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Upscaled HDG methods for Brinkman equations with high-contrast heterogeneous coefficient

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Dedicated to Bernardo Cockburn on the occasion of his 60-th anniversary

Abstract

In this paper, we present new upscaled HDG methods for Brinkman equations in the context of high-contrast heterogeneous media. The a priori error estimates are derived in terms of both fine and coarse scale parameters that depends on the high-contrast coefficient weakly. Due to the heterogeneousity of the problem, a huge global system will be produced after the numerical discretization of HDG method. Thanks to the upscaled structure of the proposed methods, we are able to reduce the huge global system onto the skeleton of the coarse mesh only while still capturing important fine scale features of this problem. The finite element space over the coarse mesh is irrelevant to the fine scale computation. This feature makes our proposed method very attractive. Several numerical examples are presented to support our theoretical findings.

1 Introduction

In this paper, we present a projection-based analysis of an upscaled hybridizable discontinuous Galerkin ($\mathbf{HDG}$) method for Brinkman equations. The Brinkman model can be written as

$$\begin{aligned} \nabla p - \nabla\cdot(\nu\nabla\boldsymbol{u}) + \kappa^{-1}\boldsymbol{u} &= \boldsymbol{f} \quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \partial\Omega \end{aligned}$$

with $\Omega \in \mathbb{R}^n$ being a bounded polygonal domain if $n = 2$ or a Lipschitz polyhedral domain if $n = 3$. Here, $\boldsymbol{f}$ and $\boldsymbol{g}$ are given functions on Ω satisfying the compatible condition $\int_{\partial\Omega}\boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{d}s(x) = 0$, p is the fluid pressure, $\boldsymbol{u}$ represents the velocity, ν is the viscosity and κ is a heterogeneous multiscale coefficient that models the permeability of the porous medium. The coefficient $\kappa^{-1} \in L^\infty(\Omega)$ satisfies $0 < \alpha \le \kappa^{-1} \le \beta$ for some positive constants α, β. We assume that the variations of κ occur within a very fine scale and therefore a direct simulation of this model is costly.

The Brinkman equation has wide applications in the mathematical modeling of flows in heterogeneous fields due to its flexibility in changing between a slow flow region and a fast flow region through the Darcy drag term $\kappa^{-1}u$, for example, vuggy carbonate reservoirs, low porosity filtration devices and biomedical hydrodynamic studies [28, 29]. When $\kappa \to \infty$, this Darcy drag term can be ignored and we can obtain Stokes flow. On the contrary, if $\kappa \to 0$, one can retrieve Darcy flow. In comparison, the simple Darcy model is only capable of describing slow flow problems [30].

The upscaled HDG methods were introduced and studied in [26, 27] for Darcy flows. Therein, the authors rigorously analyze the methods and optimal orders of convergence are obtained for both pressure and the gradient variables. One feature of the upscaled HDG framework is that the only globally coupled unknowns are the numerical trace and the average of pressure on the coarse skeleton. In addition, with the help of the HDG *stabilization*

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technique, the choice of the coarse space for the numerical trace is independent of the fine scale computations. Several choices based on different spectral decomposition have been studied and tested in [27] for Darcy flows in porous media.

In this paper, we propose to develop the upscaled HDG framework for the more challenging Brinkman equations. In [25], the authors proposed a mixed multiscale FEM for the Brinkman equations. The partition of unity functions are used in order to construct a stable pair of the scheme. A main drawback of this approach is that the approximation of the pressure is relatively weak due to the fact only piecewise constant functions on the coarse mesh are used to approximate the pressure. Yet it is not clear how to enrich the space for the pressure without destroying the stability of the method. With the HDG approach, we can improve the approximation of the pressure significantly. Roughly speaking, in our approach, piecewise constant functions are used on coarse mesh for the *average* pressure within each coarse cell only. The pressure itself is approximated by fine scale polynomials via local solvers.

The main contribution in this paper are three folds. Firstly, this proposed upscaled HDG method for Brinkman flow is nontrivial compared to the previous work on the elliptic problems [26, 27] due to its complicity involving the balance between the pressure space and the velocity space. Secondly, comparing with our previous work in [25], we have better error estimates for both velocity and pressure in the sense that the resulting error estimate depends on the L^2 norm of κ^{-1} instead of L^∞ norm as employed in the previous study for the velocity. In addition, the pressure approximation is significantly improved. Indeed, in [25] the approximation for the pressure is extremely weak because of the constraint of the discrete *inf-sup* condition. Thirdly, we have arrived at a general error estimate that allows for using any conforming or nonconforming coarse edge space $\boldsymbol{M}_H$. This feature makes the method much more robust due to the absence of the partition of unity functions, cf. [25]. Numerical simulations are shown to verify our theoretical findings.

The remainder of this paper is organized as follows. The upscaled HDG methods are presented in Section 2 along with auxiliary notations. Its well-posedness is studied in Section 3. In Section 4, the upscaling structure and local solvers of this upscaled HDG method are derived. The main error estimate and numerical results are presented in Sections 5 and 6. Finally, we end our paper with a conclusion in Section 7.

2 Preliminary

We present in this section the main algorithm of our proposed upscaled HDG methods. To proceed, we start writing the Brinkman equations as a system of first order equations

$$
\begin{aligned}
\mathbb{L} - \nabla \boldsymbol{u} &= 0 \qquad \text{in } \Omega && (2.1a)\\
-\nabla \cdot \nu \mathbb{L} + \nabla p + \kappa^{-1}\boldsymbol{u} &= \boldsymbol{f} \qquad \text{in } \Omega && (2.1b)\\
\nabla \cdot \boldsymbol{u} &= 0 \qquad \text{in } \Omega && (2.1c)\\
\boldsymbol{u} &= \boldsymbol{g} \qquad \text{on } \partial\Omega && (2.1d)\\
\int_\Omega p &= 0. && (2.1e)
\end{aligned}
$$

The two-scale finite element spaces are defined as below. Let $\mathcal{T}_H$ denote a decomposition of the domain Ω into non-overlapping shape-regular coarse elements with maximal mesh size H and coarse edges $\mathcal{E}_H := \cup_{T\in\mathcal{T}_H}\partial T$. In each coarse element $T \in \mathcal{T}_H$, let $\mathcal{T}_h(T)$ be a regular quasi-uniform fine triangulation of T with mesh-parameter h and $\mathcal{E}_h^0(T) := \cup_{K\in\mathcal{T}_h(T)}\partial K\backslash\partial T$ its interior edges. Denote the fine scale mesh $\mathcal{T}_h := \cup_{T\in\mathcal{T}_H}\mathcal{T}_h(T)$ and fine scale edges $\mathcal{E}_h := \cup_{K\in\mathcal{T}_h}\partial K$. Each fine element $K \in \mathcal{T}_h$ is a shape regular triangle in $2D$ or tetrahedral in $3D$. We also define the internal fine-scale edges $\mathcal{E}_h^0 := \{e_h \in \mathcal{E}_h | e_h \cap \mathcal{E}_H = \emptyset\}$. We assume that κ is piece wise constant over the fine mesh T_h.

The methods we consider seek an approximation to $(\mathbb{L}, \boldsymbol{u}, p, \boldsymbol{u}|_{\mathcal{E}_h})$ denoted by $(\mathbb{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_{h,H})$, which belongs to the finite dimensional space $\mathbb{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_{h,H}$ where

$$
\begin{aligned}
\mathbb{G}_h &= \{\mathbb{G} \in \mathrm{L}^2(\mathcal{T}_h) : \quad \mathbb{G}|_K \in \mathbb{G}(K) \qquad \forall\, K \in \mathcal{T}_h\} && (2.2a)\\
\boldsymbol{V}_h &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{T}_h) : \quad \boldsymbol{v}|_K \in \boldsymbol{V}(K) \qquad \forall\, K \in \mathcal{T}_h\} && (2.2b)\\
P_h &= \{q \in L^2(\mathcal{T}_h) : \quad q|_K \in P(K) \qquad \forall\, K \in \mathcal{T}_h\} && (2.2c)\\
\boldsymbol{M}_{h,H} &= \boldsymbol{M}_h^0 \oplus \boldsymbol{M}_H. && (2.2d)
\end{aligned}
$$

Here, the spaces $\boldsymbol{M}_h^0$ and $\boldsymbol{M}_H$ are defined as

$$\boldsymbol{M}_h := \{\boldsymbol{\mu} \in \boldsymbol{L}^2(\mathcal{E}_h): \quad \boldsymbol{\mu}|_F \in \boldsymbol{M}_h(F_h) \quad \forall\, F \in \mathcal{E}_h\} \tag{2.2e}$$
$$\boldsymbol{M}_h^0 := \{\boldsymbol{\mu} \in \boldsymbol{M}_h: \quad \boldsymbol{\mu}|_F = 0 \quad \forall\, F \in \mathcal{E}_H\} \tag{2.2f}$$
$$\boldsymbol{M}_H := \{\boldsymbol{\mu} \in \boldsymbol{L}^2(\mathcal{E}_H): \quad \boldsymbol{\mu}|_F \in \boldsymbol{M}_H(F) \quad \forall\, F \in \mathcal{E}_H\}. \tag{2.2g}$$

To describe how the approximation is defined, we need to introduce some notation related to integrals on the triangulation $\mathcal{T}_h$. Here, we write $(\mathbb{N}\,,\,\mathbb{Z})_{\mathcal{T}_h} := \sum_{i,j=1}^n(\mathbb{N}_{i,j}\,,\,\mathbb{Z}_{i,j})_{\mathcal{T}_h}$, $(\boldsymbol{\eta}\,,\,\boldsymbol{\zeta})_{\mathcal{T}_h} := \sum_{i=1}^n(\eta_i\,,\,\zeta_i)_{\mathcal{T}_h}$, and $(\eta\,,\,\zeta)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}(\eta,\zeta)_K$, where $(\eta,\zeta)_D$ denotes the integral of $\eta\zeta$ over $D \subset \mathbb{R}^n$. Similarly, we write $\langle\boldsymbol{\eta}\,,\,\boldsymbol{\zeta}\rangle_{\partial\mathcal{T}_h} := \sum_{i=1}^n\langle\eta_i\,,\,\zeta_i\rangle_{\partial\mathcal{T}_h}$ and $\langle\eta\,,\,\zeta\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\langle\eta\,,\,\zeta\rangle_{\partial K}$, where $\langle\eta\,,\,\zeta\rangle_D$ denotes the integral of $\eta\zeta$ over $D \subset \mathbb{R}^{n-1}$. On the coarse mesh $\mathcal{T}_H$, we also define $\langle\boldsymbol{\eta}\,,\,\boldsymbol{\zeta}\rangle_{\partial\mathcal{T}_H} := \sum_{i=1}^n\langle\eta_i\,,\,\zeta_i\rangle_{\partial\mathcal{T}_H}$ and $\langle\eta\,,\,\zeta\rangle_{\partial\mathcal{T}_H} := \sum_{K\in\mathcal{T}_H}\langle\eta\,,\,\zeta\rangle_{\partial K}$. Similarly, we can define the integral over $\mathcal{T}_h(T)$ and $\partial\mathcal{T}_h(T)$. Furthermore, we write $A \lesssim B$ if $A \le CB$ for some constant C independent of h, H and κ^{-1}.

The Galerkin approximation solution $(\mathbb{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_{h,H}) \in \mathbb{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_{h,H}$ to (2.1) can now be formulated as following

$$(\mathbb{L}_h\,,\,\mathbb{G})_{\mathcal{T}_h} + (\boldsymbol{u}_h\,,\,\nabla\cdot\mathbb{G})_{\mathcal{T}_h} - \langle\widehat{\boldsymbol{u}}_{h,H}\,,\,\mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0 \tag{2.3a}$$
$$(\nu\mathbb{L}_h\,,\,\nabla\boldsymbol{v})_{\mathcal{T}_h} - (p_h\,,\,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{u}_h\,,\,\boldsymbol{v})_{\mathcal{T}_h} - \langle\widehat{\mathbb{S}}_h\boldsymbol{n}\,,\,\boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}\,,\,\boldsymbol{v})_{\mathcal{T}_h} \tag{2.3b}$$
$$-(\boldsymbol{u}_h\,,\,\nabla q)_{\mathcal{T}_h} + \langle\widehat{\boldsymbol{u}}_{h,H}\cdot\boldsymbol{n}\,,\,q\rangle_{\partial\mathcal{T}_h} = 0 \tag{2.3c}$$
$$\langle\widehat{\mathbb{S}}_h\boldsymbol{n}\,,\,\boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0 \tag{2.3d}$$
$$(p_h\,,\,1)_{\mathcal{T}_h} = 0 \tag{2.3e}$$

for all $(\mathbb{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathbb{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_{h,H}$ where $\widehat{\mathbb{S}}_h$ denotes the *numerical flux* and

$$\widehat{\mathbb{S}}_h\boldsymbol{n} := \mathbb{S}_h\boldsymbol{n} - \tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_{h,H}) := \nu\mathbb{L}_h\boldsymbol{n} - p_h\boldsymbol{n} - \tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_{h,H}) \text{ on } \mathcal{E}_h. \tag{2.3f}$$

The flux is $\mathbb{S} := \nu\mathbb{L} - \mathbb{I}\,p$ with $\mathbb{I}$ being the identity matrix in $\mathbb{R}^n$. To complete the system, the boundary condition is imposed by

$$\widehat{\boldsymbol{u}}_{h,H}|_{\partial\Omega} = \boldsymbol{P}_H^{\partial}\boldsymbol{g}. \tag{2.3g}$$

Here, $\boldsymbol{P}_H^{\partial}$ is the L^2-projection from $\boldsymbol{L}^2(\mathcal{E}_H)$ onto $\boldsymbol{M}_H$.

