\ell=0}^\infty z_j^\ell \eta_j^{-\ell} \quad x \in \Gamma_j. \tag{5.14}$$

Further, upon assuming that $\{\eta_j\}_{j=1}^m$ are of comparable magnitude, we let

$$z^n(x) = \sum_{j=1}^m \big(\sum_{\ell=0}^n z_j^\ell \eta_j^{-\ell}\big)\delta_{\Gamma_j} \tag{5.15}$$

be the n^{th} order approximation to z. Then the n^{th} order approximation u^n to u_η is defined by

$$u^n = u_\infty + \hat{u} + \hat{\mathcal{R}}(z^n). \tag{5.16}$$

Upon substituting (5.14) into (5.13) and collecting terms according to the order in η_j, by the trace formula and Lemma 5.2, we obtain the following hierarchies:

(i) $\mathcal{O}(\eta)$ term

$$\frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^0) = 0 \quad \text{and} \quad \int_{\Gamma_j} \frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^0)\mathrm{d}s(x) = 0; \tag{5.17}$$

(ii) the $\mathcal{O}(1)$ term

$$-\frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^1 - z^0)\eta_j + \frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^0) = -\frac{\partial u_\infty}{\partial n_j^+} + \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \quad \text{and} \quad \int_{\Gamma_j} \frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^1)\mathrm{d}s(x) = 0;$$

(iii) the high-order terms, for $\ell = 1, 2, \cdots$, the $\mathcal{O}(\eta^{-\ell})$ term

$$-\frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^{\ell+1} - z^\ell)\eta_j + \frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^\ell - z^{\ell-1}) = 0 \quad \text{and} \quad \int_{\Gamma_j} \frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^{\ell+1})\mathrm{d}s(x) = 0. \tag{5.18}$$

Next we discuss these terms one by one. First, for the $\mathcal{O}(\eta)$ term, the conditions in (5.17) imply that $\hat{\mathcal{R}}(z^0) \in V_m \cap V^h$, cf. Lemma 5.2. Then an application of Theorem 4.3 yields

$$z_j^0 = 0, \quad j = 1, 2, \cdots, m.$$

Next, we solve for the second term z^1, which satisfies

$$\frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^1) = \eta_j^{-1}\Big(\frac{\partial u_\infty}{\partial n_j^+} - \eta_j \frac{\partial \hat{u}}{\partial n_j^-}\Big), \quad j = 1, 2, \cdots, m. \tag{5.19}$$

The identity (5.1) together with (5.6) yields

$$\int_{\Gamma_j} \Big(\frac{\partial u_\infty}{\partial n_j^+} - \eta_j \frac{\partial \hat{u}}{\partial n_j^-}\Big) \mathrm{d}s(x) = -\int_{D_j} f \mathrm{d}x - \Big(-\int_{D_j} f \mathrm{d}x\Big) = 0, \quad j = 1, \ldots, m.$$

Therefore, the second term $\hat{\mathcal{R}}(z^1)$ can be obtained by solving the following problem

$$\begin{cases} -\nabla \cdot (\kappa_\eta \nabla \hat{\mathcal{R}}(z^1)) = 0 & \text{in } D_j, \\ \dfrac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^1) = \eta_j^{-1}\Big(\dfrac{\partial u_\infty}{\partial n_j^+} - \eta_j \dfrac{\partial \hat{u}}{\partial n_j^-}\Big) & \text{on } \Gamma_j. \end{cases}$$

Together with (5.12), this yields $z_j^1 \in L_0^2(\Gamma_j)$.

