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INS Preprint No. 1810

September 2018

Haar system as Schauder basis in Besov spaces: The limiting cases for $0 < p \le 1$

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Abstract

We show that the d-dimensional Haar system H^d on the unit cube I^d is a Schauder basis in the classical Besov space $\mathbf{B}^s_{p,q,1}(I^d)$, $0 < p < 1$, defined by first order differences in the limiting case $s = d(1/p - 1)$, if and only if $0 < q \le p$. For $d = 1$ and $p < q < \infty$, this settles the only open case in our 1979 paper [4], where the Schauder basis property of H in $\mathbf{B}^s_{p,q,1}(I)$ for $0 < p < 1$ was left undecided. We also consider the Schauder basis property of H^d for the standard Besov spaces $B^s_{p,q}(I^d)$ defined by Fourier-analytic methods in the limiting cases $s = d(1/p-1)$ and $s = 1$, complementing results by Triebel [7].

Keywords: Haar system, Besov spaces, Schauder bases in quasi-Banach spaces, spline approximation.
2000 MSC: 42C15, 42C40, 46E35, 41A15

1. Introduction

The classical univariate Haar system $H := \{h_m\}_{m\in\mathbb{N}}$ was one of the first examples of a Schauder basis in some classical function spaces on the unit interval $I := [0, 1]$. In this note, we deal with various Besov spaces $B^s_{p,q}$ on the unit cube $I^d \subset \mathbb{R}^d$ for the parameter range

$$0 < p \le 1, \qquad 0 < q < \infty, \qquad s > 0, \tag{1}$$

and complement early results by Triebel [7] and this author [4] by settling the remaining limiting cases, where the Schauder basis property of the multivariate Haar system H^d was not known until now (for detailed definitions, we refer to the following sections).

There are many alternative definitions (Fourier-analytic, local means, atoms, approximations, differences, ...) that may lead to different Besov spaces for certain parts of the parameter range (1), see, e.g., [8] for a brief introduction to function spaces of Besov-Hardy-Sobolev spaces on $\mathbb{R}^d$ and on domains. We consider the by now standard Besov spaces $B^s_{p,q}(I^d)$ of distributions defined in terms of Littlewood-Paley type norms (or equivalently, in terms of atomic decompositions or local means), and the classical Besov spaces $\mathbf{B}^s_{p,q,1}(I^d)$ of functions defined by first-order differences (or, equivalently,

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by best approximations with dyadic step functions). In the parameter range (1), these two scales of Besov spaces coincide up to equivalent quasi-norms if and only if

$$d/(d+1) < p \leq 1, \quad 0 < q < \infty, \quad d(1/p - 1) < s < 1. \tag{2}$$

In [7] it was proved that H^d forms a Schauder basis in $B^s_{p,q}(I^d)$ in the parameter range (2), see also [8, Section 1.7.2] and [9, Section 2.5.1], where additionally the unconditionality of the Haar basis was established. Moreover, it was also shown that the Haar system cannot be a Schauder basis in $B^s_{p,q}(I^d)$, $0 < p \leq 1$, $0 < q < \infty$, if either $s < d(1/p - 1)$ or $s > 1$. Recently, there has been renewed interest in investigating low-order spline wavelet systems such as the Haar and Faber-Schauder systems and their multivariate counterparts as bases in Besov-Hardy-Sobolev spaces on $\mathbb{R}^d$ and I^d. We refer e.g. to [9, 10, 3] and the many references cited therein. However, for $B^s_{p,q}(I^d)$ the limiting cases $s = d(1/p - 1)$ and $s = 1$, which were not settled in [7], are still open. We also mention the recent paper [11] directly related to this note, where the authors study necessary and sufficient conditions on the parameters p, q, s, τ for which the map $f \to (f, \chi_{I^d})_{L_2} = \int_{I^d} f\,dx$ extends to a bounded linear functional on Besov-Morrey-Campanato-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$.

As to Besov spaces defined by differences, in [4] it was shown that for $0 < p < 1$ the univariate Haar system H is a Schauder basis in $\mathbf{B}^s_{p,q,1}(I)$ if $1/p - 1 < s < 1/p$ and $0 < q < \infty$. If $0 < s < 1/p - 1$, then $\mathbf{B}^s_{p,q,1}(I)$ has a trivial dual space, and thus cannot possess a Schauder basis. For $s \geq 1/p$, $0 < q < \infty$, the spaces $\mathbf{B}^s_{p,q,1}(I)$ degenerate to containing only constant functions. In the only remaining limiting case $s = 1/p - 1$, the proof in [4] established the Schauder basis property of the Haar system H in $\mathbf{B}^s_{p,q,1}(I)$ also for $0 < q \leq p$ while for the parameter range $p < q < \infty$ the question was left open.

Our main goal in this paper is to settle the limiting cases for both scales of Besov spaces. In Section 2 we will prove the following:

Theorem 1 *Let $d = 1, 2, \ldots$, and let p, q, s satisfy (1). The Haar system H^d (equipped with any of its natural enumerations) is a Schauder basis in the Besov space $\mathbf{B}^s_{p,q,1}(I^d)$ if either*

$$d(1/p - 1) < s < 1/p, \quad (d-1)/d < p \leq 1, \quad 0 < q < \infty. \tag{3}$$

or if

$$s = d(1/p - 1), \quad (d-1)/d < p < 1, \quad 0 < q \leq p. \tag{4}$$

In all other cases, H^d cannot be a Schauder basis in $\mathbf{B}^s_{p,q,1}(I^d)$. More precisely:

i) Let $0 < q < \infty$. If $0 < s < \min(d(1/p - 1), 1/p)$ then $\mathbf{B}^s_{p,q,1}(I^d)$ has a trivial dual, while for $s \geq 1/p$ it degenerates to containing only constant functions.

ii) If $s = d(1/p - 1)$, $(d-1)/d < p < 1$, then we have two cases:

a) If $1 < q < \infty$ then the coefficient functionals of the Haar expansion which are uniquely defined on $\mathrm{span}(H^d)$ *cannot be extended to bounded linear functionals on* $\mathbf{B}^{d(1/p-1)}_{p,q,1}(I^d)$.

b) If $p < q \leq 1$ then the partial sum operators of the Haar expansion are not uniformly bounded on $\mathbf{B}^{d(1/p-1)}_{p,q,1}(I^d)$.

For $d = 1$, the statement of Theorem 1 except for part ii) has been established in [4] using characterizations of $\mathbf{B}^s_{p,q,1}(I)$ in terms of best approximations by dyadic step functions. This approach carries over to the case $d > 1$. The assertions in part ii) are new, and follow from modifying the univariate examples used in [4] (see the lemma on p. 535 there).

As will be clear from the proofs, the formulation of Theorem 1 carries over to the Haar system on $\mathbb{R}^d$ and the Besov spaces $\mathbf{B}^s_{p,q,1}(\mathbb{R}^d)$ without change. Similar results are expected to hold for Besov spaces $\mathbf{B}^{d(1/p-1)}_{p,q,r}(I^d)$ defined in terms of r-th order differences, $r > 1$, and multivariate spline systems of higher order, such as the Franklin system. As to assertion ii) a), we do not know whether $\mathbf{B}^{d(1/p-1)}_{p,q,1}(I^d)$ has a nontrivial dual for $q > 1$ at all.

In Section 3, we deal with the standard Besov spaces $B^s_{p,q}(I^d)$ and use their characterizations in terms of atomic decompositions and local means to prove the following result.

Theorem 2 *Let $d = 1, 2, \ldots$, and let p, q, s satisfy (1), where additionally $d(1/p - 1) \leq s \leq 1$. The Haar system H^d is a Schauder basis in the Besov space $B^s_{p,q}(I^d)$ if either (2) or*

$$s = d(1/p - 1), \quad d/(d + 1) < p < 1, \quad 0 < q \leq p. \tag{5}$$

holds. In all other cases, H^d cannot be a Schauder basis in $B^s_{p,q}(I^d)$. In particular,

i) If $s = 1$, $d/(d + 1) \leq p < 1$, $0 < q < \infty$, then the Haar expansion of the smooth function $f(x) = x_1 + \ldots + x_d$ does not converge to f in $B^s_{p,q}(I^d)$

ii) If $s = d(1/p - 1)$, $d/(d + 1) \leq p < 1$, $p < q < \infty$, we have again two cases.

a) If $1 < q < \infty$, then the coefficient functionals of the Haar expansion which are uniquely defined on $\mathrm{span}(H^d)$ *cannot be extended to bounded linear functionals on* $B^{d(1/p-1)}_{p,q}(I^d)$.

b) If $p < q \leq 1$, then the partial sum operators of the Haar expansion are not uniformly bounded on $B^{d(1/p-1)}_{p,q}(I^d)$.

Compared to [7] only the proof of the Schauder basis property for the parameter range (5), the limiting case $s = 1$ in part i), and part ii) of Theorem 2 are new. The theorem holds for the suitably enumerated Haar system on $\mathbb{R}^d$ and the spaces $B^s_{p,q}(\mathbb{R}^d)$ without changes in the formulation. The result of case a) in part (ii) is also covered by [11, Corollary 2.7, (ii)].

Acknowledgments. This research grew out of discussions with T. Ullrich who shared the preprint version of the paper [3] with me, and encouraged me to reconsider the open problems in [4]. T. Ullrich also pointed out the counterexamples for part ii) in Theorem 2 which are reproduced here with his permission. His contributions and interest are gratefully acknowledged. We also thank W. Sickel who made an early version of [11] available to us.

The work on this paper started during a recent stay at the Institute for Numerical Simulation (INS) of the University of Bonn sponsored by the Hausdorff Center for Mathematics and the Deutsche Forschungsgemeinschaft (DFG). I thank my colleagues at the INS for the fruitful and friendly atmosphere, and the above named institutions for their support.

2. Proof of Theorem 1

2.1. Definitions and preparations

Recall first the definition of the L_2-normalized Haar functions. By χ_Ω we denote the characteristic function of a Lebesgue measurable set $\Omega \subset \mathbf{R}^d$, and by $\Delta_{k,i} = [(i-1)2^{-k}, i2^{-k})$, $i = 1, \ldots, 2^k$, the univariate dyadic intervals of length 2^{-k}, $k = 0, 1, \ldots$. Then the univariate Haar system $H = \{h_m\}_{m\in\mathbb{N}}$ on I is given by $h_1 = \chi_I$, and

$$h_{2^{k-1}+i} = 2^{(k-1)/2}(\chi_{\Delta_{k,2i-1}} - \chi_{\Delta_{k,2i}}), \qquad i = 1, \ldots, 2^{k-1}, \quad k \in \mathbb{N}.$$

As is well known, the Haar functions h_m with $m \geq 2$ can also be indexed by their supports, and identified with the appropriately scaled shifts and dilates of a single function, the centered Haar wavelet

$$h_0 := \chi_{[-1/2,0]} - \chi_{[0,1/2]}. \tag{6}$$

Indeed,

$$h_{\Delta_{k-1,i}} := h_{2^{k-1}+i} = |\Delta_{k-1,i}|^{-1/2} h_0(2^{k-1} \cdot -i + 1/2)$$

for $i = 1, \ldots, 2^{k-1}$, and $k \in \mathbb{N}$. The above introduced enumeration of the Haar functions h_m is the natural ordering used in the literature, however, one can also define H as the union of dyadic blocks

$$H = \cup_{k=0}^{\infty} H_k, \qquad H_0 = \{h_1\}, \quad H_k = \{h_{\Delta_{k-1,i}} : i = 1, \ldots, 2^{k-1}\}, \quad k \in \mathbb{N},$$

and allow for arbitrary orderings within each block H_k. Below, we will work with the multivariate counterparts of the spaces

$$S_k = \mathrm{span}(\{h_m\}_{m=1}^{2^k}) = \mathrm{span}(\{\chi_{\Delta_{k,i}}\}_{i=1}^{2^k}), \qquad k = 0, 1, \ldots,$$

of piecewise constant functions with respect to the uniform dyadic partition $T_k = \{\Delta_{k,i} : i = 1, \ldots, 2^k\}$ of step-size 2^{-k} on the unit interval I, which we call for short dyadic step functions of level k.