To ensure well-posedness of the problem (2.3), we make the following assumption for the parameter τ:

Assumption 2.1.

- *On each fine element $K \in \mathcal{T}_h$ and every face $F \in \partial K \cup \mathcal{E}_h$, we assume that $\tau|_F \ge 0$. Furthermore, there exists at least one face $F^* \in \partial K$ such that $\tau|_{F^*} > 0$.*
- *Any element $K \in \mathcal{T}_h$ admits at most one face shared with the coarse skeleton $\mathcal{E}_H$. Then $F^* := \partial K \cap \mathcal{E}_H \neq \emptyset$, with $\tau|_{F^*} > 0$.*

Note that, by taking particular choices of the local spaces $\mathbb{G}(K)$, $\boldsymbol{V}(K)$, $P(K)$ and $\boldsymbol{M}_{h,H}(F)$, together with the *linear local stabilization operator* τ, different mixed and **HDG** methods are obtained.

3 Solvability of the method

This section is devoted to the well-posedness of the proposed upscaling method (2.3). Let the set of local spaces $\mathbb{G}(K) \times \boldsymbol{V}(K) \times P(K) \times \boldsymbol{M}_h(F)$ be any admissible local spaces presented in [10]. On each fine element K, there exists a projection $\Pi_h(\mathbb{L}, \boldsymbol{u}, p) = (\boldsymbol{\Pi}_G\mathbb{L}, \boldsymbol{\Pi}_v\boldsymbol{u}, \mathit{\Pi}_q p)$ fulfilling the following conditions:

$$(\boldsymbol{\Pi}_G\mathbb{L}, \mathbb{G})_K = (\mathbb{L}, \mathbb{G})_K \qquad \text{for all } \mathbb{G} \in \widetilde{\mathbb{G}}(K) \subset \mathbb{G}(K) \tag{3.1a}$$
$$(\boldsymbol{\Pi}_v\boldsymbol{u}, \boldsymbol{v})_K = (\boldsymbol{u}, \boldsymbol{v})_K \qquad \text{for all } \boldsymbol{v} \in \widetilde{\boldsymbol{V}}(K) \subset \boldsymbol{V}(K) \tag{3.1b}$$
$$(\mathit{\Pi}_q p, q)_K = (p, q)_K \qquad \text{for all } q \in P(K). \tag{3.1c}$$

For each $F \in \partial K$, there holds

$$\begin{aligned} \langle \nu \boldsymbol{\Pi}_G \mathbb{L}\boldsymbol{n} - \mathit{\Pi}_q p\boldsymbol{n} - \tau(\boldsymbol{\Pi}_v \boldsymbol{u}) \, , \, \boldsymbol{\mu} \rangle_F \\ = \langle \nu \mathbb{L}\boldsymbol{n} - p\boldsymbol{n} - \tau \boldsymbol{u} \, , \, \boldsymbol{\mu} \rangle_F \text{ for all } \boldsymbol{\mu} \in \boldsymbol{M}_h(F). \end{aligned} \tag{3.1d}$$

Furthermore, the local spaces satisfy

$$\nabla \cdot \mathbb{G}(K) \cup \nabla P(K) \subset \boldsymbol{V}(K) \qquad \nabla \boldsymbol{V}(K) \subset \mathbb{G}(K) \qquad \mathbb{I}P(K) \subset \mathbb{G}(K) \tag{3.1e}$$

$$\mathbb{G}(K)\boldsymbol{n}|_F \subset \boldsymbol{M}_h(F) \quad \boldsymbol{V}(K)|_F \subset \boldsymbol{M}_h(F) \; P(K)\boldsymbol{n}|_F \subset \boldsymbol{M}_h(F). \tag{3.1f}$$

Here, $\widetilde{\mathbb{G}}(K)$ and $\widetilde{\boldsymbol{V}}(K)$ are two auxiliary spaces to make the projection well-defined.

In addition, we define $\boldsymbol{P}_h^{\partial}$ as the L^2-projection from $\boldsymbol{L}^2(\mathcal{E}_h)$ onto $\boldsymbol{M}_h$. $\boldsymbol{P}_M$ denotes the L^2-projection onto the space $\boldsymbol{M}_{h,H}$.

Finally, we have the following lifting result on the local spaces.

Lemma 3.1. *On each $K \in \mathcal{T}_h$, for any $\boldsymbol{\mu} \in \boldsymbol{M}_h(F), F \in \partial K$, if $\boldsymbol{\mu}|_{F^*} = 0$, then there exists a function $\mathbb{G} \in \mathbb{G}(K)$ such that*

$$\begin{aligned} (\mathbb{G}, \nabla \boldsymbol{v})_K = 0 \qquad & \textit{for all } \boldsymbol{v} \in \boldsymbol{V}(K) \\ \mathbb{G}\boldsymbol{n}|_F = \boldsymbol{\mu}|_F \quad & \textit{for all } F \in \partial K \backslash F^*. \end{aligned}$$

This result was originally proved in [5] for a special case where K is simplex and all the local spaces consist of polynomials of degree no more than k. We omit the proof in our general case since it is highly close to the proof in [5].

We are now ready to prove the stability of the method.

Theorem 3.1. *If the Assumption 2.1 is valid, then the system* (2.3) *admits the unique solution.*

Proof. In view that the system (2.3) is square, It suffices to show that only zero solution exists given $\boldsymbol{f} = 0$ and $\boldsymbol{g} = 0$.

Let $(\mathbb{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) := (\mathbb{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_{h,H})$ in (2.3a)-(2.3d). By adding them up, we deduce

$$(\mathbb{L}_h \, , \, \mathbb{L}_h)_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{u}_h \, , \, \boldsymbol{u}_h)_{\mathcal{T}_h} + \langle \widehat{\mathbb{S}}_h \boldsymbol{n} - \mathbb{S}_h \boldsymbol{n} \, , \, \widehat{\boldsymbol{u}}_{h,H} - \boldsymbol{u}_h \rangle_{\partial \mathcal{T}_h} = 0.$$

Owing to the definition of the numerical traces (2.3f),

$$(\mathbb{L}_h \, , \, \mathbb{L}_h)_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{u}_h \, , \, \boldsymbol{u}_h)_{\mathcal{T}_h} + \langle \tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_{h,H}) \, , \, \boldsymbol{u}_h - \widehat{\boldsymbol{u}}_{h,H} \rangle_{\partial \mathcal{T}_h} = 0.$$

This implies that

$$\mathbb{L}_h = 0, \quad \boldsymbol{u}_h = 0 \text{ and } \tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_{h,H}) = 0, \tag{3.2}$$

whereas a combination of Assumption 2.1 reveals

$$\widehat{\boldsymbol{u}}_{h,H} = 0 \text{ on } \mathcal{E}_H.$$

Consequently, (2.3b) becomes

$$\forall v \in \boldsymbol{V}_h : \quad -(p_h \, , \, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle p_h \boldsymbol{n} \, , \, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} = 0.$$

An application of integration parts leads to

$$\forall v \in \boldsymbol{V}_h : \quad (\nabla p_h \, , \, \boldsymbol{v})_{\mathcal{T}_h} = 0. \tag{3.3}$$

(3.3) implies that p_h is constant on each fine element $T \in \mathcal{T}_h$ and finally (2.3d) and (3.2) show that $p_h = 0$.

To complete, we need to show that $\widehat{\boldsymbol{u}}_{h,H}$ vanishes on $\mathcal{E}_h^0$. Taking into account that all the rest unknowns are zero, (2.3a) becomes

$$\langle \widehat{\boldsymbol{u}}_{h,H} \, , \, \mathbb{G}\boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0, \quad \text{for all } \mathbb{G} \in \mathbb{G}_h.$$

Given $K \in \mathcal{T}_h$ and for all $F \in \partial K$, due to the inclusion property $\mathbb{G}(K)\boldsymbol{n}|_F \subset \boldsymbol{M}_h(F)$, the above equation implies

$$\langle \boldsymbol{P}_h^{\partial} \widehat{\boldsymbol{u}}_{h,H} \,,\, \mathbb{G}\boldsymbol{n} \rangle_{\partial K} = 0, \quad \text{for all } \mathbb{G} \in \mathbb{G}(K). \tag{3.4}$$

By Assumption 2.1, τ is positive on at least one face F^* of K. This, together with the fact that $\tau \widehat{\boldsymbol{u}}_{h,H} = 0$ by (3.2), proves $\widehat{\boldsymbol{u}}_{h,H} = 0$ on F^*. Therefore, $\boldsymbol{P}_h^{\partial} \widehat{\boldsymbol{u}}_{h,H} = 0$ on F^*. Then Lemma 3.1 verifies the existence of $\mathbb{G} \in \mathbb{G}(K)$, which satisfies $\mathbb{G}\boldsymbol{n}|_F = \boldsymbol{P}_h^{\partial} \widehat{\boldsymbol{u}}_{h,H}$ for all $F \in \partial K \backslash F^*$. In combination with (3.4), we can deduce $\boldsymbol{P}_h^{\partial} \widehat{\boldsymbol{u}}_{h,H} = 0$. Hence, $\widehat{\boldsymbol{u}}_{h,H} = 0$ on $\mathcal{E}_h^0$ considering $\boldsymbol{P}_h^{\partial} \widehat{\boldsymbol{u}}_{h,H}|_{\mathcal{E}_h^0} = \widehat{\boldsymbol{u}}_{h,H}$. This ends the proof. $\square$

Remark 3.1. *If the stabilization term disappears, i.e., $\tau \equiv 0$, then the system can be well defined provided that the local spaces $\mathbb{G}(K) \times \boldsymbol{V}(K)$ are H_{div} conforming mixed elements. Apart from it, the following assumption between the fine and coarse spaces is valid:*

$$\forall F \in \mathcal{E}_H \text{ and } \boldsymbol{\mu} \in \boldsymbol{M}_H(F): \quad \|\boldsymbol{\mu}\|_F \lesssim \|\boldsymbol{P}_h^{\partial} \boldsymbol{\mu}\|_F.$$

This yields the mortar *spaces for Brinkman equation, cf. [1].*

4 Upscaling structure of the method

In this section, we present the upscaling structure of (2.3), which is a key to allow reducing the global computational cost. We follow the idea of hybridization technique for Stokes equation in [11]. To this end, we introduce an additional Lagrange multiplier $\bar{p}_H \in W_H$ with

$$W_H := \{q \in L_0^2(\Omega) : q|_T \in P_0(T), \text{ for all } T \in \mathcal{T}_H\}.$$

The new unknown $\bar{p}_H$ approximates the average of the pressure over each coarse element.

Given $(\widehat{\boldsymbol{u}}_H, \bar{p}_H) \in \boldsymbol{M}_H \times W_H$, we can obtain $(\mathbb{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_{h,H}) \in \mathbb{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_{h,H}$ coarse element-wise by solving the following local linear system

$$(\mathbb{L}_h \,,\, \mathbb{G})_{\mathcal{T}_h(T)} + (\boldsymbol{u}_h \,,\, \nabla \cdot \mathbb{G})_{\mathcal{T}_h(T)} - \langle \widehat{\boldsymbol{u}}_{h,H} \,,\, \mathbb{G}\boldsymbol{n} \rangle_{\partial \mathcal{T}_h(T)} = 0 \tag{4.1a}$$

$$(\mathbb{L}_h \,,\, \nabla \boldsymbol{v})_{\mathcal{T}_h(T)} - (p_h \,,\, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h(T)} + (\kappa^{-1} \boldsymbol{u}_h \,,\, \boldsymbol{v})_{\mathcal{T}_h(T)} - \langle \widehat{\mathbb{S}}_h \boldsymbol{n} \,,\, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h(T)}$$

$$(\boldsymbol{f} \,,\, \boldsymbol{v})_{\mathcal{T}_h(T)} \tag{4.1b}$$

$$-(\boldsymbol{u}_h \,,\, \nabla q)_{\mathcal{T}_h(T)} + \langle \widehat{\boldsymbol{u}}_{h,H} \cdot \boldsymbol{n} \,,\, q - \bar{q} \rangle_{\partial \mathcal{T}_h(T)} = 0 \tag{4.1c}$$

$$\langle \widehat{\mathbb{S}}_h \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h(T) \backslash T} = 0 \tag{4.1d}$$

$$\bar{p}_h = \bar{p}_H \tag{4.1e}$$

$$\widehat{\boldsymbol{u}}_{h,H}|_{\partial T} = \widehat{\boldsymbol{u}}_H \tag{4.1f}$$

for all $(\mathbb{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{\mu}}) \in \mathbb{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_{h,H}$ and $T \in \mathcal{T}_H$ with $\bar{q} := \frac{1}{|T|}(q, 1)_T$.