To explicitly construct higher-order terms $\hat{\mathcal{R}}(z^\ell)$ for $\ell = 2, 3, \cdots$, we need their Neumann data in the inclusion D_j, which in turn is related to the Neumann data of the lower order terms in D_0 by (5.18), where the Dirichlet data is available by the continuity of $\hat{\mathcal{R}}(z^\ell)$ along the interface Γ_j, for $j = 1, 2, \cdots, m$. Thus, we employ the DtN map and NtD map. We denote by $\Lambda_j^N : H^{-\frac{1}{2}}(\Gamma_j) \to H^{\frac{1}{2}}(\Gamma_j)$ the NtD map on D_j and by $\Lambda^D : H^{\frac{1}{2}}(\partial D_0) \to H^{-\frac{1}{2}}(\partial D_0)$ the DtN map on D_0. Then the Neumann data of lower orders in D_0 can be expressed as

$$\Big[\frac{\partial}{\partial n_1^+}\hat{\mathcal{R}}(z^\ell - z^{\ell-1}), \frac{\partial}{\partial n_2^+}\hat{\mathcal{R}}(z^\ell - z^{\ell-1}), \cdots, \frac{\partial}{\partial n_m^+}\hat{\mathcal{R}}(z^\ell - z^{\ell-1})\Big] = \Lambda^D(\hat{\mathcal{R}}(z^\ell - z^{\ell-1})), \quad \ell = 1, 2, \cdots.$$

The boundedness of the operators Λ_j^N and Λ^D implies

$$\begin{aligned} &\Big(\sum_{j=1}^m \|\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^\ell - z^{\ell-1})\|^2_{H^{-\frac{1}{2}}(\Gamma_j)}\Big)^{\frac{1}{2}} \\ \leq& \|\Lambda^D\| \max_{j=1,\cdots,m}\{\|\Lambda_j^N\|\}\Big(\sum_{j=1}^m \|\frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^\ell - z^{\ell-1})\|^2_{H^{-\frac{1}{2}}(\Gamma_j)}\Big)^{\frac{1}{2}}. \end{aligned} \tag{5.20}$$

Then we can obtain the higher order terms $\hat{\mathcal{R}}(z^{\ell+1})$ by solving Neumann problems in D_j:

$$-\Delta \hat{\mathcal{R}}(z^{\ell+1}) = 0 \quad \text{in } D_j,$$

together with the corresponding boundary condition

$$\begin{aligned} \frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^{\ell+1}) = \frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^\ell) + \eta_j^{-1}\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^\ell - z^{\ell-1}) \quad \text{on } \Gamma_j, \\ \text{satisfying } \int_{\Gamma_j} \frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z_j^{\ell+1}) \mathrm{d}s(x) = 0, \end{aligned} \tag{5.21}$$

which is a consequence of the higher order terms in (5.18), (5.12) and the fact that z_j^ℓ and $z_j^{\ell-1}$ belong to $L_0^2(\Gamma_j)$. Clearly, this is a well-posed problem. Next, we bound the energy error $\|u_\eta - u^n\|_{H^1_\kappa(D)}$. To this end, we first derive the expression of the flux jump of u^n.

Lemma 5.3. *Let u^n be the n^{th} order approximation to u_η defined in* (5.16) *for $n \in \mathbb{N}_+$. Then it holds*

$$\Big[\kappa_\eta \frac{\partial u^n}{\partial n_j}\Big] = \frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^n - z^{n-1}) \quad \text{on } \Gamma_j, \quad j = 1, \ldots, m.$$

Proof. By the definition of u^n in (5.16) and noting $\frac{\partial u_\infty}{\partial n_j^-} = 0$, we have

$$\eta_j \frac{\partial u^n}{\partial n_j^-} = \eta_j \frac{\partial u_\infty}{\partial n_j^-} + \eta_j \frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^n) + \eta_j \frac{\partial \hat{u}}{\partial n_j^-} = \eta_j \frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^n) + \eta_j \frac{\partial \hat{u}}{\partial n_j^-},$$

Then by rewriting $\frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^n)$ as a telescopic sum and using the flux conditions (5.19) and (5.21), we obtain

$$\begin{aligned}\eta_j\frac{\partial u^n}{\partial n_j^-} &= \eta_j\frac{\partial\hat{\mathcal{R}}(z^1)}{\partial n_j^-}+\eta_j\sum_{\ell=2}^{n}\frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^\ell-z^{\ell-1})+\eta_j\frac{\partial\hat{u}}{\partial n_j^-}\\ &= \frac{\partial u_\infty}{\partial n_j^+}+\sum_{\ell=1}^{n-1}\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^\ell-z^{\ell-1}).\end{aligned}$$