Consider now the isotropic multivariate Haar system H^d on the d-dimensional cube I^d, $d > 1$, which we define in a blockwise fashion as follows. Let the partition T_k^d be the set of all dyadic cubes of side-length 2^{-k} in I^d. Each cube in T_k^d is the d-fold product of univariate $\Delta_{k,i}$, i.e.,

$$T_k^d = \{\Delta_{k,\mathbf{i}} := \Delta_{k,i_1} \times \ldots \times \Delta_{k,i_d} : \mathbf{i} = (i_1, \ldots, i_d) \in \{1, \ldots, 2^k\}^d\}.$$

The set of all piecewise constant functions on T_k^d is denoted by S_k^d. With each $\Delta_{k-1,\mathbf{i}} \in T_{k-1}^d$, $\mathbf{i} \in \{1, \ldots, 2^{k-1}\}^d$, $k \in \mathbb{N}$, we can associate a set $H_{k,\mathbf{i}}^d \subset S_k^d$ of $2^d - 1$ multivariate Haar functions with support $\Delta_{k-1,\mathbf{i}}$, given by all possible tensor products

$$\psi_{k,i_1}(x_1) \cdot \psi_{k,i_2}(x_2) \cdot \ldots \cdot \psi_{k,i_d}(x_d), \qquad \psi_{k,i} = h_{\Delta_{k-1,i}} \text{ or } 2^{(k-1)/2}\chi_{\Delta_{k-1,i}}$$

where at least one of the $\psi_{k,i}$ equals $h_{\Delta_{k-1,i}}$.

The blocks H_k^d that define the d-dimensional Haar system

$$H^d = \cup_{k=0}^{\infty} H_k^d$$

are given as follows: The block H_0^d is exceptional, and consists of the single constant function χ_{I^d}. The block H_1^d coincides with $H_{1,\mathbf{1}}^d$ and consists of $2^d - 1$ Haar functions (we use the notation $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{2} = (2, \ldots, 2)$, etc.). For general $k \geq 2$, the block

$$H_k^d := \cup_{\Delta_{k-1,\mathbf{i}} \in T_{k-1}^d} H_{k,\mathbf{i}}^d$$

consists of $(2^d - 1)2^{(k-1)d}$ Haar functions which we call Haar functions of level k. It is obvious that

$$S_k^d = \mathrm{span}(\cup_{l=0}^{k} H_l^d),$$

and that H^d is a complete orthonormal system in $L_2(I^d)$. Since each Haar function in H^d has support on a d-dimensional cube, we call this system isotropic Haar system (in contrast to the d-dimensional tensor-product Haar system, where the supports of the Haar functions are d-dimensional dyadic rectangles). As mentioned before for the univariate case, the ordering of the multivariate Haar functions within the blocks H_k^d can be arbitrary. The statements of Theorems 1 and 2 hold for any enumeration of H^d as long as the enumeration does not violate the natural ordering by level k.

The Besov spaces $\mathbf{B}_{p,q,1}^s(I^d)$ considered in this section are defined for $s > 0$, $0 < p, q \leq \infty$, as the set of all (equivalence classes of) Lebesgue measurable functions $f : I^d \to \mathbb{R}$ for which the quasi-norm

$$\|f\|_{\mathbf{B}_{p,q,1}^s} := \|f\|_{L_p(I^d)} + \|t^{-s-1/q}\omega(t, f)_p\|_{L_q(I)}$$

is finite. Here,

$$\omega(t, f)_p := \sup_{0<|y|\leq t} \|\Delta_y f\|_{L_p(I_y^d)}, \qquad t > 0,$$

stands for the first-order L_p modulus of smoothness, where

$$\Delta_y f(x) := f(x + y) - f(x), \qquad x \in I_y^d := \{z \in I^d : z + y \in I^d\}, \quad y \in \mathbb{R}^d,$$

denotes the first-order forward difference. Here and throughout the paper, we adopt the following notational convention: If the domain is I^d, we omit the domain in the quasi-norm notation, e.g., we write $\|\cdot\|_{L_p}$ instead of $\|\cdot\|_{L_p(I^d)}$. Also, by c, C we denote generic positive constants that may change from line to line, and, unless stated otherwise, depend on p, q, s only. The notation $A \approx B$ is used if $cA \le B \le CA$ holds for two such constants c, C.

For the case $0 < p < 1$, $0 < q < \infty$ we are interested in, $\mathbf{B}^s_{p,q,1}(I^d)$ is a quasi-Banach space equipped with a γ-quasi-norm, where $\gamma = \min(p, q)$, meaning that $\|\cdot\|_{\mathbf{B}^s_{p,q,1}}$ is homogeneous and satisfies

$$\|f+g\|^\gamma_{\mathbf{B}^s_{p,q,1}} \le \|f\|^\gamma_{\mathbf{B}^s_{p,q,1}} + \|g\|^\gamma_{\mathbf{B}^s_{p,q,1}}.$$

Similarly, the L_p quasi-norm is a p-quasi-norm if $0 < p \le 1$. All spaces introduced in the sequel have γ-quasi-norms for some suitable $\gamma \in (0, 1]$.

For the parameter region (1), the spaces are nontrivial only if $s < 1/p$. Indeed, if $f \in \mathbf{B}^s_{p,q,1}(I^d)$ for some $s \ge 1/p$ then using the properties of the first-order L_p modulus of smoothness we have $\omega(t, f)_p = \mathrm{o}(t^{1/p})$, $t \to 0$, which in turn implies $\omega(t, f)_p = 0$ for all $t > 0$ and $f(x) = \xi$ for some constant $\xi \in \mathbb{R}$ almost everywhere on I^d. From now on, we can therefore restrict ourselves to $0 < s < 1/p$ in (1).

In this section we will exclusively work with an equivalent quasi-norm based on approximation techniques using piecewise constant approximation on dyadic partitions. Let

$$E_k(f)_p := \inf_{s \in S^d_k} \|f - s\|_{L_p}, \qquad k = 0, 1, \ldots,$$

denote the best approximations to $f \in L_p(I^d)$ with respect to S_k. From [5, Theorem 6] for $d = 1$, and [1, Theorem 5.1] for $d > 1$ we have that

$$\|f\|_{\mathbf{A}^s_{p,q,1}} := \|f\|_{L_p} + (\sum_{k=0}^{\infty} (2^{ks} E_k(f)_p)^q)^{1/q} \tag{7}$$

provides an equivalent quasi-norm on $\mathbf{B}^s_{p,q,1}(I^d)$ for all $0 < p < 1$, $0 < q < \infty$, $0 < s < 1/p$. This norm equivalence automatically implies that the set of all dyadic step functions

$$S^d := \mathrm{span}(H^d) = \mathrm{span}(\{S^d_k\}_{k=0}^{\infty}) \tag{8}$$

is dense in $\mathbf{B}^s_{p,q,1}(I^d)$ for all those parameter values. Note that in [1] the case $1 \le s < 1/p$ is formally excluded in the formulations but the proofs in [1] extend to this parameter range as well.

At the heart of the counterexamples used for the proof of Theorem 1 is the following simple observation which we formulate as

Lemma 1 *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, and $f \in L_p(\Omega) := L_p(\Omega, \mathcal{A}, \mu)$, $0 < p \le 1$, be supported on $\Omega' \in \mathcal{A}$, where $\mu(\Omega') \le \frac{1}{2}\mu(\Omega)$. Then*

$$\|f\|_{L_p(\Omega)} = \inf_{\xi \in \mathbb{R}} \|f - \xi\|_{L_p(\Omega)},$$

i.e., best approximation by constants in $L_p(\Omega)$ is achieved by setting $\xi = 0$.

Proof. Indeed, under the above assumptions and by the inequality $|a+b|^p \le |a|^p + |b|^p$ we have

$$\begin{aligned}\|f-\xi\|^p_{L_p(\Omega)} &= \int_{\Omega'} |f(x)-\xi|^p\,d\mu(x) + \mu(\Omega\backslash\Omega')|\xi|^p \ge \int_{\Omega'} (|f(x)-\xi|^p + |\xi|^p)\,d\mu(x) \\ &\ge \int_{\Omega'} |f(x)|^p\,d\mu(x) = \|f\|^p_{L_p(\Omega)}\end{aligned}$$

for any $\xi \in \mathbb{R}$, with equality for $\xi = 0$. This gives the statement. □

Note that the equivalence (up to constants depending on parameters but not on f) between L_p quasi-norms and best approximations by constants holds also for $p \ge 1$ and under weaker assumptions on the support size of f (e.g., $\mu(\Omega')/\mu(\Omega) \le \delta < 1$ would suffice). We will apply this lemma to our examples of dyadic step functions constructed below, and to the Lebesgue measure on dyadic cubes in I^d, where the step functions are not constant. Extensions to higher degree polynomial and spline approximation are possible as well (see the proof of the Lemma on p. 535 in [4] for $d = 1$).

If H^d is a Schauder basis in a quasi-Banach space X of functions or distributions defined on I^d then necessarily $S^d = \mathrm{span}(H^d) \subset X$ and any dyadic step function $g \in S^d$ has a unique Haar expansion given by

$$g(x) = \sum_{h\in H^d} c_h(g)h(x), \qquad c_h(g) := \int_{I^d} gh\,dx. \tag{9}$$

Since for $g \in S^d$ only finitely many coefficients $c_h(g)$ do not vanish, the summation in (9) is finite, and there are no convergence issues. Thus, for the Schauder basis property of H^d in X to hold, the coefficient functionals $c_h(g)$ must be extendable to elements in X', and the level k partial sum operators

$$(P_k g)(x) = \sum_{l=0}^{k} \sum_{h\in H^d_l} c_h(g)h(x), \qquad k = 0,1,\ldots, \tag{10}$$

must be extendable to *uniformly* bounded linear operators in X. When applied to the case $X = \mathbf{B}^s_{p,q,1}(I^d)$ considered in this section, this explains that the statements in Theorem 1 i)-ii) imply the failure of the Schauder basis property of H^d in $\mathbf{B}^s_{p,q,1}(I^d)$ for the associated parameter ranges. The same is true for $X = B^s_{p,q}(I^d)$ and Theorem 2 considered in Section 3.