The existence and uniqueness of the solution to the above system is proved in [10]. In fact, due to the linearity of the system, we can further split the system as

$$\begin{aligned}(\mathbb{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_{h,H})|_T = {} & (\mathbb{L}_h(\widehat{\boldsymbol{u}}_H), \boldsymbol{u}_h(\widehat{\boldsymbol{u}}_H), p_h(\widehat{\boldsymbol{u}}_H), \widehat{\boldsymbol{u}}_{h,H}(\widehat{\boldsymbol{u}_H})) \\ & + (\mathbb{L}_h(\bar{p}_H), \boldsymbol{u}_h(\bar{p}_H), p_h(\bar{p}_H), \widehat{\boldsymbol{u}}_{h,H}(\bar{p}_H)) + (\mathbb{L}_h(\boldsymbol{f}), \boldsymbol{u}_h(\boldsymbol{f}), p_h(\boldsymbol{f}), \widehat{\boldsymbol{u}}_{h,H}(\boldsymbol{f})).\end{aligned}$$

Here $(\mathbb{L}_h(\widehat{\boldsymbol{u}}_H), \boldsymbol{u}_h(\widehat{\boldsymbol{u}}_H), p_h(\widehat{\boldsymbol{u}}_H), \widehat{\boldsymbol{u}}_{h,H}(\widehat{\boldsymbol{u}}_H))$ is the solution of system (4.1) by setting the data $\bar{p}_H = 0,\ \boldsymbol{f} = \boldsymbol{0}$. The other two are obtained in a similar way. Moreover, it is obvious that

$$(\mathbb{L}_h(\bar{p}_H), \boldsymbol{u}_h(\bar{p}_H), p_h(\bar{p}_H), \widehat{\boldsymbol{u}}_{h,H}(\bar{p}_H)) = (\mathbb{O}, \boldsymbol{0}, \bar{p}_H, \boldsymbol{0}). \tag{4.2}$$

To eliminate the local variables and formulate a linear system over the coarse mesh $\mathcal{T}_H$ for the global variables $(\widehat{\boldsymbol{u}}_H, \bar{p}_H) \in \boldsymbol{M}_H \times W_H$, we utilize the transmission equations and boundary conditions and obtain

$$\langle \widehat{\mathbb{S}}_h \boldsymbol{n} \,,\, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_H \backslash \partial \Omega} = 0 \qquad \text{for all } \boldsymbol{\mu} \in \boldsymbol{M}_H \tag{4.3a}$$

$$\langle \widehat{\boldsymbol{u}}_H \cdot \boldsymbol{n} \,,\, \bar{q} \rangle_{\partial \mathcal{T}_H \backslash \partial \Omega} = 0 \qquad \text{for all } \bar{q} \in W_H \tag{4.3b}$$

$$\widehat{\boldsymbol{u}}_H = P_M \boldsymbol{g} \quad \text{on } \partial \Omega \tag{4.3c}$$

$$(\bar{p}_H, 1)_\Omega = 0. \tag{4.3d}$$

Take into account the equality (4.2), we deduce

$$\forall \boldsymbol{\mu} \in \boldsymbol{M}_H : -\langle \widehat{\mathbb{S}}_h(\bar{p}_H)\boldsymbol{n}\,,\, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_H\setminus\partial\Omega} = \langle \bar{p}_H\boldsymbol{n}\,,\, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_H\setminus\partial\Omega}.$$

Consequently, we can write the upscaled system as a saddle-point problem as: seeking $(\widehat{\boldsymbol{u}}_H, \bar{p}_H) \in \boldsymbol{M}_H \times W_H$ satisfying

$$\begin{aligned} a_H(\widehat{\boldsymbol{u}}_H, \boldsymbol{\mu}) + b_H(\bar{p}_H, \boldsymbol{\mu}) &= \ell_H(\boldsymbol{\mu}) && \text{for all } \boldsymbol{\mu} \in \boldsymbol{M}_H, \\ b_H(\bar{q}, \widehat{\boldsymbol{u}}_H) &= 0 && \text{for all } \bar{q} \in W_H. \end{aligned} \tag{4.4}$$

The associated bilinear forms $a_H(\cdot,\cdot)$ on $\boldsymbol{M}_H \times \boldsymbol{M}_H$, $b_H(\cdot,\cdot)$ on $W_H \times \boldsymbol{M}_H$, and linear form $\ell_H(\cdot)$ on $\boldsymbol{M}_H$ are

$$\begin{aligned} a_H(\boldsymbol{\nu}, \boldsymbol{\mu}) &:= \langle \widehat{\mathbb{S}}_h(\boldsymbol{\nu})\boldsymbol{n}\,,\, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_H\setminus\partial\Omega} \qquad b_H(\bar{q}, \boldsymbol{\mu}) := -\langle \boldsymbol{\mu}\cdot\boldsymbol{n}\,,\, \bar{q}\rangle_{\partial\mathcal{T}_H\setminus\partial\Omega} \\ \ell_H(\boldsymbol{\mu}) &:= -\langle \widehat{\mathbb{S}}_h(\boldsymbol{f})\boldsymbol{n}\,,\, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_H\setminus\partial\Omega}. \end{aligned}$$

The main feature of this method is that we only need to solve the global system (4.4) on the coarse edges $\mathcal{E}_H$ and obtain $(\widehat{\boldsymbol{u}}_H, \bar{p}_H) \in \boldsymbol{M}_H \times W_H$. The remaining unknowns can be revealed by solving the local system (4.1a)-(4.1f) coarse element-wise. Namely, on each element $T \in \mathcal{T}_H$, given the boundary data of $\widehat{\boldsymbol{u}}_H|_{\partial T}$ and average pressure $\bar{p}_H|_T$, we can solve (4.1a)-(4.1f) on T and retrieve the local variables $(\mathbb{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_{h,H})$. In this manner, we avoid solving the global system (2.3a) - (2.3e) via computation over the coarse system (4.4) on $\mathcal{E}_H$.

5 Error estimates

In this section, we present a general error estimate of the HDG method proposed in (2.3). For the sake of simplicity, we assume that the permeability coefficient κ is piecewise constant on each fine element. Also, we only consider 2-dimensional case since there is no essential difference/difficulty to extend our analysis to 3-dimensional case. We set all nonzero stabilization parameter $\tau = 1$. In what follows, we use $\|\cdot\|_{k,D}, |\cdot|_{k,D}$ to denote the standard norm and seminorm on any Sobolev space $H^k(D)$, respectively. When $k = 0$, we omit the index k and simply write $\|\cdot\|_D$. For the sake of simplicity, we assume the fine scale triangulation $\mathcal{T}_h$ is made of simplexes. In addition, we specify the local spaces as follows:

Assumption 5.1. • *Let $k \in \mathbb{N}$. The local approximation spaces are*

$$\begin{aligned} \mathbb{G}(K) &:= [P_k(K)]^{n\times n} \quad \boldsymbol{V}(K) := [P_k(K)]^n \quad P(K) := P_k(K), \\ \boldsymbol{M}_h(F) &:= [P_k(F)]^n \quad \widetilde{\mathbb{G}}(K) := [P_{k-1}(K)]^{n\times n} \quad \widetilde{\boldsymbol{V}}(K) := [P_{k-1}(K)]^n. \end{aligned}$$

- *The coarse space $\boldsymbol{M}_H$ contains the piecewise constant vectors. i.e.,*

$$\forall\, F \in \mathcal{E}_H : \boldsymbol{P}_0(F) \subset \boldsymbol{M}_H(F).$$

We also introduce a couple of notations for the analysis:

$$\begin{aligned} &\mathbb{E}_{\mathbb{L}} := \boldsymbol{\Pi}_G\mathbb{L} - \mathbb{L}_h \qquad \boldsymbol{e}_u := \boldsymbol{\Pi}_v\boldsymbol{u} - \boldsymbol{u}_h \qquad e_p := \mathit{\Pi}_q p - p_h \\ &\widehat{\boldsymbol{e}_u} := \boldsymbol{P}_M\boldsymbol{u} - \widehat{\boldsymbol{u}}_{h,H} \quad \widehat{\mathbb{E}_{\mathbb{S}}\boldsymbol{n}} := \mathbb{S}\boldsymbol{n} - \widehat{\mathbb{S}}_h\boldsymbol{n} = \nu\mathbb{L}\boldsymbol{n} - p\boldsymbol{n} - \widehat{\mathbb{S}}_h\boldsymbol{n}. \end{aligned}$$

Notice that by (3.1d) and the definition of $\boldsymbol{P}_h^{\partial}$ we have

$$\nu\boldsymbol{\Pi}_G\mathbb{L}\boldsymbol{n} - \mathit{\Pi}_q p\boldsymbol{n} - \tau\boldsymbol{\Pi}_v\boldsymbol{u} = \boldsymbol{P}_h^{\partial}(\nu\mathbb{L}\boldsymbol{n} - p\boldsymbol{n} - \tau\boldsymbol{u}) = \boldsymbol{P}_h^{\partial}(\mathbb{S}\boldsymbol{n} - \tau\boldsymbol{u}). \tag{5.1}$$

Let

$$\delta_{\mathbb{L}} := \mathbb{L} - \boldsymbol{\Pi}_G\mathbb{L} \quad \boldsymbol{\delta}_u := \boldsymbol{u} - \boldsymbol{\Pi}_v\boldsymbol{u} \quad \delta_p := p - \mathit{\Pi}_q p.$$

The projection errors $\delta_{\mathbb{L}}$, $\boldsymbol{\delta}_u$ and δ_p follow from the standard approximation properties of the projections, c.f. [10]. Furthermore, the approximation property of L^2-projection yields that for $\boldsymbol{P}_h^{\partial}$. Assume that $\mathbb{L}$, $\boldsymbol{u}$ and p have piecewise regularity up to $s \geq 1$ on $\mathcal{T}_h$ and $k = \lceil s\rceil - 1$. Then it holds

$$\begin{aligned} &\|\boldsymbol{\delta}_u\|_K \lesssim h^s(\|\boldsymbol{u}\|_{s,K} + \|\mathbb{L}\|_{s,K}) && \|\delta_{\mathbb{L}}\|_K \lesssim h^s(\|\boldsymbol{u}\|_{s,K} + \|\mathbb{L}\|_{s,K}) \\ &\|\delta_p\|_K \lesssim h^s\|p\|_{s,K} && \|\boldsymbol{u} - \boldsymbol{P}_h^{\partial}\boldsymbol{u}\|_F \lesssim h^{s-\frac{1}{2}}\|\boldsymbol{u}\|_{s,K} \\ &\|(\boldsymbol{I} - \boldsymbol{P}_h^{\partial})(\mathbb{L}\boldsymbol{n} - p\boldsymbol{n})\|_F \lesssim h^{s-\frac{1}{2}}(\|\mathbb{L}\|_{s,K} + \|p\|_{s,K}) \end{aligned} \tag{5.2}$$

for all $K \in \mathcal{T}_h$ and $F \in \partial K$. We are ready to present our main error estimate results:

Theorem 5.1 (Estimate for $\mathbb{E}_{\mathbb{L}}$ and $\boldsymbol{e}_u$)**.** *Assume that the exact solutions $\mathbb{L}$, $\boldsymbol{u}$ and p have piecewise regularity up to $s \geq 1$ on $\mathcal{T}_h$. Let the nonzero stabilization parameter $\tau = 1$ and $k = \lceil s \rceil - 1$, then there holds*

$$\begin{aligned}
&\|\mathrm{L} - \mathrm{L}_h\|_{\mathcal{T}_h} + \|\kappa^{-\frac{1}{2}}(\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathcal{T}_h} \\
&\lesssim (\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}} + h^{-\frac{1}{2}})\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}} h^s(\|\boldsymbol{u}\|_{s,\mathcal{T}_h} + \|\mathbb{L}\|_{s,\mathcal{T}_h}) \\
&+ h^{-\frac{1}{2}}(1 + H^{\frac{1}{2}}\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}})\|\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\|_{\partial\mathcal{T}_H}.
\end{aligned}$$

As for the pressure, the estimates depend on additional assumptions as:

1. *if Assumption 5.2 holds. We have*

$$\begin{aligned}
\|p - p_h\|_{\mathcal{T}_h} &\lesssim (1 + \|\kappa^{-1}\|_{\Omega})(h^{-\frac{1}{2}} + \|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}})\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}} h^s(\|\mathbb{L}\|_{s,\mathcal{T}_h} + \|\boldsymbol{u}\|_{s,\mathcal{T}_h}) \\
&+ (1 + \|\kappa^{-1}\|_{\Omega})(1 + H^{\frac{1}{2}}\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}})h^{-\frac{1}{2}}\|\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\|_{\partial\mathcal{T}_H}.
\end{aligned}$$

2. *If Assumption 5.3 holds, we can obtain*

$$\begin{aligned}
\|p - p_h\|_{\mathcal{T}_h} &\lesssim \Big(\sqrt{\frac{H}{h}}(1 + H^{\frac{1}{2}}\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}}) + \|\kappa^{-1}\|_{\Omega}\Big)(h^{-\frac{1}{2}} + \|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}})\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}} h^s \\
&\times (\|\mathbb{L}\|_{s,\mathcal{T}_h} + \|\boldsymbol{u}\|_{s,\mathcal{T}_h}) + H^{\frac{1}{2}} h^{s-\frac{1}{2}}\|p\|_{s,\mathcal{T}_h} \\
&+ (\|\kappa^{-1}\|_{\Omega} + \sqrt{\frac{H}{h}})(1 + H^{\frac{1}{2}}\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}})h^{-1}\|\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\|_{\partial\mathcal{T}_H}.
\end{aligned}$$

To estimate the errors between the exact solution from (2.1) and the numerical approximation by (2.3), an application of the projection properties (5.2) and the triangle inequality indicates that the estimates of $\mathbb{E}_{\mathbb{L}}$, $\boldsymbol{e}_u$ and e_p are sufficient. This is the aim of the following sections.

5.1 Some preliminary identities

First we establish the error equation for (2.3) that relates $\mathbb{E}_{\mathbb{L}}$, $\boldsymbol{e}_u$, $\widehat{\boldsymbol{e}_u}$ and e_p to the known projection errors $\delta_{\mathbb{L}}$ and $\boldsymbol{\delta}_u$.