Likewise, by the definition of u^n, and noting $\frac{\partial\hat{u}}{\partial n_j^+}=0$ and $\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^0)=0$ (since $\hat{\mathcal{R}}(z^0)=0$), a direct calculation leads to

$$\frac{\partial u^n}{\partial n_j^+}=\frac{\partial u_\infty}{\partial n_j^+}+\frac{\partial\hat{u}}{\partial n_j^+}+\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^n)=\frac{\partial u_\infty}{\partial n_j^+}+\sum_{\ell=1}^{n}\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^\ell-z^{\ell-1}).$$

Now the desired result follows by subtraction the preceding two identities. □

A similar argument as for (5.3) together with Lemma 5.3 yields

$$\begin{aligned}\|u_\eta-u^n\|^2_{H^1_\kappa(D)}=\langle u_\eta-u^n,u_\eta-u^n\rangle_D &= \sum_{j=1}^{m}\int_{\Gamma_j}\left[\kappa_\eta\frac{\partial u^n}{\partial n_j}\right](u_\eta-u^n)\mathrm{d}s(x)\\ &= \sum_{j=1}^{m}\int_{\Gamma_j}\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^n-z^{n-1})(u_\eta-u^n)\mathrm{d}s(x).\end{aligned}\tag{5.22}$$

Next we estimate the order of the term $\hat{\mathcal{R}}(z^n-z^{n-1})$ for $n\in\mathbb{N}_+$:

Lemma 5.4. *Let z^n be defined in* (5.15) *with $n\in\mathbb{N}_+$. There holds*

$$\sum_{j=1}^{m}\|\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^n-z^{n-1})\|^2_{H^{-\frac12}(\Gamma_j)}\lesssim\eta_{\mathit{min}}^{-2n}(C_{\mathit{poin}}(D)^2+\max\{C_{\mathit{poin}}(D_j)^2\})\|f\|^2_{L^2(D)}.\tag{5.23}$$

Proof. We prove the result by mathematical induction. First we consider the case $n=1$. In view of $\hat{\mathcal{R}}(z^0)=0$, by appealing to (5.20) and the flux condition (5.19), we have

$$\sum_{j=1}^{m}\|\frac{\partial\hat{\mathcal{R}}(z^1)}{\partial n_j^+}\|^2_{H^{-\frac12}(\Gamma_j)}\lesssim\sum_{j=1}^{m}\|\frac{\partial\hat{\mathcal{R}}(z^1)}{\partial n_j^-}\|^2_{H^{-\frac12}(\Gamma_j)}=\sum_{j=1}^{m}\eta_j^{-2}\|\frac{\partial u_\infty}{\partial n_j^+}-\eta_j\frac{\partial\hat{u}}{\partial n_j^-}\|^2_{H^{-\frac12}(\Gamma_j)}.$$

By the trace theorem and the *a priori* estimate (5.2),

$$\sum_{j=1}^{m}\|\frac{\partial u_\infty}{\partial n_j^+}\|^2_{H^{-\frac12}(\Gamma_j)}\lesssim\int_{D_0}|\nabla u_\infty|^2\mathrm{d}x\lesssim C_{\mathrm{poin}}(D)^2\|f\|^2_{L^2(D)}.$$

Likewise, by the trace theorem and the *a priori* estimate (5.7), we deduce

$$\sum_{j=1}^{m}\|\eta_j\frac{\partial\hat{u}}{\partial n_j^-}\|^2_{H^{-\frac12}(\Gamma_j)}\lesssim\sum_{j=1}^{m}\eta_j^2|\hat{u}|^2_{H^1(D_j)}\lesssim\max\{C_{\mathrm{poin}}(D_j)^2\})\|f\|^2_{L^2(D)}.$$

Combining the preceding three estimates yields

$$\sum_{j=1}^{m}\|\frac{\partial\hat{\mathcal{R}}(z^1)}{\partial n_j^+}\|^2_{H^{-\frac12}(\Gamma_j)}\lesssim\eta_{\mathrm{min}}^{-2}(C_{\mathrm{poin}}(D)^2+\max\{C_{\mathrm{poin}}(D_j)^2\})\|f\|^2_{L^2(D)}.$$