For $X = L_1(I^d)$, P_k extends to a bounded projection on $L_1(I^d)$ with range S^d_k, and with constant values on the dyadic cubes in T^d_k explicitly given by averaging. This comes in handy when computing $P_k f$ for concrete functions f. Indeed, the constant values taken by $P_k f$ on dyadic intervals in T^d_k are given by

$$P_k f(x) = av_\Delta(f), \qquad x \in \Delta, \quad \Delta \in T^d_k, \qquad f \in L_1(I^d), \tag{11}$$

where the functionals

$$av_\Delta(f) := 2^{kd}\int_{I^d} \chi_\Delta f\,dx = 2^{kd}\int_\Delta f\,dx, \qquad \Delta \in T^d_k, \quad k = 0,1,\ldots, \tag{12}$$

represent the average value of f on dyadic cubes (we will call them for short average functionals). Note that coefficient functionals c_h are finite linear combinations of average functionals as defined in (12), vice versa. Finally, for $X = L_2(I^d) \subset L_1(I^d)$ the level k partial sum operator P_k realize the orthoprojection onto S_k^d.

2.2. Proof of Theorem 1: Positive results

For $d = 1$, the cases in Theorem 1, where the Schauder basis property can be established, are covered by [4]. The proof for $d > 1$ does not hold surprises, we give it for completeness. By density of $S^d = \mathrm{span}(H^d)$ in the quasi-Banach space $\mathbf{B}^s_{p,q,1}(I^d)$, it is sufficient to establish the inequality

$$\|Pg\|_{\mathbf{A}^s_{p,q,1}} \le C\|g\|_{\mathbf{A}^s_{p,q,1}}, \qquad g \in S^d, \tag{13}$$

for any partial sum operator P of the Haar expansion (9), with a constant C independent of g and P, for the parameters satisfying (3) or (4). According to our ordering convention for H^d, any such partial sum operator P can be written, for some $k = 0, 1, \ldots$ and some subset $\bar{H}^d_{k+1} \subset H^d_{k+1}$, in the form

$$Pg = P_k g + \sum_{h \in \bar{H}^d_{k+1}} c_h(g) h \in S^d_{k+1}. \tag{14}$$

For $\bar{H}^d_{k+1} = \emptyset$, we get $P = P_k$ as partial case.

The first step for establishing (13) is the proof of the inequality

$$\|Pg\|^p_{L_p} \le C 2^{kd(p-1)} \sum_{\Delta \in T^d_k} \|g\|^p_{L_1(\Delta)}, \tag{15}$$

with the explicit constant $C = 2^d$. By (11) and (12), we have

$$\|P_k g\|^p_{L_p} = \sum_{\Delta \in T^d_k} 2^{-kd} \left(2^{kd} \int_\Delta g\, dx \right)^p \le 2^{kd(p-1)} \sum_{\Delta \in T^d_k} \|g\|^p_{L_1(\Delta)}.$$

The remaining $h \in \bar{H}^d_{k+1}$ can be grouped by their supports (which are dyadic cubes $\Delta \in T^d_k$ by construction), where each group may hold up to $2^d - 1$ Haar functions with the same $\mathrm{supp}(h) = \Delta \in T^d_k$. Individually, by the definition of the Haar coefficients $c_h(g)$ and the scaling $\|h\|_{L_\infty(\Delta)} = 2^{kd/2}$ of the Haar functions in H^d_{k+1}, we obtain for each term associated with a Haar function in such a group the estimate

$$\|c_h(g)h\|^p_{L_p} = |c_h(g)|^p \|h\|^p_{L_p(\Delta)} \le 2^{kdp/2} \|g\|^p_{L_1(\Delta)} \cdot 2^{-kd} 2^{kdp/2} = 2^{kd(p-1)} \|g\|^p_{L_1(\Delta)}.$$

Thus, applying the p-quasi-norm triangle inequality for $L_p(I^d)$ in the appropriate way, we obtain (15).

Now we apply the embedding theorem $\mathbf{B}^{d(1/p-1)}_{p,p,1}(I^d) \subset L_1(I^d)$, with the appropriate coordinate transformation, to the terms $\|g\|^p_{L_1(\Delta)}$ (the stronger, optimal embedding $\mathbf{B}^{d(1/p-1)}_{p,1,1}(I^d) \subset L_1(I^d)$ is covered by [1, Theorem 7.1]). This gives

$$\|g\|^p_{L_1(\Delta)} \leq C2^{kd(1-p)}(\|g\|^p_{L_p(\Delta)} + \sum_{l=0}^{\infty} 2^{ld(1-p)} E_{k+l}(g)^p_{p,\Delta})$$

for each $\Delta \in T^d_k$, where

$$E_{k+l}(g)_{p,\Delta} := \inf_{s \in S^d_{k+l}} \|g - s\|_{L_p(\Delta)}, \qquad l = 0, 1, \ldots,$$

denotes the local best L_p approximation by dyadic step functions restricted to cubes Δ from T^d_k. Since

$$\|g\|^p_{L_p} = \sum_{\Delta \in T^d_k} \|g\|^p_{L_p(\Delta)}, \qquad E_{k+l}(g)^p_p = \sum_{\Delta \in T^d_k} E_{k+l}(g)^p_{p,\Delta}, \quad l = 0, 1, \ldots,$$

after substitution into (15), we arrive at the estimate

$$\|Pg\|^p_{L_p} \leq C(\|g\|^p_{L_p} + 2^{kd(p-1)} \sum_{l=k}^{\infty} 2^{ld(1-p)} E_l(g)^p_p) \tag{16}$$

for the L_p quasi-norm of any partial sum Pg.

With the auxiliary estimate (16) at hand, we turn now to the estimate of the Besov quasi-norm of $g - Pg$. Since $Pg \in S^d_{k+1}$, we have

$$E_l(g - Pg)_p = E_l(g)_p, \qquad l > k,$$

while for $l \leq k$ the trivial bound

$$E_l(g - Pg)_p \leq \|g - Pg\|_{L_p(I^d)}$$

will suffice. This gives

$$\begin{aligned} \|g - Pg\|_{\mathbf{A}^s_{p,q,1}} &= \|p - Pg\|_{L_p} + (\sum_{l=0}^{\infty} (2^{ls} E_l(g - Pg)_p)^q)^{1/q} \\ &\leq C \left(2^{ks} \|g - Pg\|_{L_p} + (\sum_{l=k+1}^{\infty} (2^{ls} E_l(g)_p)^q)^{1/q} \right), \end{aligned} \tag{17}$$

uniformly for all P and $g \in S^d$.

To deal with the term $\|g - Pg\|_{L_p}$, we introduce the element $s_k \in S^d_k$ of best L_p approximation, i.e.,

$$\|g - s_k\|_{L_p} = E_k(g)_p,$$

and estimate with (16) and $Ps_k = P_k s_k = s_k$ as follows:

$$\begin{aligned}\|g-Pg\|_{L_p}^p &\le \|g-s_k\|_{L_p}^p + \|P(g-s_k)\|_{L_p}^p \\ &\le \|g-s_k\|_{L_p}^p + C(\|g-s_k\|_{L_p(I^d)}^p + 2^{kd(p-1)}\sum_{l=k+1}^{\infty} 2^{ld(1-p)}E_l(g-s_k)_p^p) \\ &\le C2^{kd(p-1)}\sum_{l=k}^{\infty} 2^{ld(1-p)}E_l(g)_p^p. \qquad (18)\end{aligned}$$

Thus, since $s = d(1/p-1)$, $(d-1)/d < p < 1$, and $q/p \le 1$ according to (4), for the first term in the right-hand side of (17) we get

$$2^{kd(1/p-1)}\|g-Pg\|_{L_p} \le C\left(\left(\sum_{l=k}^{\infty} 2^{ld(1-p)}E_l(g)_p^p\right)^{q/p}\right)^{1/q} \le C\left(\sum_{l=k}^{\infty}(2^{ld(1/p-1)}E_l(f)_p)^q\right)^{1/q},$$

where the inequality

$$\sum_{l=0}^{\infty} a_l \le (\sum_{l=0}^{\infty} a_l^{\gamma})^{1/\gamma}, \qquad a_l \ge 0, \quad 0 < \gamma \le 1, \qquad (19)$$

has been used with $\gamma = q/p$, $a_l = 2^{ld(1-p)}E_l(g)_p^p$ for $l \ge k$, and $a_l = 0$ for $l < k$. After substitution into (43) we arrive at

$$\|g-Pg\|_{\mathbf{A}_{p,q,1}^{d(1/p-1)}} \le C\left(\sum_{l=k}^{\infty}(2^{ls}E_l(g)_p)^q\right)^{1/q} \le C\|g\|_{\mathbf{A}_{p,q,1}^{d(1/p-1)}} \qquad (20)$$

for all $g \in S^d$ if the parameters satisfy (4). Since the quasi-norm in $\mathbf{A}_{p,q,1}^s(I^d)$ is a $\min(p,q)$-quasi-norm, (20) is equivalent with (13). This proves the Schauder basis property for H^d in $\mathbf{B}_{p,q,1}^s(I^d)$ for the parameters satisfying (4).

For the parameter range (3), i.e., when $d(1/p-1) < s < 1/p$, $(d-1)/d < p \le 1$, $0 < q < \infty$, we can apply the Hardy-type inequality

$$(\sum_{l=k}^{\infty} a_l^p)^{1/p} \le C_{\epsilon,q/p}2^{-k\epsilon}(\sum_{l=k}^{\infty}(2^{l\epsilon}a_l)^q))^{1/q}, \quad \epsilon > 0, \quad k = 0,1,\ldots,$$

valid for non-negative sequences $\{a_l\}$ and all $0 < p,q < \infty$. Setting $\epsilon = s - d(1/p-1)$ and $a_l = 2^{ld(1/p-1)}E_l(f)_p$, from (18) we obtain

$$2^{ks}\|g-Pg\|_{L_p} \le C2^{k(s-d(1/p-1))}(\sum_{l=k}^{\infty} 2^{ld(1-p)}E_l(g)_p^p)^{1/p} \le C(\sum_{l=k}^{\infty} 2^{lds}E_l(g)_p^q)^{1/q}.$$

It remains to substitute this into (17) and proceed as above for the parameter range (4). This concludes the proof of the Schauder basis property for all parameters satisfying (3) or (4).

2.3. Proof of Theorem 1: Negative results

We first deal with part i), and follow the proof given in [4] for $d = 1$. The case $s \geq 1/p$ has been discussed before. Let $0 < s < d(1/p - 1)$, $0 < p < 1$, $0 < q < \infty$, and assume that F is a bounded linear functional on $\mathbf{B}^s_{p,q,1}(I^d)$. For any dyadic cube $\Delta \in T^d_k$ of side-length 2^{-k} we have by Lemma 1

$$\|\chi_\Delta\|_{L_p} = E_l(\chi_\Delta)_p = 2^{-kd/p}, \quad l = 0, \ldots, k-1, \qquad E_l(\chi_\Delta)_p = 0, \quad l = k, k+1, \ldots,$$

and consequently

$$\|\chi_\Delta\|_{\mathbf{A}^s_{p,q,1}} = 2^{-kd/p}(1 + (\sum_{l=0}^{k-1} 2^{lsq})^{1/q} \leq C2^{k(s-d/p)}, \qquad \Delta \in T^d_k, \quad k = 0, 1, \ldots.$$

By linearity and boundedness of F this implies

$$\begin{aligned} |F(\chi_\Delta)| &= |\sum_{\Delta' \in T^d_l :\, \Delta' \subset \Delta} F(\chi_{\Delta'}| \leq \sum_{\Delta' \in T^d_l :\, \Delta' \subset \Delta} |F(\chi_{\Delta'}| \\ &\leq C \sum_{\Delta' \in T^d_l :\, \Delta' \subset \Delta} \|\chi_{\Delta'}\|_{\mathbf{A}^s_{p,q,1}} \leq C2^{(l-k)d}2^{l(s-d/p)} = C2^{-kd}2^{l(s-d(1/p-1))}, \qquad l > k, \end{aligned}$$

for any given $\Delta \in T^d_k$. Here C also depends on F. If we let $l \to \infty$, we obtain $F(\chi_\Delta) = 0$ for all dyadic cubes, and by linearity $F(g) = 0$ for all $g \in S^d = \mathrm{span}(H^d)$. Since the dyadic step functions are dense in $\mathbf{B}^s_{p,q,1}(I^d)$, this shows $F = 0$, i.e., the dual of $\mathbf{B}^s_{p,q,1}(I^d)$ is trivial.