Lemma 5.1. *The following identies are valid*

$$\begin{aligned}
(\mathbb{E}_{\mathbb{L}}, \mathbb{G})_{\mathcal{T}_h} + (\boldsymbol{e}_u, \nabla\cdot\mathbb{G})_{\mathcal{T}_h} - \langle\widehat{\boldsymbol{e}_u}, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= -(\delta_{\mathbb{L}}, \mathbb{G})_{\mathcal{T}_h} \\
&\quad + \langle\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_H} && (5.3a)\\
(\nu\mathbb{E}_{\mathbb{L}} - \mathbb{I}e_p, \nabla\boldsymbol{v})_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{e}_u, \boldsymbol{v})_{\mathcal{T}_h} - \langle\widehat{\mathbb{E}_{\mathbb{S}}\boldsymbol{n}}, \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} &= -(\kappa^{-1}\boldsymbol{\delta}_u, \boldsymbol{v})_{\mathcal{T}_h} && (5.3b)\\
-(\boldsymbol{e}_u, \nabla q)_{\mathcal{T}_h} + \langle\widehat{\boldsymbol{e}_u}, q\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= -\langle\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}, q\boldsymbol{n}\rangle_{\partial\mathcal{T}_H} && (5.3c)\\
\langle\widehat{\mathbb{E}_{\mathbb{S}}\boldsymbol{n}}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} &= 0 && (5.3d)\\
\langle\widehat{\boldsymbol{e}_u}, \boldsymbol{\mu}\rangle_{\partial\Omega} &= 0 && (5.3e)\\
(e_p, 1)_{\mathcal{T}_h} &= 0 && (5.3f)
\end{aligned}$$

for all $(\mathbb{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathbb{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_{h,H}$. In addition, we have

$$\widehat{\mathbb{E}_{\mathbb{S}}\boldsymbol{n}} = \nu\mathbb{E}_{\mathbb{L}}\boldsymbol{n} - e_p\boldsymbol{n} - \tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}) + (\boldsymbol{I} - \boldsymbol{P}_h^{\partial})(\nu\mathbb{L}\boldsymbol{n} - p\boldsymbol{n}) + \tau(\boldsymbol{P}_h^{\partial}\boldsymbol{u} - \boldsymbol{P}_M\boldsymbol{u}). \tag{5.4}$$

Proof. Taking into account that the exact solution satisfies the same equations (2.3) combined with the property of the projections (3.1), we can equip the exact solution with the projection and obtain

$$\begin{aligned}
(\mathbb{E}_{\mathbb{L}}, \mathbb{G})_{\mathcal{T}_h} + (\boldsymbol{e}_u, \nabla\cdot\mathbb{G})_{\mathcal{T}_h} - \langle\boldsymbol{u} - \widehat{\boldsymbol{u}}_{h,H}, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= -(\delta_{\mathbb{L}}, \mathbb{G})_{\mathcal{T}_h} && (5.5a)\\
(\nu\mathbb{E}_{\mathbb{L}} - \mathbb{I}e_p, \nabla\boldsymbol{v})_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{e}_u, \boldsymbol{v})_{\mathcal{T}_h} - \langle\widehat{\mathbb{E}_{\mathbb{S}}\boldsymbol{n}}, \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} &= -(\kappa^{-1}\boldsymbol{\delta}_u, \boldsymbol{v})_{\mathcal{T}_h} && (5.5b)\\
-(\boldsymbol{e}_u, \nabla q)_{\mathcal{T}_h} + \langle\boldsymbol{u} - \widehat{\boldsymbol{u}}_{h,H}, q\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0 && (5.5c)\\
\langle\widehat{\mathbb{E}_{\mathbb{S}}\boldsymbol{n}}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} &= 0\\
\langle\widehat{\boldsymbol{e}_u}, \boldsymbol{\mu}\rangle_{\partial\Omega} &= 0\\
(e_p, 1)_{\mathcal{T}_h} &= 0
\end{aligned}$$

for all $(\mathbb{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathbb{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_{h,H}$.

The combination of $\boldsymbol{P}_M|_{\mathcal{E}_h^0} = \boldsymbol{P}_h^\partial|_{\mathcal{E}_h^0}$, (3.1f) and $\boldsymbol{P}_M = \boldsymbol{P}_H^\partial$ on $\mathcal{E}_H$, yields

$$\begin{aligned}\langle \boldsymbol{u} - \widehat{\boldsymbol{u}}_{h,H}\,,\, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \langle \widehat{\boldsymbol{e}_u}\,,\, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{u} - \boldsymbol{P}_M\boldsymbol{u}\,,\, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\\ &= \langle \widehat{\boldsymbol{e}_u}\,,\, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{u} - \boldsymbol{P}_M\boldsymbol{u}\,,\, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_H}.\end{aligned}$$

Consequently,

$$\langle \boldsymbol{u} - \widehat{\boldsymbol{u}}_{h,H}\,,\, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = \langle \widehat{\boldsymbol{e}_u}\,,\, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\,,\, \mathbb{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_H}.$$

This reveals (5.3a) after inserting the above equation into (5.5a). In a similar manner, we can show that

$$\langle \boldsymbol{u} - \widehat{\boldsymbol{u}}_{h,H}\,,\, q\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = \langle \widehat{\boldsymbol{e}_u}\,,\, q\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\,,\, q\boldsymbol{n}\rangle_{\partial\mathcal{T}_H}.$$

Together with (5.5c), (5.3c) is derived.

To prove (5.4), by utilizing the identity (5.1) and the definition of the numerical trace (2.3f), we derive that

$$\begin{aligned}\widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n} = \mathbb{S}\boldsymbol{n} - \widehat{\mathbb{S}}_h\boldsymbol{n} &= \nu\mathbb{L}\boldsymbol{n} - p\boldsymbol{n} - (\nu\mathbb{L}_h\boldsymbol{n} - p_h\boldsymbol{n} - \tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_{h,H}))\\ &= \nu\mathbb{E}_{\mathbb{L}}\boldsymbol{n} - e_p\boldsymbol{n} + \tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_{h,H}) + (\boldsymbol{I} - \boldsymbol{P}_h^\partial)(\nu\mathbb{L}\boldsymbol{n} - p\boldsymbol{n})\\ &+ \boldsymbol{P}_h^\partial(\nu\mathbb{L}\boldsymbol{n} - p\boldsymbol{n}) - (\nu\boldsymbol{\Pi}_G\mathbb{L}\boldsymbol{n} - \mathit{\Pi}_q p\boldsymbol{n})\\ &= \nu\mathbb{E}_{\mathbb{L}}\boldsymbol{n} - e_p\boldsymbol{n} + \tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_{h,H}) + (\boldsymbol{I} - \boldsymbol{P}_h^\partial)(\nu\mathbb{L}\boldsymbol{n} - p\boldsymbol{n})\\ &+ \tau(\boldsymbol{P}_h^\partial\boldsymbol{u} - \boldsymbol{\Pi}_v\boldsymbol{u}).\end{aligned}$$

Finally, the proof is completed by invoking the definition of $\boldsymbol{e}_u$ and $\widehat{\boldsymbol{e}_u}$. □

To arrive at the error estimate, we next provide an auxiliary equality.

Lemma 5.2. *We have*

$$\begin{aligned}&(\nu\mathbb{E}_{\mathbb{L}}\,, \mathbb{E}_{\mathbb{L}})_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{e}_u\,,\, \boldsymbol{e}_u)_{\mathcal{T}_h} + \langle \tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\,,\, \boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h}\\ &\quad = -(\nu\delta_{\mathbb{L}}\,,\, \mathbb{E}_{\mathbb{L}})_{\mathcal{T}_h} - (\kappa^{-1}\boldsymbol{\delta}_u\,,\, \boldsymbol{e}_u)_{\mathcal{T}_h} + \langle \boldsymbol{P}_h^\partial\boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\,,\, \tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\rangle_{\partial\mathcal{T}_H}\\ &\quad + \langle \boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\,,\, \mathbb{E}_{\mathbb{L}}\boldsymbol{n} - e_p\boldsymbol{n}\rangle_{\partial\mathcal{T}_H}.\end{aligned}$$

Proof. The combination of (5.3d) and (5.3e) implies

$$\langle \widehat{\boldsymbol{e}_u}\,,\, \widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0.$$

Adding up (5.3a) - (5.3c) by taking $(\mathbb{G}, \boldsymbol{v}, q) := (\mathbb{E}_{\mathbb{L}}, \boldsymbol{e}_u, e_p)$, together with the above equation and some algebraic manipulation, leads to

$$\begin{aligned}&(\nu\mathbb{E}_{\mathbb{L}}\,,\, \mathbb{E}_{\mathbb{L}})_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{e}_u\,,\, \boldsymbol{e}_u)_{\mathcal{T}_h} + \langle \mathbb{E}_{\mathbb{L}}\boldsymbol{n} - e_p\boldsymbol{n} - \widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n}\,,\, \boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h}\\ &\quad = -(\delta_{\mathbb{L}}\,,\, \mathbb{E}_{\mathbb{L}})_{\mathcal{T}_h} - (\kappa^{-1}\boldsymbol{e}_u\,,\, \boldsymbol{e}_u)_{\mathcal{T}_h} + \langle \boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\,,\, \nu\mathbb{E}_{\mathbb{L}}\boldsymbol{n} - e_p\boldsymbol{n}\rangle_{\partial\mathcal{T}_H}.\end{aligned}$$

By (5.4), the last term on the left hand side can be manipulated as follows:

$$\begin{aligned}&\langle \mathbb{E}_{\mathbb{L}}\boldsymbol{n} - e_p\boldsymbol{n} - \widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n}\,,\, \boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h}\\ &\quad = \langle \tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\,,\, \boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h} - \langle \tau(\boldsymbol{P}_h^\partial\boldsymbol{u} - \boldsymbol{P}_M\boldsymbol{u})\,,\, \boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h}\\ &\quad - \langle (\boldsymbol{I} - \boldsymbol{P}_h^\partial)\mathbb{S}\boldsymbol{n}\,,\, \boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

The last term vanishes. Indeed,

$$\langle (\boldsymbol{I} - \boldsymbol{P}_h^\partial)\mathbb{S}\boldsymbol{n}\,,\, \boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h} = \langle (\boldsymbol{I} - \boldsymbol{P}_h^\partial)\mathbb{S}\boldsymbol{n}\,,\, \boldsymbol{e}_u\rangle_{\partial\mathcal{T}_h} - \langle (\boldsymbol{I} - \boldsymbol{P}_h^\partial)\mathbb{S}\boldsymbol{n}\,,\, \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h}.$$

Consequently, the first term of the above expression vanishes due to the fact that $\boldsymbol{e}_{\boldsymbol{u}}|_{\partial\mathcal{T}_h} \in \boldsymbol{M}_h$ and the second term vanishes because that $\widehat{\boldsymbol{e}_u}$ is single valued on $\mathcal{E}_h$ and $\mathbb{S}\boldsymbol{n} = \nu\mathbb{L}\boldsymbol{n} - p\boldsymbol{n}$ is continuous across all the interfaces, so that

$$\langle (\boldsymbol{I} - \boldsymbol{P}_h^\partial)\mathbb{S}\boldsymbol{n}\,,\, \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h} = \langle\, (\boldsymbol{I} - \boldsymbol{P}_h^\partial)\mathbb{S}\boldsymbol{n}\,,\, \widehat{\boldsymbol{e}_u}\rangle_{\partial\Omega} = 0.$$

Finally, we complete the proof by utilizing $\boldsymbol{P}_M = \boldsymbol{P}_h^\partial$ on $\mathcal{E}_h^0$. □

5.2 Estimate for e_p

In this section we present an intermediate error estimate for e_p in terms of other errors. To this end, we first apply a local estimate on each subdomain $T \in \mathcal{T}_H$ for $e_p - (e_p)_T$, where $(e_p)_T$ denotes the average of e_p on T. Let (e_p) be the average function of e_p over the coarse mesh $\mathcal{T}_H$, i.e., $(e_p)|_T = (e_p)_T$.

By virtue of [31, Corollary I.2.4], there exists $\boldsymbol{w} \in \boldsymbol{H}_0^1(T)$ s.t.

$$\nabla \cdot \boldsymbol{w} = e_p - (e_p)_T \text{ in } T \qquad \text{and} \qquad \|\nabla \boldsymbol{w}\|_T \leq C_T^{-1} \|e_p - (e_p)_T\|_T \tag{5.6}$$

An immediate outcome of (5.6) is

$$\sup_{\boldsymbol{w} \in \boldsymbol{H}_0^1(T) \setminus \{\boldsymbol{0}\}} \frac{(e_p - (e_p)_T \, , \, \nabla \cdot \boldsymbol{w})_T}{\|\nabla \boldsymbol{w}\|_T} \geq C_T \|e_p - (e_p)_T\|_T. \tag{5.7}$$

Now our next step involves a bound for the left hand side of this inequality. To this end, we introduce a special projection operator $\boldsymbol{P}_\tau : \ \boldsymbol{H}^1(K) \to \boldsymbol{V}(K)$ defined as the unique polynomial, satisfies

$$(\boldsymbol{P}_\tau \boldsymbol{w} \, , \, \boldsymbol{v})_K = (\boldsymbol{w} \, , \, \boldsymbol{v})_K \quad \text{for all } \boldsymbol{v} \in \boldsymbol{P}_{k-1}(K), \tag{5.8a}$$

$$\langle \boldsymbol{P}_\tau \boldsymbol{w} \, , \, \boldsymbol{\mu} \rangle_{F^*} = \langle \boldsymbol{w} \, , \, \boldsymbol{\mu} \rangle_{F^*} \quad \text{for all } \boldsymbol{\mu} \in \boldsymbol{P}_k(F^*). \tag{5.8b}$$

Here F^* denotes the face defined in Assumption 2.1 where the parameter $\tau|_{F^*} > 0$.