This verifies (5.23) for $n = 1$. Now assume that the estimate (5.23) holds for some $n = \ell > 1$, and we show that (5.23) holds for $n = \ell + 1$. Appealing to (5.20) and (5.21) yields

$$\begin{aligned}\sum_{j=1}^{m}\|\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^{\ell+1}-z^{\ell})\|^2_{H^{-\frac{1}{2}}(\Gamma_j)} &\lesssim \sum_{j=1}^{m}\|\frac{\partial}{\partial n_j^-}\hat{\mathcal{R}}(z^{\ell+1}-z^{\ell})\|^2_{H^{-\frac{1}{2}}(\Gamma_j)}\\ &= \sum_{j=1}^{m}\eta_j^{-2}\|\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^{\ell}-z^{\ell-1})\|^2_{H^{-\frac{1}{2}}(\Gamma_j)}\\ &\lesssim \eta_{\text{min}}^{-2(\ell+1)}(C_{\text{poin}}(D)^2+\max\{C_{\text{poin}}(D_j)^2\})\|f\|^2_{L^2(D)},\end{aligned}$$

where the last line follows from the induction hypothesis, and this completes the proof. □

By the elliptic regularity theory and Lemma 5.4, the following assertion holds:

Proposition 5.1. *Let the n-th order approximation z^n be defined in* (5.15) *for $n \in \mathbb{N}_+$. Then there holds*

$$\|\hat{\mathcal{R}}(z^n - z^{n-1})\|_{H^1_\kappa(D)} \lesssim \eta_{min}^{-n+\frac{1}{2}}(C_{poin}(D) + \max\{C_{poin}(D_j)\})\|f\|_{L^2(D)}. \tag{5.24}$$

Proof. By the elliptic regularity in the domain D_0 and each inclusion D_i and Lemma 5.4, we deduce

$$\begin{aligned}\|\hat{\mathcal{R}}(z^n-z^{n-1})\|^2_{H^1_\kappa(D)} &= |\hat{\mathcal{R}}(z^n-z^{n-1})|^2_{H^1(D_0)} + \sum_{i=1}^{m}\eta_i|\hat{\mathcal{R}}(z^n-z^{n-1})|^2_{H^1(D_i)}\\ &\lesssim \sum_{i=1}^{m}\|\frac{\partial}{\partial n_i^+}\hat{\mathcal{R}}(z^n-z^{n-1})\|^2_{H^{-\frac{1}{2}}(\Gamma_i)} + \sum_{i=1}^{m}\eta_i\|\frac{\partial}{\partial n_i^-}\hat{\mathcal{R}}(z^n-z^{n-1})\|^2_{H^{-\frac{1}{2}}(\Gamma_i)}\\ &\lesssim \eta_{\text{min}}^{-2n+1}(C_{\text{poin}}(D)^2+\max\{C_{\text{poin}}(D_j)^2\})\|f\|^2_{L^2(D)}.\end{aligned}$$

The assertion follows by taking the square root of both sides. □

Finally, we are ready to state an energy error estimate by combining (5.22) with Lemma 5.4.

Theorem 5.1. *Let u^n be the n^{th} order approximation to u_η defined in* (5.16). *There holds*

$$\|u_\eta - u^n\|_{H^1_\kappa(D)} \lesssim \eta_{min}^{-n}(C_{poin}(D) + \max\{C_{poin}(D_j)\})\|f\|_{L^2(D)}.$$

Proof. By (5.22) and the trace theorem

$$\begin{aligned}\|u_\eta - u^n\|^2_{H^1_\kappa(D)} &= \sum_{j=1}^{m}\int_{\Gamma_j}\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^n-z^{n-1})(u_\eta-u^n)\mathrm{d}s(x)\\ &\lesssim \sum_{j=1}^{m}\|\frac{\partial}{\partial n_j^+}\hat{\mathcal{R}}(z^n-z^{n-1})\|_{H^{-\frac{1}{2}}(\Gamma_j)}\|u_\eta-u^n\|_{H^{\frac{1}{2}}(\Gamma_j)}.\end{aligned}$$

Then the desired result follows from Lemma 5.4, Hölder's inequality and the fact that $\eta_{\text{min}} \geq 1$. □

The asymptotic expansion for high-contrast problems when $\eta \to \infty$ was studied earlier [7, 6]. However, our result contains a much better zeroth-order approximation, i.e., the solution u_∞ to the perfect conductivity problem (5.1), which is the weak limit of u_η in $H^1(D)$ as $\eta \to \infty$, and thus, also a much sharp error estimate.