The counterexamples proving the statement in part ii) of Theorem 1 are new even for $d = 1$ (actually, subcase b) disproves our conjecture concerning the parameter range $p < q \leq 1$ made in [4]). Consider first a), i.e., assume that $s = d(1/p - 1)$, $(d-1)/d < p < 1$, and $1 < q < \infty$. Proving that the coefficient functionals in (9) cannot be extended boundedly from S^d to $\mathbf{B}^{d(1/p-1)}_{p,q,1}(I^d)$ is the same as proving this for the average functionals av_Δ defined in (12) for arbitrary dyadic cubes and $g \in S^d \subset L_1(I^d)$. Since for the above parameter range the Besov space $B^{d(1/p-1)}_{p,q,1}(I^d)$ is not embedded into $L^1(I^d)$, see [4, 5, 1] for the corresponding embedding theorems, we see the problem.

We provide the argument for the average functional av_{I^d} associated with the dyadic cube $\Delta = I^d \in T^d_0$, by showing that there is a sequence $g_k \in S^d$ of dyadic step functions which is uniformly bounded in $\mathbf{B}^{d(1/p-1)}_{p,q,1}(I^d)$, and such that

$$av_{I^d}(g_k) = \int_{I^d} g_k \, dx \to \infty, \qquad k \to \infty. \tag{21}$$

By dilating and shifting these g_k to fit their supports into an arbitrarily given dyadic cube Δ, similar examples can be obtained for all average functionals in (12). Since we do not attempt to show quantitative lower bounds for the divergence in (21), the following construction suffices for $d > 1$, the modification for $d = 1$ is stated below. Set

$$g_k = \sum_{j=1}^{k} a_j \chi_{\Delta_{j,\mathbf{2}}}, \qquad a_j = 2^{jd} j^{-1}, \quad j \in \mathbb{N}. \tag{22}$$

This is a dyadic step function which takes positive values a_j on a sequence of non-overlapping dyadic cubes $\Delta_{j,\mathbf{2}}$, $j = 1, \ldots, k$, located along the main diagonal of I^d, and is otherwise zero. Thus,

$$\int_{I^d} g_k \, dx = \sum_{j=1}^{k} 2^{-jd} a_j = \sum_{j=1}^{k} j^{-1} \geq c \log(k+1),$$

and (21) is established.

For $d > 1$ these g_k also satisfy the assumptions of Lemma 1 with respect to any dyadic cube Δ, where g_k is not constant (if g_k is constant on a dyadic cube, its best L_p approximation by constants on this cube is obviously zero). This allows us to compute the best approximations of g_k exactly:

$$\begin{aligned} E_l(g_k)_p &= \| \sum_{j=l+1}^{k} a_j \chi_{\Delta_{j,\mathbf{2}}} \|_{L_p} = (\sum_{j=l+1}^{k} a_j^p 2^{-jd})^{1/p} \\ & \begin{cases} \leq C 2^{-ld(1/p-1)} (l+1)^{-1}, & l = 0, \ldots, k-1 \\ = 0, & l = k, k+1, \ldots \end{cases} \end{aligned}$$

Substituted into the expression for the $\mathbf{A}^{d(1/p-1)}_{p,q,1}$ quasi-norm, this gives

$$\|g_k\|^q_{\mathbf{A}^{d(1/p-1)}_{p,q,1}} \leq C (\sum_{l=0}^{k-1} (l+1)^{-q})^{1/q}, \qquad k = 1, 2, \ldots,$$

which shows the uniform boundedness of the sequence g_k in $\mathbf{A}^{d(1/p-1)}_{p,q,1}$ since $q > 1$ in case a). Here, we have silently used that $\|g_k\|_{L_p} = E_0(g_k)_p$ by Lemma 1. For $d = 1$, to enable the application of Lemma 1 also in this case, a modified definition of the g_k, e.g.,

$$g_k = \sum_{j=1}^{k} a_{j+1} \chi_{\Delta_{j+1,3}},$$

will do, the details are left to the reader. Note that the above sequences g_k converge to a limit function $f \in \mathbf{B}^{d(1/p-1)}_{p,q,1}(I^d)$ which does not belong to $L_1(I^d)$, for $d = 1$ similar examples were used in [4].

In case b), i.e., when $s = d(1/p-1)$, $(d-1)/d < p < 1$, $p < q \leq 1$, the coefficient functionals in (9) and the dyadic averaging functionals (12) can be extended to bounded linear functionals on $\mathbf{B}^{d(1/p-1)}_{p,q,1}(I^d)$. Thus, the level k partial sum operators P_k defined in (10) can be extended to bounded operators acting in $\mathbf{B}^{d(1/p-1)}_{p,q,1}(I^d)$. However, they are not uniformly bounded as will be shown by a different type of examples. Fix $k = 1, 2, \ldots$, and consider the $2^{(k-1)d}$ dyadic cubes $\Delta_{k,\mathbf{i}}$ in T^d_k for which all entries of $\mathbf{i}$ are odd. Select dyadic subcubes $\tilde{\Delta}_{k+j}$ of shrinking side-length 2^{-k-j}, $j = 1, 2, \ldots, 2^{(k-1)d}$, one in each of them. Then we define

$$g_k = \sum_{j=1}^{2^{(k-1)d}} b_{k,j} \chi_{\tilde{\Delta}_k}, \qquad b_{k,j} = 2^{(k+j)d} j^{-1/p}, \quad j = 1, \ldots, 2^{(k-1)d}.$$

will do. The construction of this g_k is such that Lemma 1 is again applicable, locally on each dyadic cube where g_k is not constant. This allows us to compute the best approximations $E_l(g_k)_p$ as follows: For $l = 0, \dots, k$, we have

$$E_l(g_k)_p^p = \|g_k\|_{L_p}^p = \sum_{j=1}^{2^{(k-1)d}} 2^{-(j+k)^d} b_{k,j}^p \le C2^{-kd(1-p)} = C2^{-ksp}.$$

For $l = k+1, \dots, k + 2^{(k-1)d} - 1$, we get similarly

$$E_l(g_k)_p^p = \sum_{j=1+l-k}^{2^{(k-1)d}} 2^{-(k+j)^d} b_{k,j}^p \le C2^{-ld(1-p)}(l-k)^{-p} = C2^{-ksp} \cdot 2^{-(l-k)sp}(l-k)^{-p}(l-k)^{-1},$$

while $E_l(g_k)_p^p = 0$ for $l \ge k + 2^{(k-1)d}$. Thus, with these formulas for the best approximations $E_l(g_k)_p$ and the substitution $j = l - k$ for $l > k$, one arrives at

$$\begin{aligned} \|g_k\|_{\mathbf{A}_{p,q,1}^{d(1/p-1)}} &\le C2^{-ks}\left(1 + (\sum_{l=0}^{k} 2^{lsq} + 2^{ksq} \sum_{j=1}^{2^{(k-1)d}-1} 2^{jsq} \cdot 2^{-jsq} j^{-q/p})^{1/q}\right) \\ &\le C(1 + (\sum_{j=1}^{\infty} j^{-q/p})^{1/q}) \le C < \infty, \qquad k = 1, 2, \dots, \end{aligned}$$

since $q > p$.

On the other hand, by (11) the level k partial sum $P_k g_k$ of g_k is constant on dyadic cubes in T_k^d, and equals $2^{kd} \cdot 2^{-(k+j)d} b_{k,j} = 2^{kd} j^{-1/p}$ on the cube $\Delta_{k,\mathbf{i}}$ containing $\tilde{\Delta}_{k+j}$, $j = 1, \dots, 2^{(k-1)d}$, and vanishes on all cubes $\Delta_{k,\mathbf{i}}$ for which at least one entry in $\mathbf{i}$ is even. The latter property ensures that Lemma 1 is also applicable to $P_k g_k$, and gives

$$E_l(P_k g_k)_p^p = \|P_k g_k\|_{L_p}^p = \sum_{j=1}^{2^{(k-1)d}} 2^{-kd} \cdot 2^{kdp} j^{-1} = 2^{-ksp} \sum_{j=1}^{2^{(k-1)d}} j^{-1} \ge c2^{-ksp} k$$

for $l = 0, 1, \dots, k-1$. Consequently,

$$\|g_k\|_{\mathbf{A}_{p,q,1}^{d(1/p-1)}} \ge c2^{-ks} k^{1/p} (\sum_{l=0}^{k-1} 2^{lsq})^{1/q} \ge ck^{1/p},$$

which shows that the partial sum operators P_k are not uniformly bounded on $\mathbf{B}_{p,q,1}^{d(1/p-1)}(I^d)$ for $(d-1)/d < p < q$. This concludes the proof of Theorem 1.

We have not made any attempt to obtain the exact growth of norms of partial sum operators in part b) of Theorem 1. For $s = d(1/p - 1)$ and $(d-1)/d < p < q \le 1$, the above considerations give the lower bound

$$\|P_k\|_{\mathbf{B}_{p,q,1}^{d(1/p-1)} \to \mathbf{B}_{p,q,1}^{d(1/p-1)}} \ge ck^{1/p}, \qquad j = 1, 2, \dots,$$

which is certainly not optimal. If one takes $b_{k,j} = 2^{(k+j)d} j^{-\alpha}$ with $\alpha = 1/q + \epsilon$ and small enough $\epsilon > 0$ in the above definition of g_k then the better estimate

$$\|P_k\|_{\mathbf{B}^{d(1/p-1)}_{p,q,1} \to \mathbf{B}^{d(1/p-1)}_{p,q,1}} \geq c 2^{k(1/q-1/p-\epsilon)}, \qquad k = 1, 2, \dots,$$

results, where $c > 0$ depends also on ϵ.