The approximation property of $\boldsymbol{P}_\tau$ plays an important role in our further error analysis, which is asserted below.

Lemma 5.3 (Approximation properties of $\boldsymbol{P}_\tau$)**.** *For each $K \in \mathcal{T}_h$, we have*

$$\begin{aligned} \|\boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w}\|_K &\lesssim h \|\nabla \boldsymbol{w}\|_K \\ \|\boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w}\|_{\partial K} &\lesssim h^{\frac{1}{2}} \|\nabla \boldsymbol{w}\|_{\omega_K}. \end{aligned}$$

with $\omega_K := \cup\{\tilde{K} \in \mathcal{T}_h : \tilde{K} \cap K \neq \emptyset\}$.

Proof. The first inequality was proved in [5] Proposition 2.1 (vii). For the second inequality, let $\mathcal{I}\boldsymbol{w}$ denote the Clément interpolant of $\boldsymbol{w}$ (component-by-component) on K, we have

$$\begin{aligned} \|\boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w}\|_{\partial K} &\leq \|\boldsymbol{w} - \mathcal{I}\boldsymbol{w}\|_{\partial K} + \|\mathcal{I}\boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w}\|_{\partial K} \\ &\leq C_1 h^{\frac{1}{2}} \|\nabla \boldsymbol{w}\|_{\omega_K} + C_2 h^{-\frac{1}{2}} \|\mathcal{I}\boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w}\|_K \\ &\lesssim h^{\frac{1}{2}} \|\nabla \boldsymbol{w}\|_{\omega_K}. \end{aligned}$$

Here, C_1 and C_2 are constant depending on the regularity parameter of $\mathcal{T}_h$ only. In the last two steps we applied approximation property of Clément interpolation, inverse inequality, the triangle inequality and the first inequality in the lemma. ☐

The following lemma provides a crucial identity to bound the error of the pressure.

Lemma 5.4. *Let $T \in \mathcal{T}_H$ and $\boldsymbol{w} \in \boldsymbol{H}_0^1(T)$, there holds*

$$\begin{aligned} (e_p \, , \, \nabla \cdot \boldsymbol{w})_T = {} & (\nu \mathbb{E}_{\mathbb{L}} \, , \, \nabla \boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1} \boldsymbol{e}_u \, , \, \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1} \boldsymbol{\delta}_u \, , \, \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} \\ & - \langle \tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}) \, , \, \boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w} \rangle_{\partial \mathcal{T}_h(T)}. \end{aligned}$$

Proof. A combination of the integration by parts and the property (5.8) indicate

$$\begin{aligned} (e_p \, , \, \nabla \cdot \boldsymbol{w})_{\mathcal{T}_h(T)} &= (e_p \, , \, \nabla \cdot \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} + (e_p \, , \, \nabla \cdot (\boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w}))_{\mathcal{T}_h(T)} \\ &= (e_p \, , \, \nabla \cdot \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} + \langle e_p \boldsymbol{n} \, , \, \boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w} \rangle_{\partial \mathcal{T}_h(T)}. \end{aligned}$$

By utilizing the error equation (5.3b) with $\boldsymbol{v} = \boldsymbol{P}_\tau \boldsymbol{w}$ restricted over T, we arrive at

$$\begin{aligned} (e_p \, , \, \nabla \cdot \boldsymbol{w})_{\mathcal{T}_h(T)} = {} & (\nu \mathbb{E}_{\mathbb{L}} \, , \, \nabla \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1} \boldsymbol{e}_u \, , \, \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1} \boldsymbol{\delta}_u \, , \, \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} \\ & + \langle e_p \boldsymbol{n} \, , \, \boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w} \rangle_{\partial \mathcal{T}_h(T)} - \langle \widehat{\mathbb{E}_{\mathbb{S}}} \boldsymbol{n} \, , \, \boldsymbol{P}_\tau \boldsymbol{w} \rangle_{\partial \mathcal{T}_h(T)} \\ = {} & (\nu \mathbb{E}_{\mathbb{L}} \, , \, \nabla \boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1} \boldsymbol{e}_u \, , \, \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1} \boldsymbol{\delta}_u \, , \, \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} \\ & + \langle e_p \boldsymbol{n} \, , \, \boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w} \rangle_{\partial \mathcal{T}_h(T)} - \langle \widehat{\mathbb{E}_{\mathbb{S}}} \boldsymbol{n} \, , \, \boldsymbol{P}_\tau \boldsymbol{w} \rangle_{\partial \mathcal{T}_h(T)} \\ & - (\nu \mathbb{E}_{\mathbb{L}} \, , \, \nabla (\boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w}))_{\mathcal{T}_h(T)}. \end{aligned}$$

Then integrating by parts upon the last term and utilizing equality (5.8) lead to

$$
\begin{aligned}
(e_p\,,\,\nabla\cdot\boldsymbol{w})_{\mathcal{T}_h(T)} = & ((\nu\mathbb{E}_{\mathbb{L}}\,,\,\nabla\boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{P}_\tau\boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1}\boldsymbol{\delta}_u\,,\,\boldsymbol{P}_\tau\boldsymbol{w})_{\mathcal{T}_h(T)} \\
& + \langle e_p\boldsymbol{n}\,,\,\boldsymbol{w}-\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)} - \langle\widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n}\,,\,\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)} \\
& - \langle(\nu\mathbb{E}_{\mathbb{L}}\boldsymbol{n}\,,\,\boldsymbol{w}-\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)}.
\end{aligned}
$$

Considering (2.3d) and (2.3f), and that $\mathbb{S}$ is continuous across all interior face $F\in\mathcal{E}_h^0$, together with (3.1f) and the fact that $w=0$ on ∂T, we deduce that

$$
\langle\widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n}\,,\,\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)} := \langle\widehat{\mathbb{S}}_h\boldsymbol{n}-\mathbb{S}\boldsymbol{n}\,,\,\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)} = \langle\widehat{\mathbb{S}}_h\boldsymbol{n}-\mathbb{S}\boldsymbol{n}\,,\,\boldsymbol{w}\rangle_{\partial T} = 0. \tag{5.9}
$$

After inserting this term into the previous expression, we arrive at

$$
\begin{aligned}
(e_p\,,\,\nabla\cdot\boldsymbol{w})_{\mathcal{T}_h(T)} = & (\nu\mathbb{E}_{\mathbb{L}}\,,\,\nabla\boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{P}_\tau\boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1}\boldsymbol{\delta}_u\,,\,\boldsymbol{P}_\tau\boldsymbol{w})_{\mathcal{T}_h(T)} \\
& - \langle\nu\mathbb{E}_{\mathbb{L}}\boldsymbol{n}-e_p\boldsymbol{n}\,,\,\boldsymbol{w}-\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)}.
\end{aligned}
$$

An application of the trace identity (5.4) and (5.9), yields

$$
\begin{aligned}
(e_p\,,\,\nabla\cdot\boldsymbol{w})_{\mathcal{T}_h(T)} = & (\nu\mathbb{E}_{\mathbb{L}}\,,\,\nabla\boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{P}_\tau\boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1}\boldsymbol{\delta}_u\,,\,\boldsymbol{P}_\tau\boldsymbol{w})_{\mathcal{T}_h(T)} \\
& - \langle\tau(\boldsymbol{e}_u-\widehat{\boldsymbol{e}_u})\,,\,\boldsymbol{w}-\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)} \\
& + \langle(\boldsymbol{I}-\boldsymbol{P}_\partial^h)\mathbb{S}\boldsymbol{n}\,,\,\boldsymbol{w}-\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)} \\
& + \langle\tau(\boldsymbol{P}_\partial^h\boldsymbol{u}-\boldsymbol{P}_M\boldsymbol{u})\,,\,\boldsymbol{w}-\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)}.
\end{aligned}
$$

To complete the proof, we will estimate the last two terms. The continuity of $\mathbb{S}$ across all interior faces $F\in\mathcal{E}_h^0$ shows the first term vanishes.

$$
\begin{aligned}
&\langle(\boldsymbol{I}-\boldsymbol{P}_\partial^h)(\nu\mathbb{L}\boldsymbol{n}-p\boldsymbol{n})\,,\,\boldsymbol{w}-\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)} \\
&\quad=\langle(\boldsymbol{I}-\boldsymbol{P}_\partial^h)((\nu\mathbb{L}\boldsymbol{n}-p\boldsymbol{n})\,,\,\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)} - \langle(\boldsymbol{I}-\boldsymbol{P}_\partial^h)((\nu\mathbb{L}\boldsymbol{n}-p\boldsymbol{n})\,,\,\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)} \\
&\quad=\langle(\boldsymbol{I}-\boldsymbol{P}_\partial^h)((\nu\mathbb{L}\boldsymbol{n}-p\boldsymbol{n})\,,\,\boldsymbol{w}\rangle_{\partial T} = 0.
\end{aligned}
$$

For the last term, notice that $\boldsymbol{P}_\partial^h\boldsymbol{u}=\boldsymbol{P}_M\boldsymbol{u}$ on $\partial\mathcal{T}_h(T)\backslash\partial T$, therefore, we have

$$
\langle\tau(\boldsymbol{P}_\partial^h\boldsymbol{u}-\boldsymbol{P}_M\boldsymbol{u})\,,\,\boldsymbol{w}-\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial\mathcal{T}_h(T)} = \langle\tau(\boldsymbol{P}_\partial^h\boldsymbol{u}-\boldsymbol{P}_M\boldsymbol{u})\,,\,\boldsymbol{w}-\boldsymbol{P}_\tau\boldsymbol{w}\rangle_{\partial T} = 0.
$$

The last step is due to the fact that $\boldsymbol{w}\in\boldsymbol{H}_0^1(T)$, Assumption 2.1 and (5.8). □

As a consequence of the *inf-sup* condition (5.7) and Lemma 5.4, we obtain the upper bound to pressure error $\|e_p-(e_p)_T\|_T$:

Lemma 5.5 (Estimate for $\|e_p-(e_p)_T\|_T$)**.** *Given $T\in\mathcal{T}_H$, there holds*

$$
\begin{aligned}
\|e_p-(e_p)_T\|_{\mathcal{T}_h(T)} \lesssim & \|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h(T)} + H^{\frac{1}{2}}\|\kappa^{-1}\|_T^{\frac{1}{2}}\,\|\kappa^{-1/2}\boldsymbol{e}_u\|_{\mathcal{T}_h(T)} \\
& + h^{\frac{1}{2}}\|\tau^{\frac{1}{2}}(\boldsymbol{e}_u-\widehat{\boldsymbol{e}_u})\|_{\partial\mathcal{T}_h(T)} + h^s\|\kappa^{-1}\|_T\,(\|\boldsymbol{u}\|_{s,\mathcal{T}_h(T)} + \|\mathbb{L}\|_{s,\mathcal{T}_h(T)}).
\end{aligned}
$$

Proof. We bound the terms on the right hand side of the identity in Lemma 5.4 individually. An application of the Hölder's inequality to the first term yields

$$
(\mathbb{E}_{\mathbb{L}}\,,\,\nabla\boldsymbol{w})_{\mathcal{T}_h(T)} \le \|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h(T)}\|\nabla\boldsymbol{w}\|_T.
$$

Take into account that κ^{-1} is piece wise constant over the fine mesh $\mathcal{T}_h$, we derive

$$
(\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{P}_\tau\boldsymbol{w})_{\mathcal{T}_h(T)} = (\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{w})_{\mathcal{T}_h(T)} + (\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{P}_\tau\boldsymbol{w}-\boldsymbol{P}_k\boldsymbol{w})_{\mathcal{T}_h(T)} := A+B \tag{5.10}
$$

with $\boldsymbol{P}_{\boldsymbol{k}}$ being the L^2-projection onto $\boldsymbol{P}_k(K)$ for $K\in\mathcal{T}_h$. The generalized Hölder's inequality implies

$$
A := (\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{w})_{\mathcal{T}_h(T)} \le \|\kappa^{-\frac{1}{2}}\boldsymbol{e}_u\|_{\mathcal{T}_h(T)}\|\kappa^{-1}\|_{\mathcal{T}_h(T)}^{\frac{1}{2}}\|\boldsymbol{w}\|_{L^4(T)}. \tag{5.11}
$$

By virtue of the Sobolev embedding, there exists a positive constant C_{imb} depending on T satisfying

$$\|\boldsymbol{w}\|_{L^4(T)} \leq C_{\text{imb}} H^{\frac{1}{2}} \|\nabla \boldsymbol{w}\|_T.$$

Inserting the above result into (5.11), we deduce

$$A \lesssim H^{\frac{1}{2}} \|\kappa^{-1}\|_T^{\frac{1}{2}} \|\kappa^{-\frac{1}{2}} \boldsymbol{e}_u\|_{\mathcal{T}_h(T)} \|\nabla \boldsymbol{w}\|_T. \tag{5.12}$$