Proposition 5.2. *Let $\eta \to \infty$. There holds*

$$\|u_\eta - u_\infty\|_{H^1_\kappa(D)} \lesssim \eta_{min}^{-\frac{1}{2}}(C_{poin}(D) + \max\{C_{poin}(D_j)\})\|f\|_{L^2(D)}.$$

Proof. This result follows from Theorem 5.1, Proposition 5.1 for $n = 1$, the *a priori* estimate (5.7) and an application of the triangle inequality. □

Last, we examine the connection between the n^{th} approximant u^n in (5.16) and the orthogonal decomposition (4.7) more closely. One observes that $u_\infty \in V_m \oplus V_0^b$, $\hat{u} \in V^b$ and $\hat{\mathcal{R}}(z^n) \in V^h$. The zeroth approximant u_∞ is related to the force term f via the component w_0, the second term $\hat{u}$ also depends on f, cf. (5.6), and the dependence of $\hat{\mathcal{R}}(z^n)$ on f is due to the normal flux (5.18). In order to obtain a low-rank approximation to u_η that are independent of the force term f, cf. (1.3), we require Assumption 4.1. An application of Propositions 2.1 and 5.2, and Theorem 4.2 yields the next result on Kolmogorov n-width.

Proposition 5.3. *Let $d = 2$, and let Assumption 4.1 be valid. Assume that $\eta \to \infty$ and $\delta_j \gg \epsilon_i$, for $j = 1, 2, \cdots, m$. There holds*

$$d_i(\mathcal{S}(W); V) \begin{cases} \geq C_{poin}(D)\sqrt{\dfrac{|D_{i+1}|}{\pi}} \text{ for } i \leq m-1; \\ \lesssim \sup\limits_{u \in \mathcal{U}} \inf\limits_{v \in V_m} \|u - v\|_{H^1_\kappa(D)} \lesssim \eta_{min}^{-\frac{1}{2}} (C_{poin}(D) + \max\{C_{poin}(D_j)\}) \text{ for } i = m. \end{cases}$$

Remark 5.1. *First, Proposition 5.3 implies the assertion* (4.9)*. Further, it indicates that there is a spectral gap in the high-contrast limit, i.e., as $\eta \to \infty$, if Assumption 4.1 holds. Moreover, there are precisely m dominant eigenmodes, where m is the number of inclusions. Such a gap implies the existence of an effective low-rank approximation, and can and should be effectively employed in the numerical treatment of high-contrast problems.*

6 Conclusion

In this work, we have investigated the low-rank approximation properties to heterogeneous elliptic problems, and provided their optimal approximation rate via the concept of Kolmogorov n-width, which is essentially related to the eigenvalue decay rate of the solution map. To illustrate the important role the structure of the coefficient plays in the low-rank property of the solution, we provided one sufficient conditions for low-rank approximation, which directly motivates the use of harmonic functions. In order to derive the eigenvalue decay rate, we discussed realistic assumptions on the permeability field κ, e.g., the values, the locations of the inclusions and the pairwise distances, which would hugely influence the eigenvalues. Further, we have provided a new eigenvalue estimate for elliptic operators with high-contrast coefficient and derived a new asymptotic expansion with respect to the high-contrast, which are of independent interest. These results show the existence of a low-rank structure of the solution manifold for certain heterogeneous problems, and thereby provide the theoretical justifications of multiscale model reduction techniques.

This work represents a first step towards the complete theoretical understanding of multiscale model reduction algorithms. There are a few lines for future research, e.g., general L^∞ coefficient and optimal approximation rate.

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