On a final note: In the literature (with the exception of [4, 5]), for $0 < p < 1$ the Besov spaces $\mathbf{B}^s_{p,q,r}(I^d)$ defined as subspaces of $L_p(I^d)$ using r-th order moduli of smoothness are only considered for the values $0 < s < r$ (see, e.g., [1, 8]). The reason is two-fold: It is known that for $r \leq s < r + 1/p - 1$, $0 < q < \infty$, the spaces $\mathbf{B}^s_{p,q,r}(I^d)$ are strange: With the exception of polynomials of degree $< r$, smooth functions from $C^r(I^d)$ cannot belong to $\mathbf{B}^s_{p,q,r}(I^d)$, while C^{r-2}-smooth dyadic splines of degree $r-1$ are dense in these spaces. This is counter-intuitive, and makes their usefulness in applications doubtful. Moreover, the spaces $B^s_{p,q}(I^d)$ defined by the Fourier-analytic approach that dominate the scene coincide with $\mathbf{B}^s_{p,q,r}(I^d)$ (in the sense of having equivalent quasi-norms) only in the range $d(1/p-1) < s < r$. In other words, our new results on the properties of the Haar system H^d in Besov spaces $\mathbf{B}^s_{p,q,1}(I^d)$ for the limiting case $s = d(1/p-1)$ do not automatically answer the same question for the scale $B^s_{p,q}(I^d)$. The latter will be considered in the next section.

3. Proof of Theorem 2

3.1. Definitions and preparations

The role of the Haar system as Schauder basis in the Besov spaces $B^s_{p,q}(I^d)$ with $0 < p \leq 1$, $0 < q < \infty$, $s \in \mathbb{R}$, defined in Fourier-analytic terms has been examined by Triebel [7] (see also [8, Theorem 1.58]) who settled all but the limiting cases $s = d(1/p-1)$, $d/(d+1) < p \leq 1$, $0 < q < \infty$, and $s = 1$, $d/(d+1) \leq p \leq 1$, $0 < q < \infty$. Theorem 2 gives now answers in the limiting cases as well.

The definition of $B^s_{p,q}(I^d)$ is reduced by restriction to the definition of $B^s_{p,q}(\mathbb{R}^d)$:

$$B^s_{p,q}(I^d) = \{f = \tilde{f}|_{I^d} : \ \tilde{f} \in B^s_{p,q}(\mathbb{R}^d)\}, \qquad \|f\|_{B^s_{p,q}} = \inf_{\tilde{f}:\, f = \tilde{f}|_{I^d}} \|\tilde{f}\|_{B^s_{p,q}(\mathbb{R}^d)}. \qquad (23)$$

The definition of $B^s_{p,q}(\mathbb{R}^d)$ will be given in terms of atoms, for the equivalent definition in Fourier-analytic terms and a short review of the various definitions for spaces on $\mathbb{R}^d$ and on domains we refer to [8, Chapter 1]. Since we are only interested in the limiting cases of low smoothness $s = d(1/p-1) < 1$ and $s = 1$ in (1), some simplifications are possible. Let us go to the details. For the parameter range $d(1/p-1) \leq s \leq 1$ of interest, take any $\sigma > s$ (note that for $s < 1$ it is always possible to take $s < \sigma \leq 1$), and consider the set of all Hölder class functions $a \in \mathcal{C}^\sigma(\mathbb{R}^d)$ with support in a fixed cube of side-length $C_0 > 1$ centered at the origin, and with $\mathcal{C}^\sigma(\mathbb{R}^d)$ norm bounded by C_0. Denote this set for short by $\mathcal{C}^\sigma_{C_0}$. Functions of the form

$$a_{j,\mathbf{i}}(x) = a(2^j x - \mathbf{i}), \qquad a \in \mathcal{C}^\sigma_{C_0}, \qquad \mathbf{i} \in \mathbb{Z}^d, \qquad (24)$$

are called atoms of level 0 if $j = 0$, and atoms of level $j = 1, 2, \dots$ if additionally

$$\int_{\mathbb{R}^d} a_{j,\mathbf{i}}\, dx = \int_{\mathbb{R}^d} a_{j,\mathbf{i}}\, dx = 0, \qquad j \in \mathbb{N}.$$

This latter additional condition is necessary for the following statement only if $s = d(1/p - 1)$, the case we are most interested in.

Lemma 2 *Let $d(1/p-1) \le s < \sigma$, $d/(d+1) < p < 1$, $0 < q < \infty$, and $c_0 > 1$ be fixed. Then $\tilde{f} \in B^s_{p,q}(\mathbb{R}^d)$ if and only if*

$$\tilde{f}(x) = \sum_{j=0}^{\infty} \sum_{\mathbf{i} \in \mathbb{Z}^d} c_{j,\mathbf{i}} a_{j,\mathbf{i}}(x) \tag{25}$$

(unconditional convergence in $S'(\mathbb{R}^d)$) for some atoms $a_{j,\mathbf{i}}$ specified by (24) and with coefficients such that

$$\sum_{j=0}^{\infty} 2^{j(s-d/p)q} \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} |c_{j,\mathbf{i}}|^p \right)^{q/p} < \infty.$$

Moreover,

$$\|\tilde{f}\|^{+}_{B^s_{p,q}(\mathbb{R}^d)} := \inf \left(\sum_{j=0}^{\infty} 2^{j(s-d/p)q} (\sum_{\mathbf{i} \in \mathbb{Z}^d} |c_{j,\mathbf{i}}|^p)^{q/p} \right)^{1/q} \approx \|f\|_{B^s_{p,q}(\mathbb{R}^d)}, \tag{26}$$

where the infimum is taken with respect to all possible representations (25), is an equivalent quasi-norm on $B^s_{p,q}(\mathbb{R}^d)$. The constants in the norm equivalence depend on σ, C_0, and p, q, s.

This statement is covered by [8, Corollary 1.23 (i)], where references to the history of atomic characterizations of function spaces can be found. Note that our atoms correspond to the 1_σ-atoms ($j = 0$) and $(s,p)_{\sigma,1}$-atoms ($j = 1, 2, \dots$) of Definition 1.21 in [8] but are scaled differently. Instead, the necessary scaling has been incorporated in the definition of the atomic quasi-norm (26). Below, we will apply this lemma with values $\sigma > 1$, and appropriately fixed C_0, to obtain upper bounds for (atomic) Besov norms.

In some cases, especially for obtaining lower bounds for $B^s_{p,q}(\mathbb{R}^d)$ quasi-norms, it is more convenient to use characterizations in terms of local means

$$\kappa(t, \tilde{f})(x) = (\kappa^t * \tilde{f})(x) := \int_{\mathbb{R}^d} \kappa^t(x-y) \tilde{f}(y)\, dy, \qquad \kappa^t(x) := t^{-d} \kappa(t^{-1}x), \quad t > 0, \tag{27}$$

where the kernel $\kappa \in C^\infty(\mathbb{R}^d)$ has support in the cube $[-1/2, 1/2]^d$, and satisfies

$$\kappa^\vee(\xi) \neq 0, \quad 0 < |\xi| < \epsilon, \qquad (D^\alpha \kappa^\vee)(0) = 0 \text{ if } |\alpha| \le s, \tag{28}$$

for some $\epsilon > 0$. Here, $\kappa^\vee$ denotes the Fourier transform of κ. For $s < 1$, the moment condition in (28) reduces to requiring

$$\int_{\mathbb{R}^d} \kappa(x)\, dx = 0,$$

while for $s = 1$ we additionally need also orthogonality to linear polynomials:

$$\int_{\mathbb{R}^d} x_i \kappa(x)\,dx = 0, \qquad i = 1, \ldots, d.$$

We also fix another kernel $\kappa_0 \in C^\infty(\mathbb{R}^d)$ with support in the cube $[-1/2, 1/2]^d$ which, instead of (28), satisfies

$$\kappa_0^\vee(0) = \int_{\mathbb{R}^d} \kappa_0(x)\,dx > 0.$$

By applying [8, Theorem 1.10], we have the following characterization in the range of parameters of interest to us.

Lemma 3 *Let $0 < s \le 1$, $0 < p \le 1$, $0 < q < \infty$, and let the kernels κ, κ_0 satisfy the above conditions. Then*

$$\|\tilde{f}\|^*_{B^s_{p,q}(\mathbf{R}^d)} := \left(\|\kappa_0(1, \tilde{f})\|^q_{L_p(\mathbb{R}^d)} + \sum_{j=1}^{\infty} 2^{jsq} \|\kappa(2^{-j}, \tilde{f})\|^q_{L_p(\mathbb{R}^d)} \right)^{1/q} \approx \|\tilde{f}\|_{B^s_{p,q}(\mathbf{R}^d)}.$$

The constants in the norm equivalence depend on κ, κ_0, and p, q, s.

We conclude this subsection by a technical result which shows how to reduce estimates for partial sum operators P associated with the Haar expansion (9) of functions on I^d to estimates for similar operators acting on functions defined on $\mathbb{R}^d$. To this end, to any P given by (14) we associate its extension

$$(\tilde{P}\tilde{f})(x) = (\tilde{P}_k\tilde{f})(x) + \sum_{h \in \bar{H}^d_{k+1}} c_h(f)h = \begin{cases} (P(\tilde{f}|_{I^d}))(x), & x \in I^d, \\ av_\Delta(\tilde{f}), & x \in \Delta \not\subset I^d, \end{cases} \tag{29}$$

for $\tilde{f} \in L_{1,loc}(\mathbb{R}^d)$, where Δ runs through all dyadic cubes Δ of side-length 2^{-k} in $\mathbb{R}^d \backslash I^d$. In other words, we define $\tilde{P}$ outside I^d by $\tilde{P}_k$, the natural extension of the level k partial sum operator P_k to functions on $\mathbb{R}^d$. Other extensions are possible, this one simplifies some considerations below. In particular, $\tilde{P}\tilde{f}$ has the following properties which we use throughout the rest of this subsection. First of all, it is piecewise constant on dyadic cubes $\Delta \in \tilde{T}^d_k$ outside I^d and $\Delta \in T^d_{k+1}$ inside I^d. Here, and in the following, $\tilde{T}^d_k$ denotes the collection of all dyadic cubes of side-length 2^{-k} in $\mathbb{R}^d$ (thus, $T^d_k = \tilde{T}^d_k \cap I^d$). Moreover, in analogy to (11), we have

$$\tilde{P}_k\tilde{f}(x) = av_\Delta(\tilde{f}); \qquad x \in \Delta \in \tilde{T}^d_k, \tag{30}$$

while by the definition of the Haar functions for each $h \in H^d_{k+1}$ we have

$$|c_h(\tilde{f})h| = \left| \sum_{\Delta^* \in T^d_{k+1}:\, \Delta^* \subset \mathrm{supp}(h)} \alpha_{h,\Delta^*} av_{\Delta^*}(\tilde{f}) \right|, \qquad \sum_{\Delta^* \in T^d_{k+1}:\, \Delta^* \subset \mathrm{supp}(h)} \alpha_{h,\Delta^*} = 0. \tag{31}$$

Because of shift-invariance, depending on the type of Haar function h and the location of the cube Δ^* relative to the cube $\mathrm{supp}(h)$ containing it, there appear only finitely many different coefficient sets $\{\alpha_{h,\Delta^*}\}$ in (31). Consequently, the restriction of the difference

$$|(\tilde{P}-\tilde{P}_k)\tilde{f}| \le \sum_{h\in\bar{H}^d_{k+1}} |c_h(\tilde{f})h|$$

to any $\Delta^* \in T^d_{k+1}$ can be bounded by the sum of *differences* $|av_{\Delta'}(\tilde{f}) - av_{\Delta''}(\tilde{f})|$ of averages with respect to neighboring dyadic cubes $\Delta', \Delta'' \in T^d_{k+1}$ belonging to the same dyadic cube of side-length 2^k as Δ^*. This will be used in subsection 3.3 .