With respect to B in (5.10), a combination of the generalized Hölder's inequality and the inverse estimate [24, Section 4.5] leads to

$$\begin{aligned} B &:= (\kappa^{-1}\boldsymbol{e}_u\,,\, \boldsymbol{P}_\tau \boldsymbol{w} - \boldsymbol{P}_k \boldsymbol{w})_{\mathcal{T}_h(T)} \leq \|\kappa^{-\frac{1}{2}} \boldsymbol{e}_u\|_{\mathcal{T}_h(T)} \|\kappa^{-1}\|_T^{\frac{1}{2}} \|\boldsymbol{P}_\tau \boldsymbol{w} - \boldsymbol{P}_k \boldsymbol{w}\|_{L^4(\mathcal{T}_h(T))} \\ &\lesssim h^{-\frac{1}{2}} \|\kappa^{-1}\|_T^{\frac{1}{2}} \|\kappa^{-\frac{1}{2}} \boldsymbol{e}_u\|_{\mathcal{T}_h(T)} \|\boldsymbol{P}_\tau \boldsymbol{w} - \boldsymbol{P}_k \boldsymbol{w}\|_{\mathcal{T}_h(T)} \end{aligned}$$

Then the triangle inequality, together with the approximation property of $\boldsymbol{P}_k$ and $\boldsymbol{P}_\tau$, yields

$$\begin{aligned} B &\lesssim h^{-\frac{1}{2}} \|\kappa^{-1}\|_T^{\frac{1}{2}} \|\kappa^{-\frac{1}{2}} \boldsymbol{e}_u\|_{\mathcal{T}_h(T)} \big(\|\boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w}\|_{\mathcal{T}_h(T)} + \|\boldsymbol{w} - \boldsymbol{P}_k \boldsymbol{w}\|_{\mathcal{T}_h(T)}\big) \\ &\lesssim h^{\frac{1}{2}} \|\kappa^{-1}\|_T^{\frac{1}{2}} \, \|\kappa^{-\frac{1}{2}} \boldsymbol{e}_u\|_{\mathcal{T}_h(T)} \, \|\nabla \boldsymbol{w}\|_T. \end{aligned} \tag{5.13}$$

A combination of the estimates (5.10), (5.12) and (5.13) indicates

$$(\kappa^{-1}\boldsymbol{e}_u\,,\, \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} \lesssim H^{\frac{1}{2}} \|\kappa^{-1}\|_T^{\frac{1}{2}} \, \|\kappa^{-\frac{1}{2}} \boldsymbol{e}_u\|_{\mathcal{T}_h(T)} \, \|\nabla \boldsymbol{w}\|_T.$$

In view that κ^{-1} is piecewise constant on $\mathcal{T}_h$, $s > 1$ and (5.2), analogously, we can derive the estimate for the third term,

$$\begin{aligned} (\kappa^{-1}\boldsymbol{\delta}_u\,,\, \boldsymbol{P}_\tau \boldsymbol{w})_{\mathcal{T}_h(T)} = (\kappa^{-1}\boldsymbol{\delta}_u\,,\, \boldsymbol{P}_\tau \boldsymbol{w} - \boldsymbol{P}_{\boldsymbol{0}} w)_{\mathcal{T}_h(T)} &\lesssim \|\kappa^{-1}\|_T \|\boldsymbol{\delta}_u\|_{\mathcal{T}_h(T)} \, \|\nabla \boldsymbol{w}\|_T \\ &\lesssim h^s \|\kappa^{-1}\|_T \, (\|\boldsymbol{u}\|_{s,\mathcal{T}_h(T)} + \|\mathbb{L}\|_{s,\mathcal{T}_h(T)}) \cdot \|\nabla \boldsymbol{w}\|_T. \end{aligned}$$

An application of the Cauchy-Schwarz inequality to the last term leads to

$$\begin{aligned} \langle \tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\,,\, \boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w} \rangle_{\partial\mathcal{T}_h(T)} &\leq \|\tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\|_{\partial\mathcal{T}_h(T)} \|\boldsymbol{w} - \boldsymbol{P}_\tau \boldsymbol{w}\|_{\partial\mathcal{T}_h(T)} \\ &\lesssim h^{\frac{1}{2}} \|\tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\|_{\partial\mathcal{T}_h(T)} \|\nabla \boldsymbol{w}\|_T. \end{aligned}$$

Finally, if we combine the estimates of the four terms, together with (5.7), then we obtain the estimate as shown in the lemma. □

To the end of estimating e_p, two estimate on (e_p) are presented under a stronger Assumption 5.2 and a weaker Assumption 5.3, respectively.

Assumption 5.2. *(a) There exists a coarse triangulation $\widetilde{\mathcal{T}_H}$ such that each face $F \in \mathcal{E}_H$ is a face of an element $K_H \in \widetilde{\mathcal{T}_H}$. $\mathcal{T}_h$ is a refinement of $\widetilde{\mathcal{T}_H}$.*

(b) $\boldsymbol{P}_2(K) \subset \boldsymbol{V}(K) \quad \forall K \in \mathcal{T}_h,$ *and* $\boldsymbol{P}_2(F) \subset \boldsymbol{M}_H(F), \quad \forall\, F \in \mathcal{E}_H.$

Assumption 5.3.

$$\boldsymbol{P}_1(K) \subset \boldsymbol{V}(K) \quad \forall K \in \mathcal{T}_h, \quad \textit{and} \quad \boldsymbol{P}_1(F) \subset \boldsymbol{M}_H(F) \quad \forall\, F \in \mathcal{E}_H.$$

Theorem 5.2 (Estimate for (e_p))**.** *Assume that $h \ll H$ and $\tau = \mathcal{O}(1)$.*

1 If Assumption 5.2 holds. Then we can obtain

$$\|(e_p)\|_{\mathcal{T}_h} \lesssim \|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h} + \|e_p - (e_p)\|_{\mathcal{T}_h} + \|\kappa^{-1}\|_\Omega \Big(\|\kappa^{-\frac{1}{2}} \boldsymbol{e}_u\|_{\mathcal{T}_h} + h^{s-\frac{1}{2}} \|\boldsymbol{u}\|_{s,\mathcal{T}_h} \Big). \tag{5.14}$$

2 If the spaces $\boldsymbol{V}_h$ and $\boldsymbol{M}_H$ satisfy Assumption 5.3, then we have

$$\begin{aligned}\|(e_p)\|_{\mathcal{T}_h} \lesssim & H^{\frac{1}{2}}\Big(h^{-\frac{1}{2}}(\|e_p-(e_p)\|_{\mathcal{T}_h}+\|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h})+\|\tau^{\frac{1}{2}}(\boldsymbol{e}_u-\widehat{\boldsymbol{e}_u})\|_{\partial\mathcal{T}_h}\Big)\\ &+\|\kappa^{-1}\|_{\Omega}\Big(\|\kappa^{-\frac{1}{2}}\boldsymbol{e}_u\|_{\mathcal{T}_h}+h^{s-\frac{1}{2}}\|\boldsymbol{u}\|_{s,\mathcal{T}_h}\Big)\\ &+H^{\frac{1}{2}}\Big(h^{s-\frac{1}{2}}(\|\mathbb{L}\|_{s,\mathcal{T}_h}+\|p\|_{s,\mathcal{T}_h})+\|\boldsymbol{u}-\boldsymbol{P}_H^{\partial}\boldsymbol{u}\|_{\partial\mathcal{T}_H}\Big).\end{aligned} \tag{5.15}$$

Proof. Denote the classical P_2-P_1 Taylor-Hood element on $\widetilde{\mathcal{T}_H}$ as $\boldsymbol{V}_H\times Q_H\subset \boldsymbol{H}_0^1(\Omega)\times L_0^2(\Omega)$. By virtue of [31, Corollary I.2.4], there exists $\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega)$ s.t.

$$\nabla\cdot\boldsymbol{w}=(e_p) \text{ in } \Omega \qquad \text{and} \qquad \|\nabla\boldsymbol{w}\|_{\Omega}\le C_T^{-1}\|e_p\|_{\Omega}$$

We can define a Fortin projection $\boldsymbol{\Pi}_1\boldsymbol{w}\in\boldsymbol{V}_H$ On $\widetilde{T}_H$, cf. [4], satisfying

$$(\nabla\cdot\boldsymbol{\Pi}_1\boldsymbol{w},q)_{\Omega}=(\nabla\cdot\boldsymbol{w},q)_{\Omega} \quad \text{for all} \quad q\in Q_H \tag{5.16a}$$

$$\|\nabla\boldsymbol{\Pi}_1\boldsymbol{w}\|_{\Omega}\le C\|\nabla\boldsymbol{w}\|_{\Omega}. \tag{5.16b}$$

Assumptions 5.2 implies that $\boldsymbol{\Pi}_1\boldsymbol{w}\in\boldsymbol{V}_h$ and $\boldsymbol{\Pi}_1\boldsymbol{w}|_{\mathcal{E}_H}\subset\boldsymbol{M}_H$. After taking $\boldsymbol{v}:=\boldsymbol{\Pi}_1\boldsymbol{w}$ in error equation (5.3b), we arrive at

$$(\nu\mathbb{E}_{\mathbb{L}}-\mathbb{I}e_p\,,\,\nabla\boldsymbol{\Pi}_1\boldsymbol{w})_{\mathcal{T}_h}+(\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{\Pi}_1\boldsymbol{w})_{\mathcal{T}_h}-\langle\widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n}\,,\,\boldsymbol{\Pi}_1\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}=-(\kappa^{-1}\boldsymbol{\delta}_u\,,\,\boldsymbol{\Pi}_1\boldsymbol{w})_{\mathcal{T}_h}. \tag{5.17}$$

By application of error equation (5.3e), this leads to

$$\langle\widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n}\,,\,\boldsymbol{\Pi}_1\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}=0.$$

Owing to (5.16a), we can obtain

$$\begin{aligned}(\nu\mathbb{E}_{\mathbb{L}}-\mathbb{I}e_p\,,\,\nabla\boldsymbol{\Pi}_1\boldsymbol{w})_{\mathcal{T}_h}&=(\nu\mathbb{E}_{\mathbb{L}}-\mathbb{I}(e_p-(e_p))\,,\,\nabla\boldsymbol{\Pi}_1\boldsymbol{w})_{\mathcal{T}_h}-((e_p)\,,\,\nabla\cdot\boldsymbol{\Pi}_1\boldsymbol{w})_{\mathcal{T}_h}\\&=(\nu\mathbb{E}_{\mathbb{L}}-\mathbb{I}(e_p-(e_p))\,,\,\nabla\boldsymbol{\Pi}_1\boldsymbol{w})_{\mathcal{T}_h}-\|(e_p)\|_{\Omega}^2.\end{aligned}$$

Inserting the above two identities into (5.17) in combination with some algebriac rearrangement, yields

$$\|(e_p)\|^2=(\nu\mathbb{E}_{\mathbb{L}}-\mathbb{I}(e_p-(e_p))\,,\,\nabla\boldsymbol{\Pi}_1\boldsymbol{w})_{\mathcal{T}_h}+(\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{\Pi}_1\boldsymbol{w})_{\mathcal{T}_h}+(\kappa^{-1}\boldsymbol{\delta}_u\,,\,\boldsymbol{\Pi}_1\boldsymbol{w})_{\mathcal{T}_h}.$$

Finally, an application of the generalized Hölder's inequality, Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^4(\Omega)$ and Lemma 5.5, shows

$$\begin{aligned}\|(e_p)\|^2&\le\|\nabla\boldsymbol{\Pi}_1\boldsymbol{w}\|_{\Omega}(\nu\|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h}+\|e_p-(e_p)\|_{\mathcal{T}_h})\\&\quad+(\|\kappa^{-\frac{1}{2}}\boldsymbol{e}_u\|_{\mathcal{T}_h}+\|\kappa^{-\frac{1}{2}}\boldsymbol{\delta}_u\|_{\mathcal{T}_h})\|\kappa^{-1}\|_{\Omega}\|\boldsymbol{\Pi}_1\boldsymbol{w}\|_{L^4(\Omega)}\\&\lesssim\|\nabla\boldsymbol{\Pi}_1\boldsymbol{w}\|_{\Omega}\Big(\|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h}+\|e_p-(e_p)\|_{\mathcal{T}_h}+\|\kappa^{-1}\|_{\Omega}(\|\kappa^{-\frac{1}{2}}\boldsymbol{e}_u\|_{\mathcal{T}_h}+h^{-\frac{1}{2}}\|\boldsymbol{\delta}_u\|_{\mathcal{T}_h})\Big)\\&\lesssim\|(e_p)\|_{\Omega}\Big(\|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h}+\|e_p-(e_p)\|_{\mathcal{T}_h}+\|\kappa^{-1}\|_{\Omega}(\|\kappa^{-\frac{1}{2}}\boldsymbol{e}_u\|_{\mathcal{T}_h}+h^{s-\frac{1}{2}}\|\boldsymbol{u}\|_{s,\mathcal{T}_h})\Big).\end{aligned}$$

Here, the last step is due to (5.16b) and the approximation property (5.2). This verifies (5.14).