From now on, the notation $\tilde{f} \in Y$ is reserved for functions in $B^s_{p,q}(\mathbb{R}^d)$ such that $\tilde{f} \in S'(\mathbf{R}^d)$ is represneted by an atomic decomposition (25) satisfying

$$c_{j,\mathbf{i}} = 0 \quad \text{if} \quad \mathrm{supp}(a_{j,\mathbf{i}}) \subset \mathbb{R}^d \backslash I^d. \tag{32}$$

For a given f belonging to a Besov space $B^s_{p,q}(I^d)$ for which Lemma 2 holds, we write $\tilde{f} \in Y_f$ if $\tilde{f} \in Y$ and $\tilde{f}|_{I^d} = f$. Then, by the definition of atomic quasi-norms we have

$$\|f\|_{B^s_{p,q}} \ge c \inf_{\tilde{f}:\, f=\tilde{f}|_{I^d}} \|\tilde{f}\|^+_{B^s_{p,q}(\mathbb{R}^d)} = c \inf_{\tilde{f}\in Y_f\cap B^s_{p,q}(\mathbb{R}^d)} \|\tilde{f}\|^+_{B^s_{p,q}(\mathbb{R}^d)}. \tag{33}$$

Lemma 4 *Assume that the parameters p, q, s are such that Lemma 2 and 3 hold. Then the operator P defined in (14) satisfies*

$$\|Pf\|_{B^s_{p,q}} \le C\|f\|_{B^s_{p,q}}, \qquad f \in B^s_{p,q}(I^d), \tag{34}$$

with a constant independent of P, if its extension $\tilde{P}$ defined in (29) satisfies

$$\|\tilde{P}\tilde{f}\|^*_{B^s_{p,q}(\mathbb{R}^d)} \le C\|\tilde{f}\|^+_{B^s_{p,q}(\mathbb{R}^d)}, \qquad \tilde{f} \in Y \cap B^s_{p,q}(\mathbb{R}^d), \tag{35}$$

with a constant independent of P.

Proof. This follows by the locality properties of partial sum operators. By (35), $\tilde{P}\tilde{f} \in B^s_{p,q}(\mathbb{R}^d)$ is meaningfully defined for $\tilde{f} \in Y_f \cap B^s_{p,q}(\mathbb{R}^d)$. Since $\tilde{f}|_{I^d} = f$, by definition of $\tilde{P}$ we also have

$$\tilde{P}\tilde{f}|_{I^d} = Pf.$$

Thus, $\tilde{P}\tilde{f}$ is an extension of Pf, and by the definition of the $B^s_{p,q}$ quasi-norm, by Lemma 3, and by (35) we get

$$\|Pf\|_{B^s_{p,q}} \le \|\tilde{P}\tilde{f}\|_{B^s_{p,q}(\mathbb{R}^d)} \le C\|\tilde{P}\tilde{f}\|^*_{B^s_{p,q}(\mathbb{R}^d)} \le C\|\tilde{f}\|^+_{B^s_{p,q}(\mathbb{R}^d)}.$$

It remains to take the infimum with respect to $\tilde{f} \in Y_f \cap B^s_{p,q}(\mathbb{R}^d)$, and to apply (33). Lemma 4 is proved. □

3.2. The limiting case $s = 1$

We first deal with the case $s = 1$, $d/(d+1) \le p \le 1$, $0 < q < \infty$, and show that for $f(x) = x_1 + \ldots + x_d$ we have

$$\|f - P_k f\|_{B^1_{p,q}} \ge c > 0, \qquad k = 0, 1, \ldots, \tag{36}$$

for some positive constant c. Since $f \notin \mathrm{span}(H^d)$, we have $f - P_k f \ne 0$, and it suffices to consider large enough k.

To obtain the lower bounds needed for (36), we compute lower estimates for the L_p quasi-norm of $\kappa(2^{-(k+1)}, \tilde{G}_k)(x)$ for any extension $\tilde{G}_k$ of $f - P_k f$ with a kernel κ as defined in subsection 3.1 for $s = 1$ (in particular, κ is orthogonal to linear polynomials (28)). To this end, we observe that inside I^d the difference $\tilde{G}_k(x) = (f - P_k f)(x)$ coincides a.e. with the restriction to I^d of a suitably dilated and scaled single integer-shift invariant function $f_0(x)$ given by

$$f_0(x + \mathbf{i}) = f(x) - d/2, \qquad x \in (0,1]^d, \quad \mathbf{i} \in \mathbb{Z}.$$

Indeed, we have

$$(f - P_k f)(x) = 2^{-k} f_0(2^k x), \qquad x \in I^d,$$

which can be checked from the formula

$$(f - P_k f)(x) = f(x) - av_\Delta(f) = f(x) - f(x_\Delta) = 2^{-k} f(2^k(x - x_\Delta)), \qquad x \in \Delta,$$

where Δ is an arbitrary cube in T^d_k, and x_Δ denotes the center of Δ.

Using the invariance of $f - P_k f$ with respect to shifts of the form $2^{-k}\mathbf{j}$ inside I^d, and the fact that $\kappa^{2^{-(k+1)}}$ has support in a cube of side-length $2^{-(k+1)}$ centered at the origin, we see that

$$\kappa(2^{-(k+1)}, \tilde{G}_k)(x) = 2^{-k} \kappa(2^{-(k+1)}, f_0(2^k \cdot))(x) = 2^{-k} \kappa(1/2, f_0)(2^k x)$$

holds for all $x \in [2^{-(k+1)}, 1 - 2^{-(k+1)}]^d$. Thus, for $k > 1$ we obtain

$$\begin{aligned} \|\kappa(2^{-(k+1)}, \tilde{G}_k)\|_{L_p(\mathbb{R}^d)} &\ge 2^{-k} \|\kappa(1/2, f_0)(2^k \cdot)\|_{L_p([2^{-(k+1)}, 1 - 2^{-(k+1)}]^d)} \\ &\ge 2^{-k} ((2^k - 2)/2^k)^{d/p} \|\kappa(1/2, f_0)\|_{L_p} \ge c 2^{-k}, \end{aligned}$$

for some constant $c > 0$ depending on the kernel and p. By Lemma 3 we conclude that

$$\|f - P_k f\|_{B^1_{p,q}} \ge c 2^{k+1} \inf_{\tilde{G}_k} \|\kappa(2^{-(k+1)}, \tilde{G}_k)\|_{L_p(\mathbb{R}^d)} \ge c, \qquad k > 1,$$

which proves (36).

We finish the consideration for $s = 1$ with a remark concerning the special case $p = d/(d+1)$. In the proof of (36) we have not made explicit use of the restriction $d/(d+1) \le p \le 1$. That (36) contradicts the Schauder basis property of H^d in $B^1_{p,q}(I^d)$ as claimed in Theorem 2 is clear if $d/(d+1) < p \le 1$ since for this parameter range

$B^1_{p,q}(I^d)$, $0 < q < \infty$, is continuously embedded into $L_1(I^d)$ which ensures that the P_k are the right candidate to be considered for partial sum operators. Moreover, the embedding also implies together with (36) that the set $S^d = \mathrm{span}(H^d)$ of dyadic step functions cannot be dense in $B^1_{p,q}(I^d)$, thus extending the similar statement for $1 < s < 1/p$ proved in [7] to the case $s = 1$.

For $p = d/(d+1)$, we have $d(1/p - 1) = 1 = s$, and the continuous embedding $B^1_{d/(d+1),q}(I^d) \subset L_1(I^d)$ holds only if $q \le 1$. For $q > 1$, we cannot automatically exclude the possibility that there are Haar series other than (9) representing the above f in $B^1_{d/(d+1),q}(I^d)$. Nor do we know for sure if S^d is dense in $B^1_{d/(d+1),q}(I^d)$. However, even in this special case H^d cannot be a Schauder basis since case a) in part ii) of Theorem 2 applies (for the proof, see the next subsection).

We finally note that an example similar to $f(x) = x_1 + \ldots + x_d$ has been used in [3, Section 4] for showing lower bounds for $B^s_{p,q}(\mathbb{R}^d)$ quasi-norm of the level k partial sum operators $\tilde{P}_k$ if $\max(d(1/p-1), 1) < s < 1/p$. This implies that the Haar system on $\mathbb{R}^d$ is not a basic sequence in $B^s_{p,q}(\mathbb{R}^d)$ for this parameter range, and strengthens the result of Triebel [7]. As far as we know, the uniform boundedness of the partial sum operators $\tilde{P}_k$ on $B^1_{p,q}(\mathbb{R}^d)$ has not been settled. This question is also open for $B^1_{p,q}(I^d)$, as we only showed that $f - P_k f$ does not converge to zero in the $B^1_{p,q}(I^d)$ quasi-norm for some $f \in B^1_{p,q}(I^d)$ but did not provide upper bounds for $s = 1$).

3.3. The limiting case $s = d(1/p - 1)$

3.3.1. Counterexamples for $p < q < \infty$

Throughout this subsection, we fix $s = d(1/p-1)$, $d/(d+1) \le p < 1$, and $p < q < \infty$. In particular, this implies $s \le 1$ (with equality for $p = d/(d+1)$).

We start with the statement in case b) in part ii) of Theorem 2. The corresponding counterexamples have been suggested to me by T. Ullrich. They are similar to the counterexamples for Theorem 1 but are now defined by linear combinations of special atoms. For the latter, we fix a function $a(x) = \prod_{i=1}^d \phi(x_i)$, where $\phi(x) \in C^\infty(\mathbb{R})$ is a univariate odd function, supported in $[-1, 1]$, positive for $x \in (0, 1)$, and such that $\|a\|_{\mathcal{C}^\sigma} = 1$ for some $\sigma > 1$. Obviously, if we define the functions $a_{j,\mathbf{i}}$, $\mathbf{i} \in \mathbb{Z}^d$ for $j \ge 1$ as in (24) from this a, then, with C_0 suitably fixed, they represent atoms of level $j \ge 1$, and we can apply Lemma 2 for any $s \le 1$ to estimate $B^s_{p,q}(\mathbb{R}^d)$ quasi-norms of their linear combinations.

Consider the family of functions

$$g_k(x) = \sum_{j=1}^{n_k} j^{-1/p} 2^{(k+j)d} a_{k+j,\mathbf{i}_{k,j}}(x), \qquad n_k = (2^{k-2} - 1)^d, \quad k = 3, 4, \ldots,$$

where the multi-indices $\mathbf{i}_{k,j}$, $j = 1, \ldots, n_k$, are chosen such that the support centers $x_{k,j} := x_{k+j,\mathbf{i}_{k,j}}$ of the atoms $a_{k+j,\mathbf{i}_{k,j}}(x)$ are different, and coincide with the n_k interior vertices of T^d_{k-2}. Note that $n_k \approx 2^{kd}$ as $k \to \infty$, and that $g_k(x)$ is a finite linear combination of atoms with different scale parameters whose supports are well-separated.