Now we prove (5.15) with a weaker assumption (5.3) on the local spaces. In this case, the above argument is no longer available due to the fact that $\boldsymbol{\Pi}_1\boldsymbol{w}$ does not exist if the local spaces only contain piecewise linear polynomials. Instead, we take $\boldsymbol{v}=\boldsymbol{\Pi}_{\text{sz}}\boldsymbol{w}$ in (5.3b), where $\boldsymbol{\Pi}_{\text{sz}}\boldsymbol{w}\in\boldsymbol{V}_h\cap\boldsymbol{H}_0^1(\Omega)$ denotes the Scott-Zhang interpolation of $\boldsymbol{w}$, cf. [21]. The following approximation properties hold

$$|\boldsymbol{w}-\boldsymbol{\Pi}_{\text{sz}}\boldsymbol{w}|_{m,\Omega}\lesssim h^{1-m}\|\nabla\boldsymbol{w}\|_{\Omega}, \quad \text{for} \quad m=0,1, \tag{5.18a}$$

$$|\boldsymbol{w}-\boldsymbol{\Pi}_{\text{sz}}\boldsymbol{w}|_{\partial\mathcal{T}_h}\lesssim h^{\frac{1}{2}}\|\nabla\boldsymbol{w}\|_{\Omega}. \tag{5.18b}$$

We start by the following identity

$$\begin{aligned}\|(e_p)\|_{\Omega}^2=((e_p)\,,\,\nabla\cdot\boldsymbol{w})_{\mathcal{T}_h}&=((e_p)\,,\,\nabla\cdot\boldsymbol{\Pi}_{\text{sz}}\boldsymbol{w})_{\mathcal{T}_h}+((e_p)\,,\,\nabla\cdot(\boldsymbol{w}-\boldsymbol{\Pi}_{\text{sz}}\boldsymbol{w}))_{\mathcal{T}_h}\\&=((e_p)\,,\,\nabla\cdot\boldsymbol{\Pi}_{\text{sz}}\boldsymbol{w})_{\mathcal{T}_h}+\langle(e_p)\,,\,(\boldsymbol{w}-\boldsymbol{\Pi}_{\text{sz}}\boldsymbol{w})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_H}\end{aligned}$$

By taking $\boldsymbol{v} = \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}$ in error equation (5.3b), we obtain

$$(\nu\mathbb{E}_{\mathbb{L}} - \mathbb{I}e_p\,,\,\nabla\boldsymbol{\Pi}_{\rm sz}\boldsymbol{w})_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{\Pi}_{\rm sz}\boldsymbol{w})_{\mathcal{T}_h} - \langle\widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n}\,,\,\boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\rangle_{\partial\mathcal{T}_h} = -(\kappa^{-1}\boldsymbol{\delta}_u\,,\,\boldsymbol{\Pi}_{\rm sz}\boldsymbol{w})_{\mathcal{T}_h}.$$

As a consequence, we can rewrite this error equation as

$$\begin{aligned}\|(e_p)\|_\Omega^2 &= (\nu\mathbb{E}_{\mathbb{L}} - \mathbb{I}(e_p - (e_p))\,,\,\nabla\boldsymbol{\Pi}_{\rm sz}\boldsymbol{w})_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{\Pi}_{\rm sz}\boldsymbol{w})_{\mathcal{T}_h} - \langle\widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n}\,,\,\boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &\quad + (\kappa^{-1}\boldsymbol{\delta}_u\,,\,\boldsymbol{\Pi}_{\rm sz}\boldsymbol{w})_{\mathcal{T}_h} + \langle(e_p)\,,\,(\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_H}\\ &:= T_1 + \cdots + T_5.\end{aligned} \tag{5.19}$$

We will estimate T_1 to T_5 one by one. Analogous to the first case, we can bound T_1, T_2 and T_4 by

$$\begin{aligned}T_1 &\lesssim \|(e_p)\|_\Omega(\|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h} + \|e_p - (e_p)\|_{\mathcal{T}_h}),\\ T_2 + T_4 &\lesssim \|(e_p)\|_\Omega\|\kappa^{-1}\|_\Omega\Big(\|\kappa^{-\frac12}\boldsymbol{e}_u\|_{\mathcal{T}_h} + \kappa^{s-\frac12}\|\boldsymbol{u}\|_{s,\mathcal{T}_h}\Big).\end{aligned}$$

Invoking the Cauchy-Schwarz inequality, together with (5.18b) and an application of the inverse estimate, results in

$$\begin{aligned}T_5 &\le \|(e_p)\|_{\partial\mathcal{T}_H}\|(\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w})\cdot\boldsymbol{n}\|_{\partial\mathcal{T}_H} \lesssim H^{-\frac12}h^{\frac12}\|(e_p)\|_\Omega\|\nabla\boldsymbol{w}\|_\Omega\\ &\lesssim H^{-\frac12}h^{\frac12}\|(e_p)\|_\Omega^2.\end{aligned}$$

Utilizing (5.3d) and (5.4), T_3 can be written as

$$\begin{aligned}T_3 &= \langle\widehat{\mathbb{E}_{\mathbb{S}}}\boldsymbol{n}\,,\,\boldsymbol{P}_M\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &= \langle\nu\mathbb{E}_{\mathbb{L}}\boldsymbol{n} - (e_p - (e_p))\boldsymbol{n}\,,\,\boldsymbol{P}_M\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\rangle_{\partial\mathcal{T}_h} - \langle\tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\,,\,\boldsymbol{P}_M\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &+ \langle(\boldsymbol{I} - \boldsymbol{P}_h^\partial)(\nu\mathbb{L}\boldsymbol{n} - p\boldsymbol{n}) + \tau(\boldsymbol{P}_h^\partial\boldsymbol{u} - \boldsymbol{P}_M\boldsymbol{u})\,,\,\boldsymbol{P}_M\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &- \langle(e_p)\,,\,\boldsymbol{P}_M\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &:= T_{31} + T_{32} + T_{33} + T_{34}.\end{aligned}$$

We will estimate the four terms in the last equality. In order to estimate T_{31} to T_{33}, we first will first bound $\|\boldsymbol{P}_M\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\|_{\partial\mathcal{T}_h}$ by

$$\begin{aligned}\|\boldsymbol{P}_M\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\|_{\partial\mathcal{T}_h} &\le \|\boldsymbol{w} - \boldsymbol{P}_M\boldsymbol{w}\|_{\partial\mathcal{T}_h} + \|\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\|_{\partial\mathcal{T}_h}\\ &\le \|\boldsymbol{w} - \boldsymbol{P}_h^\partial\boldsymbol{w}\|_{\partial\mathcal{T}_h} + \|\boldsymbol{w} - \boldsymbol{P}_H^\partial\boldsymbol{w}\|_{\partial\mathcal{T}_H} + \|\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\|_{\partial\mathcal{T}_h}\\ &\lesssim (H^{\frac12} + h^{\frac12})\|\boldsymbol{w}\|_{1,\Omega} \lesssim H^{\frac12}\|(e_p)\|_\Omega.\end{aligned}$$

Next we can apply Cauchy-Schwarz inequality, inverse inequality and (5.2) on top of the above estimate to arrive at

$$\begin{aligned}T_{31} &\lesssim H^{\frac12}h^{-\frac12}(\|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h} + \|e_p - (e_p)\|_{\mathcal{T}_h})\|(e_p)\|_\Omega,\\ T_{32} &\lesssim H^{\frac12}\|\tau^{\frac12}(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\|_{\partial\mathcal{T}_h}\|(e_p)\|_\Omega,\\ T_{33} &\lesssim H^{\frac12}\Big(\|(\boldsymbol{I} - \boldsymbol{P}_h^\partial)(\nu\mathbb{L}\boldsymbol{n} - p\boldsymbol{n})\|_{\partial\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{P}_h^\partial\boldsymbol{u}\|_{\partial\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\|_{\partial\mathcal{T}_H}\Big)\|(e_p)\|_\Omega\\ &\lesssim H^{\frac12}\Big(h^{s-\frac12}(\|\mathbb{L}\|_{s,\mathcal{T}_h} + \|\boldsymbol{u}\|_{s,\mathcal{T}_h} + \|p\|_{s,\mathcal{T}_h}) + \|\boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\|_{\partial\mathcal{T}_H}\Big)\|(e_p)\|_\Omega.\end{aligned}$$

Finally, noticing that (e_p) is piecewise constant over $\mathcal{T}_H$, we have

$$\begin{aligned}T_{34} &= -\langle(e_p)\,,\,\boldsymbol{P}_M\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\rangle_{\partial\mathcal{T}_H} = -\langle(e_p)\,,\,\boldsymbol{P}_H^\partial\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\rangle_{\partial\mathcal{T}_H}\\ &= -\langle(e_p)\,,\,\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\rangle_{\partial\mathcal{T}_H} \le \|(e_p)\|_{\partial\mathcal{T}_H}\|\boldsymbol{w} - \boldsymbol{\Pi}_{\rm sz}\boldsymbol{w}\|_{\partial\mathcal{T}_H}\\ &\lesssim H^{-\frac12}h^{\frac12}\|(e_p)\|_\Omega\|\boldsymbol{w}\|_{1,\Omega}\\ &\lesssim H^{-\frac12}h^{\frac12}\|(e_p)\|_\Omega^2.\end{aligned}$$

In view that if $h \ll H$, the bounds for T_5 and T_{34} can be absorbed by the left hand side of (5.19). The proof is completed after combining the estimates from T_1 to T_5. □

5.3 Main error estimates

In this section, we are concerned with the main error estimate for the proposed upscaling HDG method (2.3) in terms of two results. The first results shows the estimate of $\mathbb{E}_{\mathbb{L}}$, $\boldsymbol{e}_u$ and $\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}$. Whereas the second one is on the pressure estimate.

Theorem 5.3 (Estimate for $\mathbb{E}_{\mathbb{L}}$ and $\boldsymbol{e}_u$)**.** *Assume that the exact solutions $\mathbb{L}$, $\boldsymbol{u}$ and p have piecewise regularity up to $s \geq 1$ on $\mathcal{T}_h$. Let the nonzero stabilization parameter $\tau = 1$ and $k = \lceil s \rceil - 1$, then there holds*

$$
\begin{aligned}
&\|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h} + \|\kappa^{-\frac{1}{2}}\boldsymbol{e}_u\|_{\mathcal{T}_h} + \|\tau^{\frac{1}{2}}(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\|_{\partial\mathcal{T}_h} \\
&\lesssim (\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}} + h^{-\frac{1}{2}})\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}} h^s(\|\boldsymbol{u}\|_{s,\mathcal{T}_h} + \|\mathbb{L}\|_{s,\mathcal{T}_h}) \\
&+ h^{-\frac{1}{2}}(1 + H^{\frac{1}{2}}\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}})\|\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\|_{\partial\mathcal{T}_H}.
\end{aligned}
$$

Proof. By application of Lemma 5.2,

$$
\begin{aligned}
&(\nu\mathbb{E}_{\mathbb{L}}\,,\mathbb{E}_{\mathbb{L}})_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{e}_u\,,\,\boldsymbol{e}_u)_{\mathcal{T}_h} + \langle\tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\,,\,\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h} \\
&= -(\nu\delta_{\mathbb{L}}\,,\,\mathbb{E}_{\mathbb{L}})_{\mathcal{T}_h} - (\kappa^{-1}\boldsymbol{\delta}_u\,,\,\boldsymbol{e}_u)_{\mathcal{T}_h} + \langle\boldsymbol{P}_h^{\partial}\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\,,\,\tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\rangle_{\partial\mathcal{T}_H} \\
&+ \langle\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\,,\,\mathbb{E}_{\mathbb{L}}\boldsymbol{n} - e_p\boldsymbol{n}\rangle_{\partial\mathcal{T}_H}.
\end{aligned}
\tag{5.20}
$$

We will estimate the terms on the right hand side one by one.

By virtue of the Hölder's inequality, the first term can be bounded by

$$
-(\nu\delta_{\mathbb{L}}\,,\,\mathbb{E}_{\mathbb{L}})_{\mathcal{T}_h} \leq \frac{1}{4}(\nu\mathbb{E}_{\mathbb{L}}\,,\,\mathbb{E}_{\mathbb{L}})_{\mathcal{T}_h} + (\nu\delta_{\mathbb{L}}\,,\,\delta_{\mathbb{L}})_{\mathcal{T}_h}. \tag{5.21}
$$

The generalized Hölder's inequality and the inverse inequality lead to

$$
-(\kappa^{-1}\boldsymbol{\delta}_u\,,\,\boldsymbol{e}_u)_{\mathcal{T}_h} \leq C_{\text{inv2}}h^{-\frac{1}{2}}\|\kappa^{-1}\|_{\Omega}^{\frac{1}{2}}\|\boldsymbol{\delta}_u\|_{\mathcal{T}_h}\|\kappa^{-\frac{1}{2}}\boldsymbol{e}_u\|_{\mathcal{T}_h}
$$

with C_{inv2} being a positive constant depending only on the shape regularity parameter of $\mathcal{T}_h$. An application of the Young's inequality yields

$$
-(\kappa^{-1}\boldsymbol{\delta}_u\,,\,\boldsymbol{e}_u)_{\mathcal{T}_h} \leq C_{\text{inv2}}h^{-1}\|\kappa^{-1}\|_{\Omega}\|\boldsymbol{\delta}_u\|_{\mathcal{T}_h}^2 + \frac{1}{4}\|\kappa^{-\frac{1}{2}}\boldsymbol{e}_u\|_{\mathcal{T}_h}^2. \tag{5.22}
$$

By application of the Hölder's inequality, the third term can be estimated by

$$
\begin{aligned}
&\langle\boldsymbol{P}_h^{\partial}\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\,,\,\tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\rangle_{\partial\mathcal{T}_H} \\
&= \langle\boldsymbol{P}_h^{\partial}\boldsymbol{u} - u\,,\,\tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\rangle_{\partial\mathcal{T}_H} + \langle u - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\,,\,\tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\rangle_{\partial\mathcal{T}_H} \\
&\leq (\|\boldsymbol{u} - \boldsymbol{P}_h^{\partial}\boldsymbol{u}\|_{\partial\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\|_{\partial\mathcal{T}_H})\|\tau^{1/2}(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\|_{\partial\mathcal{T}_h}.
\end{aligned}
$$

Then the projection property (5.2) and the Young's inequality lead to

$$
\begin{aligned}
&\langle\boldsymbol{P}_h^{\partial}\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\,,\,\tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\rangle_{\partial\mathcal{T}_H} \\
&\leq h^{2s-1}\|\boldsymbol{u}\|_{\mathcal{T}_h}^2 + \|\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\|_{\partial\mathcal{T}_H} + \frac{1}{4}\langle\tau(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\,,\,\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
\tag{5.23}
$$