By (23) and Lemma 2 we have $g_k(x) \in B^{d(1/p-1)}_{p,q}(I^d)$, $p < q < \infty$, with uniformly bounded quasi-norm for all $k \geq 3$ since

$$\begin{aligned}
\|g_k\|^q_{B^{d(1/p-1)}_{p,q}} &\leq \|g_k\|^q_{B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)} \leq C(\|g_k\|^+_{B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)})^q \\
&\leq C\sum_{j=1}^{n_k} 2^{(k+j)(d(1/p-1)-d/p)}(j^{-1/p}2^{(k+j)d})^q \\
&= C\sum_{j=1}^{n_k} j^{-q/p} \leq C\sum_{j=1}^{n_k} j^{-q/p} < \infty.
\end{aligned}$$

For convenience, we use the same notation g_k for the extension by zero of g_k to $\mathbb{R}^d$.

On the other hand, since the centers $x_{k,j}$ of the atoms $a_{k+j,\mathbf{i}_{k,j}}$ are located at the interior vertices of T^d_{k-2} and have supports in cubes of side-length $2^{-(k+j-1)}$, their supports are well-separated. Moreover, they have the same symmetry properties with respect to their centers as the function

$$h^d_0(x) := h_0(x_1) \cdot \ldots \cdot h_0(x_d)$$

has with respect to the origin. Here, h_0 is the univariate centralized Haar wavelet defined in (6). Therefore, the Haar projection $P_k g_k$ onto S^d_k can easily be computed in explicit form:

$$(P_k g_k)(x) = \sum_{j=1}^{n_k} b_{k,j} j^{-1/p} 2^{(k+j)d} h^d_0(2^k(x - x_{k,j}), \quad k = 3, 4, \ldots,$$

where $b_{k,j}$ is the average value of $a_{k+j,\mathbf{i}_{k,j}}$ over the cube in T^d_k whose lowest vertex coincides with $x_{k,j}$. This average value can easily be computed as

$$b_{k,j} = 2^{kd}\int_{[0,2^{-k}]^d} a(2^{k+j}y)\,dy = 2^{kd}2^{-(k+j)d}b_0 = b_0 2^{-jd}, \qquad j = 1, \ldots, n_k,$$

where $b_0 = (\int_0^1 \phi(x)\,dx)^d > 0$ is a fixed constant. Thus, the formula for $P_k g_k$ simplifies to

$$(P_k g_k)(x) = b_0 2^{kd} \sum_{j=1}^{n_k} j^{-1/p} h^d_0(2^k(x - x_{k,j})), \quad k = 3, 4, \ldots. \tag{37}$$

In order to get a lower bound for the $B^{d(1/p-1)}_{p,q}$ quasi-norm of $P_k g_k$, we next compute a lower bound for its local mean $\kappa(2^{-k}, P_k g_k)(x)$, where the kernel κ has the properties required for Lemma 3 to hold. E.g., we could set $\kappa = a(2\cdot)$ with the above function a because $s < 1$ and the dilation factor 2 ensures that $\mathrm{supp}(\kappa) \subset [-1/2, 1/2]^d$. Since the support cubes (denoted by $I_{k,j}$) of the terms $h^d_0(2^k(x - x_{k,j}))$ in the representation (37) have side-length 2^{-k+1} and are centered at $x_{k,j} \in T^d_{k-2}$, they are still well-separated, and we have

$$\begin{aligned}
\kappa(2^{-k}, P_k g_k)(x) &= 2^{(k-1)d} j^{-1/p}(\kappa^{2^{-k}} * h^d_0(2^k(\cdot - x_{k,j})))(x) \\
&= 2^{(k-1)d} j^{-1/p} 2^{kd}(\kappa(2^j\cdot) * h^d_0(2^k(\cdot))(x - x_{k,j}) \\
&= 2^{(k-1)d} j^{-1/p}(\kappa * h^d_0)(2^k(x - x_{k,j}), \qquad x \in I_{k,j},
\end{aligned}$$

where we have used (24). Since the cubes $I_{k,j}$ are also well-separated from the boundary of I^d, this lower bound holds for any extension $\tilde{G}_k \in S'(\mathbb{R}^d)$ of $P_k g_k$. Thus, since the C^∞ function $\kappa * h_0^d$ is non-vanishing in a neighborhood of the origin by construction, we obtain

$$\|\kappa(2^{-k}, \tilde{G}_k)\|^p_{L_p(\mathbb{R}^d)} \geq c2^{(k-1)dp} \sum_{j=1}^{n_k} j^{-1} 2^{-kd} \geq c2^{kd(p-1)} \log(n_k) \geq c2^{kd(p-1)} k.$$

By definition of Besov quasi-norms on domains and by Lemma 2 we arrive at

$$\|P_k g_k\|_{B^{d(1/p-1)}_{p,q}} \geq c \inf_{\tilde{G}_k} \|\tilde{G}_k\|^*_{B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)} \geq c2^{kd(1/p-1)} \|\kappa(2^{-k}, G_k)\|_{L_p(\mathbb{R}^d)} \geq ck^{1/p}.$$

This shows that for $p < q \leq 1$ the partial sum operators P_k, which are well-defined on the set of dyadic step-functions S^d and extend by continuity to $B^{d(1/p-1)}_{p,q}(I^d)$ due to the continuous embedding $B^{d(1/p-1)}_{p,q}(I^d) \subset L_1(I^d)$, cannot be uniformly bounded on $B^{d(1/p-1)}_{p,q}(I^d)$. This contradicts the Schauder basis property for $s = d(1/p-1)$, $p < q \leq 1$, and finishes the argument for case b).

For case a) of part ii) of Theorem 2, we provide examples analogous to (22) in subsection 2.3 which show that the average functionals av_Δ defined by (12) on the set of dyadic step functions S^d cannot be extended to bounded linear functionals on $B^{d(1/p-1)}_{p,q}(I^d)$ if $q > 1$. For simplicity, consider $\Delta = I^d$, and define $g_k = \tilde{g}_k|_{I^d}$ by the atomic decomposition

$$\tilde{g}_k = \sum_{j=1}^{k} b_j a_{j,\mathbf{0}}, \qquad b_j = 2^{jd} j^{-1}$$

This is, up to different coefficient notation and the replacement of characteristic functions $\chi_{\Delta_{j,\mathbf{2}}}$ by atoms $a_{j,\mathbf{0}}$ defined in (24) with the above function a, the same construction as in (22). Obviously, by construction

$$av_{I^d}(g_k) = \sum_{j=1}^{k} b_j av_{I^d}(a(2^j \cdot)) = \sum_{j=1}^{k} b_j 2^{-jd} b_0 = b_0 \sum_{j=1}^{k} j^{-1} \geq c \log(k) \to \infty$$

as $k \to \infty$, while

$$\begin{aligned} \|g_k\|^q_{B^{d(1/p-1)}_{p,q}} &\leq \|g_k\|^q_{B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)} \leq C(\|g_k\|^+_{B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)})^q \\ &\leq C \sum_{j=1}^{k} (2^{j(d(1/p-1)-d/p)} j^{-1} 2^{jd})^q = C \sum_{j=1}^{n_k} j^{-q} \leq C < \infty, \end{aligned}$$

since $1 < q < \infty$.

3.3.2. Proof of the Schauder basis property for $0 < q \leq p$

We now turn to case a) in part ii) of Theorem 2, where $d/(d+1) < p < 1$, $0 < q \leq p$, and $0 < s = d(1/p-1) < 1$ can be assumed. Since for this parameter range the set of all dyadic step functions is dense in $B^s_{p,q}(I^d)$, it suffices to prove the uniform boundedness of the partial sum operators

$$\tilde{P} = \tilde{P}_k + \sum_{h \in \bar{H}^d_{k+1}} c_h(\cdot) h$$

using Lemma 4, i.e., to establish (35) for all $\tilde{f} \in Y \cap B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)$. We proceed in several steps.

Step 1. Using the properties of $\tilde{P}\tilde{f}$, and in particular (30) and (31), we show that

$$\sum_{j=k}^{\infty} 2^{jd(1/p-1)q} \|\kappa(2^{-j}, \tilde{P}\tilde{f})\|^q_{L_p(\mathbb{R}^d)} \le C s^q_{k+1}, \tag{38}$$

where s_{k+1} is given by

$$s_{k+1} := 2^{-kd} \left(\sum_{\Delta', \Delta'' \in \tilde{T}^d_{k+1} : \mathrm{dist}_\infty(\Delta', \Delta'')=0} |av_{\Delta'}(\tilde{f}) - av_{\Delta''}(\tilde{f})|^p \right)^{1/p}. \tag{39}$$

In the case $j \ge k$, consider any cube $\Delta \in \tilde{T}^d_k$, and denote the set of its neighbors in T^d_k by

$$n(\Delta) = \{\tilde{\Delta} \in T^d_k : \ \Delta \cap \tilde{\Delta} \neq \emptyset\}.$$

Recall from (30) that $\tilde{P}_k \tilde{f}|_\Delta = av_\Delta(\tilde{f})$. Since $\kappa^{2^{-j}}(x - \cdot)$ is supported in a cube of side-length 2^{-j} centered at x and is orthogonal to constants due to the assumed moment condition for the kernel κ, for $j \ge k$ and $x \in \Delta$ we have

$$\begin{aligned} |\kappa(2^{-j}, \tilde{P}\tilde{f})(x)| &= |\kappa(2^{-j}, \tilde{P}\tilde{f}(\cdot) - av_\Delta(\tilde{f}))(x)| \\ &\le |\kappa(2^{-j}, \tilde{P}_k\tilde{f}(\cdot) - av_\Delta(\tilde{f}))(x)| + |\kappa(2^{-j}, \tilde{P}\tilde{f} - \tilde{P}_k\tilde{f})(x)|. \end{aligned}$$

Here, both terms in the right-hand side vanish only if the support cube of $\kappa^{2^{-j}}(x - \cdot)$ intersects with the boundary of any of the dyadic cubes in $\tilde{T}^d_{k+1}$, where the piecewise constant functions $\tilde{P}\tilde{f}$, $\tilde{P}_k\tilde{f}$ may have jumps. The set of these $x \in \Delta$ has measure $\le C 2^{-j} 2^{-k(d-1)}$, and due to (30) we have the bound

$$|\kappa(2^{-j}, \tilde{P}_k\tilde{f}(\cdot) - av_\Delta(\tilde{f}))(x)| \le C \sum_{\tilde{\Delta} \in n(\Delta)} |av_\Delta(\tilde{f}) - av_{\tilde{\Delta}}(\tilde{f})|,$$

which in turn can be estimated by the sum of differences $|av_{\Delta'}(\tilde{f}) - va_{\Delta''}(\tilde{f})|$ appearing in (38) with neighboring $\Delta', \Delta'' \in \tilde{T}^d_{k+1}$ belonging to the union of cubes in $n(\Delta)$. The other term is similarly bounded since for $x \in \Delta$

$$|\kappa(2^{-j}, \tilde{P}\tilde{f} - \tilde{P}_k\tilde{f})(x)| \le C \| \sum_{h \in \bar{H}^d_{k+1}:} |c_h(\tilde{f}) h(\cdot)| \|_{L_\infty(\cup_{\tilde{\Delta} \in n(\Delta)} \tilde{\Delta})},$$

and we can apply (31). Thus, altogether we arrive at

$$|\kappa(2^{-j},\tilde P\tilde f)(x)| \le C{\sum}'_{\Delta',\Delta''}|av_{\Delta'}(\tilde f)-av_{\Delta''}(\tilde f)|, \qquad x\in\Delta\in\tilde T^d_k, \qquad j\ge k, \tag{40}$$

where $\sum'$ indicates that the summation extends to all those neighboring dyadic cubes Δ',Δ'' in $\tilde T^d_{k+1}$ which belong to the union of all cubes in $n(\Delta)$. This bound is only needed on a subset of Δ of measure $\le C2^{-j}2^{-k(d-1)}$.