Now we estimate the last term in (5.20). By Assumption 5.1, $\boldsymbol{M}_H$ contains piecewise constant. Then we can write the last term in the Lemma 5.2 as

$$
\begin{aligned}
&\langle\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\,,\,\mathbb{E}_{\mathbb{L}}\boldsymbol{n} - e_p\boldsymbol{n}\rangle_{\partial\mathcal{T}_H} \\
&= \langle\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\,,\,\mathbb{E}_{\mathbb{L}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_H} - \langle\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\,,\,(e_p - (e_p)_T)\boldsymbol{n}\rangle_{\partial\mathcal{T}_H} \\
&\lesssim h^{-1/2}\|\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\|_{\partial\mathcal{T}_H}(\|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h} + \|e_p - (e_p)_T\|_{\mathcal{T}_h}).
\end{aligned}
$$

An application of Lemma 5.5 and the Young's inequality result in

$$
\begin{aligned}
&\langle\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\,,\,\mathbb{E}_{\mathbb{L}}\boldsymbol{n} - e_p\boldsymbol{n}\rangle_{\partial\mathcal{T}_H} \leq \frac{1}{4}(\|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h}^2 + \|\kappa^{-\frac{1}{2}}\boldsymbol{e}_u\|_{\mathcal{T}_h}^2) \\
&+ h^{-1}(1 + H\|\kappa^{-1}\|_{\Omega})\|\boldsymbol{u} - \boldsymbol{P}_H^{\partial}\boldsymbol{u}\|_{\partial\mathcal{T}_H}^2 + \|\kappa^{-1}\|_{\Omega}^2 h^{2s}(\|\boldsymbol{u}\|_{s,\mathcal{T}_h} + \|\mathbb{L}\|_{s,\mathcal{T}_h})
\end{aligned}
\tag{5.24}
$$

Inserting the estimates above (5.21), (5.22), (5.23) and (5.24) into (5.20), by Cauchy-Schwarz inequality and trace inequality, we get

$$\begin{aligned}\|\mathbb{E}_{\mathbb{L}}\|_{\mathcal{T}_h}^2 + \|\kappa^{-\frac{1}{2}}\boldsymbol{e}_u\|_{\mathcal{T}_h}^2 + \|\tau^{\frac{1}{2}}(\boldsymbol{e}_u - \widehat{\boldsymbol{e}_u})\|_{\partial\mathcal{T}_h}^2 &\lesssim \|\delta_{\mathbb{L}}\|_{\mathcal{T}_h}^2 + h^{-1}\|\kappa^{-1}\|_\Omega\|\boldsymbol{\delta}_u\|_{\mathcal{T}_h}^2 \\ &+ \|\boldsymbol{P}_h^\partial\boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\|_{\partial\mathcal{T}_H}^2 + h^{-1}(1+H\|\kappa^{-1}\|_\Omega)\|\boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\|_{\partial\mathcal{T}_H}^2 \\ &+ h^{2s-1}\|\boldsymbol{u}\|_{\mathcal{T}_h}^2 + \|\kappa^{-1}\|_\Omega^2 h^{2s}(\|\boldsymbol{u}\|_{s,\mathcal{T}_h} + \|\mathbb{L}\|_{s,\mathcal{T}_h}).\end{aligned}$$

Finally, by the approximation property of the projections (5.2) with some algebraic manipulation, we can obtain the estimate shown in the lemma. This completes the proof. ☐

In combination with Lemma 5.5, Theorem 5.2 and Theorem 5.3, we can derive the pressure estimate.

Theorem 5.4 (Estimate for e_p)**.** *Let the assumptions in Theorem 5.3 be fulfilled.*

1. *if Assumption 5.2 holds. We have*

$$\begin{aligned}\|e_p\|_{\mathcal{T}_h} &\lesssim (1+\|\kappa^{-1}\|_\Omega)(h^{-\frac{1}{2}} + \|\kappa^{-1}\|_\Omega^{\frac{1}{2}})\|\kappa^{-1}\|_\Omega^{\frac{1}{2}} h^s(\|\mathbb{L}\|_{s,\mathcal{T}_h} + \|\boldsymbol{u}\|_{s,\mathcal{T}_h}) \\ &+ (1+\|\kappa^{-1}\|_\Omega)(1+H^{\frac{1}{2}}\|\kappa^{-1}\|_\Omega^{\frac{1}{2}})h^{-\frac{1}{2}}\|\boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\|_{\partial\mathcal{T}_H}.\end{aligned}$$

2. *If Assumption 5.3 holds, we can obtain*

$$\begin{aligned}\|e_p\|_{\mathcal{T}_h} &\lesssim \Big(\sqrt{\frac{H}{h}}(1+H^{\frac{1}{2}}\|\kappa^{-1}\|_\Omega^{\frac{1}{2}}) + \|\kappa^{-1}\|_\Omega\Big)(h^{-\frac{1}{2}} + \|\kappa^{-1}\|_\Omega^{\frac{1}{2}})\|\kappa^{-1}\|_\Omega^{\frac{1}{2}} h^s \\ &\times(\|\mathbb{L}\|_{s,\mathcal{T}_h} + \|\boldsymbol{u}\|_{s,\mathcal{T}_h}) + H^{\frac{1}{2}}h^{s-\frac{1}{2}}\|p\|_{s,\mathcal{T}_h} \\ &+ (\|\kappa^{-1}\|_\Omega + \sqrt{\frac{H}{h}})(1+H^{\frac{1}{2}}\|\kappa^{-1}\|_\Omega^{\frac{1}{2}})h^{-1}\|\boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\|_{\partial\mathcal{T}_H}.\end{aligned}$$

Proof. An application of Theorem 5.3 and Lemma 5.5 leads to

$$\begin{aligned}\|e_p - (e_p)\|_{\mathcal{T}_h} &\lesssim (1+H^{\frac{1}{2}}\|\kappa^{-1}\|_\Omega^{1/2})(h^{-\frac{1}{2}} + \|\kappa^{-1}\|_\Omega^{\frac{1}{2}})\|\kappa^{-1}\|_\Omega^{\frac{1}{2}} h^s\ (\|\boldsymbol{u}\|_{s,\mathcal{T}_h} + \|\mathbb{L}\|_{s,\mathcal{T}_h}) \\ &+ (1+H\|\kappa^{-1}\|_\Omega)h^{-\frac{1}{2}}\|\boldsymbol{u} - \boldsymbol{P}_H^\partial\boldsymbol{u}\|_{\partial\mathcal{T}_H}.\end{aligned}$$

An application of Theorem 5.2 shows the results. ☐

Finally the main result Theorem 5.1 can be obtained by triangle inequality, Theorem 5.3 and Theorem 5.4 and the approximation properties of the projections (5.2).

6 Numerical results

In our experiments, we take the domain $\Omega = [0,1]\times[0,1]$, $\nu = 1$, $\mathrm{Re} = \frac{1}{\nu}$, $\lambda = \mathrm{Re}/2 - \sqrt{\mathrm{Re}^2/4 + 4\pi^2}$ and the source term $\boldsymbol{f}$ and boundary data $\boldsymbol{g}$ are

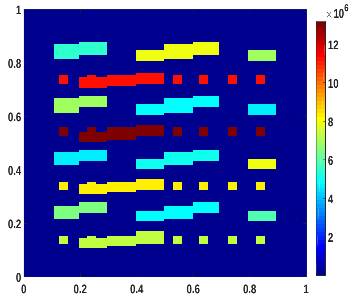
$$\boldsymbol{f} = \begin{pmatrix}\nu(\lambda^2 - (2\pi)^2)\mathrm{e}^{\lambda x}\cos(2\pi y) + \lambda\mathrm{e}^{2\lambda x} \\ \nu(2\pi\lambda - \lambda^3/(2\pi))\mathrm{e}^{\lambda x}\sin(2\pi y)\end{pmatrix} \qquad \boldsymbol{g} = \begin{pmatrix}1 - \mathrm{e}^{\lambda x}\cos(2\pi y) \\ \lambda/(2\pi)\mathrm{e}^{\lambda x}\sin(2\pi y)\end{pmatrix}.$$

Let $\mathcal{T}_H$ be a regular quasi-uniform rectangular mesh over Ω with maximal mesh size H and let $\mathcal{T}_h$ be a regular quasi-uniform triangulation over each element $T \in \mathcal{T}_H$ with maximal mesh size h. Take $h = \sqrt{2}\times 2^{-7}$. Note that the fine scale h can resolve the coefficient κ^{-1}. The regularization parameter $\tau := 1$. For $K \in \mathcal{T}_h$, $F_h \in \mathcal{E}_h$ and $F_H \in \mathcal{E}_H$, the local fine element spaces are

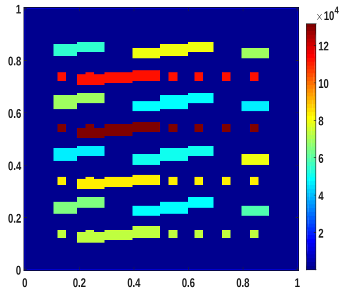
$$\begin{aligned}&\mathbb{G}(K) := P_1^4(K) &&\boldsymbol{V}(K) := P_1^2(K) &&P(K) := P_1(K)\\ &\boldsymbol{M}_h(F_h) := P_1^2(F_h) &&\boldsymbol{M}_H(F_H) := P_2^2(F_H).\end{aligned}$$

Four numerical results are shown with coefficient κ^{-1} of different contrast values (i.e. $\frac{\max\kappa^{-1}}{\min\kappa^{-1}}$), which are depicted in Figure 1. Their contrast values are of orders $1e7$, $1e5$, $1e3$ and $1e1$, respectively. The coefficient κ^{-1} takes low values in the background, which represents the fast flow region.

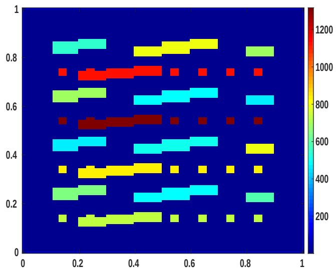
The relative errors of the gradient velocity ($\mathbb{L}$), velocity ($\boldsymbol{u}$) and pressure (p) are depicted in Table 6 for $H := \sqrt{2}2^{-2}, \sqrt{2}2^{-3}, \sqrt{2}2^{-4}$ and $\sqrt{2}2^{-5}$, respectively. Theorem 5.3 implies a convergent rate of $3/2$ for this example. One can observe from Table 6 that the convergent rate improves as the contrast in the coefficent κ^{-1} decreases. The convergent rate for coefficient in Figure 1(a) is slowest that equals 1.01, 1.32 and 1.88 for $\mathbb{L}$, $\boldsymbol{u}$ and p, respectively. The convergent rate in Figure 1(d) is the fastest with rate of 1.69, 1.42 and 1.83 for $\mathbb{L}$, $\boldsymbol{u}$ and p, respectively.



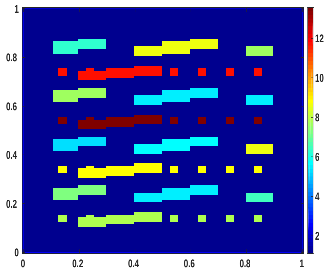
(a) Permeability field 1



(b) Permeability field 2



(c) Permeability field 3



(d) Permeability field 4

Figure 1: Permeability fields

	Fig.1(a)			Fig. 1(b)		
H	$\mathbb{L}$	$\boldsymbol{u}$	p	$\mathbb{L}$	$\boldsymbol{u}$	p
$\sqrt{2}2^{-2}$	180.59	91.95	513.27	79.26	97.85	104.27
$\sqrt{2}2^{-3}$	70.91	78.27	73.61	50.51	45.33	27.13
$\sqrt{2}2^{-4}$	38.75	26.47	24.94	23.20	15.20	5.55
$\sqrt{2}2^{-5}$	17.39	12.61	5.43	11.08	5.66	2.24
	Fig.1(c)			Fig. 1(d)		
H	$\mathbb{L}$	$\boldsymbol{u}$	p	$\mathbb{L}$	$\boldsymbol{u}$	p
$\sqrt{2}2^{-2}$	43.94	7.31	7.74	7.84	0.39	28.47
$\sqrt{2}2^{-3}$	33.10	3.41	4.67	1.77	0.05	5.55
$\sqrt{2}2^{-4}$	14.11	0.76	1.60	0.47	0.01	1.28
$\sqrt{2}2^{-5}$	5.32	0.16	0.52	0.17	0.007	0.44

Table 1: Convergence history in percentage (%)

7 Conclusion

In this paper, we have proposed new upscaled HDG methods for Brinkman equations with high contrast coefficient. The main goal is to solve Brinkman equations on the coarse mesh where the coefficient is not well resolved. Nevertheless, the numerical solution can achieve a certain convergence rate measured by the coarse-scale mesh size. To this end, on top of the classical HDG finite element space, an extra coarse-scale space over the skeleton of the fine mesh is designed. In application, this extra space is arbitrary and can be the reduced space or the multiscale space. The convergence of this upscaled HDG method relies on the approximation property of this extra space. Furthermore, the convergence is proved which depends weakly on the heterogeneous high-contrast coefficient. Numerical results are shown to verify our theoretical findings. Future work related to this project is to apply this framework to more complicated heterogeneous permeability field which might involve more scales and difficult geometry. To this end, we may consider to use offline spectral decomposition to construct the coarse space M_H.

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