From (40) we get for $j\ge k$

$$\begin{aligned}
\|\kappa(2^{-j},\tilde P\tilde f)\|^p_{L_p(\mathbb{R}^d)} &= \sum_{\Delta\in\tilde T^d_k}\|\kappa(2^{-j},\tilde P\tilde f)\|^p_{L_p(\Delta)} \\
&\le C\sum_{\Delta\in\tilde T^d_k}2^{-j-k(d-1)}{\sum}'_{\Delta',\Delta''}|av_{\Delta'}(\tilde f)-av_{\Delta''}(\tilde f)|^p \\
&\le C2^{k-j}2^{-kd(1-p)}\sum_{\Delta',\Delta''\in\tilde T^d_{k+1}:\,\mathrm{dist}_\infty(\Delta',\Delta'')=0}|av_{\Delta'}(\tilde f)-av_{\Delta''}(\tilde f)|^p \\
&= C2^{-j}2^{-k(d-1)}s^p_{k+1},
\end{aligned}$$

where we have used that each term $|av_{\Delta'}(\tilde f)-av_{\Delta''}(\tilde f)|^p$ belongs to at most 3^d neighborhoods $n(\Delta)$. Taking the previous estimate to the power q/p and substituting the result into the left-hand side of (38) leads to the desired estimate in (38). Indeed,

$$\sum_{j=k}^\infty 2^{jd(1/p-1)q}\|\kappa(2^{-j},\tilde P\tilde f)\|^q_{L_p(\mathbb{R}^d)} \le C2^{k(1/p-d/p)q}s^q_{k+1}\sum_{j=k}^\infty 2^{-j(1/p-d(1/p-1))q}\le C2^{-kdq}s^q_{k+1},$$

since $1/p-d(1/p-1)<0$ for $d/(d+1)<p<1$.

Step 2. For $1\le j<k$, we start with

$$\|\kappa(2^{-j},\tilde P\tilde f)\|^p_{L_p(\mathbb{R}^d)}\le\|\kappa(2^{-j},\tilde f-\tilde P\tilde f)\|^p_{L_p(\mathbb{R}^d)}+\|\kappa(2^{-j},\tilde f)\|^p_{L_p(\mathbb{R}^d)},$$

(similarly for $j=0$ and $\kappa_0(1,\tilde P\tilde f)$), and proceed with estimates for the term corresponding to $\tilde f-\tilde P\tilde f$ (after substitution into the expression for the local means quasi-norm, the other term will be automatically bounded by the right-hand side in (35)). This time we use the fact that $\tilde f-\tilde P\tilde f$ has zero average on each dyadic cube $\tilde\Delta\in\tilde T^d_k$, and that the kernel κ is smooth. With the short-hand notation Δ^j_x for the support cube of $\kappa^{2^{-j}}(x-\cdot)$, this yields

$$
\begin{aligned}
|\kappa(2^{-j},\tilde f-\tilde P\tilde f)(x)| &\le \sum_{\tilde\Delta\in\tilde T_k^d:\,\tilde\Delta\cap\Delta_x^j\neq\emptyset}\left|\int_{\tilde\Delta}2^{jd}\kappa^{2^{-j}}(x-y)(\tilde f-\tilde P\tilde f)(y)\,dy\right| \\
&\le \sum_{\tilde\Delta\in\tilde T_k^d:\tilde\Delta\cap\Delta_x^j\neq\emptyset}\inf_{\xi\in\mathbb{R}}\|\kappa^{2^{-j}}(x-\cdot)-\xi\|_{L_\infty(\Delta_x^j\cap\tilde\Delta)}\int_{\tilde\Delta}|(\tilde f-\tilde P\tilde f)(y)|\,dy \\
&\le C2^{jd}2^{j-k}\sum_{\tilde\Delta\in\tilde T_k^d:\tilde\Delta\cap\Delta_x^j\neq\emptyset}\int_{\tilde\Delta}|\tilde f-\tilde P\tilde f)(y)|\,dy \\
&\le C2^{jd}2^{j-k}\sum_{\tilde\Delta\in\tilde T_k^d:\,\mathrm{dist}_\infty(\tilde\Delta,\Delta)\le C_12^{-j}}\int_{\tilde\Delta}|\tilde f-\tilde P\tilde f|\,dy, \qquad x\in\Delta\in\tilde T_k^d,
\end{aligned}
$$

if the constant C_1 is suitably chosen depending on d. Since moment conditions of the kernel κ did not play a role in this part, the estimate will also hold for $j=0$ and $\kappa_0(1,\tilde f-\tilde P\tilde f)$.

We next compute the $L_p(\mathbb{R}^d)$ quasi-norm of $\kappa(2^{-j},\tilde f-\tilde P\tilde f)$:

$$
\begin{aligned}
\|\kappa(2^{-j},\tilde f-\tilde P\tilde f)\|^p_{L_p(\mathbb{R}^d)} &= \sum_{\Delta\in\tilde T_k^d}\|\kappa(2^{-j},\tilde f-\tilde P\tilde f)\|^p_{L_p(\Delta)} \\
&\le C2^{jdp}2^{(j-k)p}\sum_{\Delta\in\tilde T_k^d}2^{-kd}\left(\sum_{\tilde\Delta\in\tilde T_k^d:\,\mathrm{dist}_\infty(\tilde\Delta,\Delta)\le C_12^{-j}}\int_{\tilde\Delta}|\tilde f-\tilde P\tilde f|\,dy\right)^p \\
&\le C2^{jdp}2^{(j-k)p}\sum_{\Delta\in\tilde T_k^d}2^{-kd}\sum_{\tilde\Delta\in\tilde T_k^d:\,\mathrm{dist}_\infty(\tilde\Delta,\Delta)\le C_12^{-j}}(\int_{\tilde\Delta}|\tilde f-\tilde P\tilde f|\,dy)^p \\
&\le C2^{jd(p-1)}2^{(j-k)p}\sum_{\tilde\Delta\in\tilde T_k^d}(\int_{\tilde\Delta}|\tilde f-\tilde P\tilde f|\,dy)^p \\
&= C2^{jd(p-1)}2^{(j-k)p}\bar s_k^p, \qquad j=1,\dots,k-1,
\end{aligned}
$$

where in the change of summation step we used that the number appearances of integrals over any fixed $\tilde\Delta\in\tilde T_k^d$ is bounded by $C2^{d(k-j)}$. The notation

$$
\bar s_k := \left(\sum_{\tilde\Delta\in\tilde T_k^d}(\int_{\tilde\Delta}|\tilde f-\tilde P\tilde f|\,dy)^p\right)^{1/p} \tag{41}
$$

is introduced for convenience. The estimate also holds for $j=0$ with κ replaced by κ_0.

This eventually gives

$$
\|\kappa_0(1,\tilde P\tilde f)\|^q_{L_p(\mathbb{R}^d)}+\sum_{j=1}^{k-1}2^{jd(1/p-1)q}\|\kappa(2^{-j},\tilde P\tilde f)\|^q_{L_p(\mathbb{R}^d)}\le C2^{-kq}\bar s_k^q\sum_{j=0}^{k-1}2^{jq}\le C\bar s_k^q.
$$

Together with (38), we arrive at

$$\|\tilde{P}\tilde{f}\|^*_{B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)} \le C(s_{k+1} + \bar{s}_k + \|\tilde{f}\|^*_{B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)}). \tag{42}$$

Step 3. It remains to deal with the terms s_{k+1} and $\bar{s}_k$ in (42) which do not depend on q. This task is reminiscent of the estimation of the right-hand side in (15) in the proof of Theorem 1. We show all details for $\bar{s}_k$, the estimates for s_{k+1} are analogous, we only indicate the changes in the argument.

We explore the atomic decomposition (25) of $\tilde{f} \in Y \cap B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)$, and observe that for $0 < q \le p$ we have

$$\|\tilde{f}\|^+_{B^{d(1/p-1)}_{p,p}(\mathbb{R}^d)} \le C\|\tilde{f}\|^+_{B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)}.$$

Therefore, it suffices to set $q = p$ and to show that

$$\bar{s}^p_k \le C\sum_{j=0}^{\infty} 2^{-jdp} c_j^p, \qquad c_j := \left(\sum_{\mathbf{i}\in\mathbb{Z}^d} |c_{j,\mathbf{i}}|^p\right)^{1/p}, \tag{43}$$

since, after taking the infimum in (43) with respect to all atomic decompositions representing the same $\tilde{f}$, we get the desired bound

$$\bar{s}_k \le C\|\tilde{f}\|^+_{B^{d(1/p-1)}_{p,p}(\mathbb{R}^d)} \le C\|\tilde{f}\|^+_{B^{d(1/p-1)}_{p,q}(\mathbb{R}^d)}, \qquad 0 < q \le p.$$

For each integral over a dyadic cube $\tilde{\Delta} \in \tilde{T}^d_k$ in $\bar{s}^p_k$, we estimate

$$\int_{\tilde{\Delta}} |\tilde{f} - \tilde{P}\tilde{f}|\,dx \le \sum_{j=0}^{\infty} \sum_{\mathbf{i}:\,\mathrm{supp}(a_{j,\mathbf{i}})\cap\tilde{\Delta}\neq\emptyset} |c_{j,\mathbf{i}}| \int_{\tilde{\Delta}} |a_{j,\mathbf{i}} - \tilde{P}a_{j,\mathbf{i}}|\,dx.$$

For $j > k$, we can estimate the relevant terms in the sum by

$$\int_{\tilde{\Delta}} |a_{j,\mathbf{i}} - \tilde{P}a_{j,\mathbf{i}}|\,dx \le C\int_{\tilde{\Delta}} |a_{j,\mathbf{i}}|\,dx \le C2^{-jd},$$

and each such term may appear only for $\le C$ different $\tilde{\Delta}$ (this C depends on C_0 and d). For $j \le k$, we explore the C^1 continuity of the atoms (recall that we assumed $a \in \mathcal{C}^\sigma_{C_0}$ with $\sigma > 1$ in (24)) which gives

$$\begin{aligned}|a_{j,\mathbf{i}}(x) - \tilde{P}a_{j,\mathbf{i}}(x)| &\le |a_{j,\mathbf{i}}(x) - av_{\tilde{\Delta}}(a_{j,\mathbf{i}})| + \sum_{h\in\bar{H}^d_{k+1}:\,\mathrm{supp}(h)=\tilde{\Delta}} 2^{kd/2}|c_h(a_{j,\mathbf{i}})| \\ &\le C2^{j-k}, \qquad x \in \tilde{\Delta},\end{aligned}$$

where C depends on C_0, σ, and d. Thus, in this case we get

$$\int_{\tilde{\Delta}} |a_{j,\mathbf{i}} - \tilde{P}a_{j,\mathbf{i}}|\,dx \le C2^{j-k}2^{-kd},$$

where each such term appears for $\leq C2^{(k-j)d}$ different $\tilde{\Delta}$.

Substitution into the expression (41) for $\bar{s}_k^p$ in gives

$$\begin{aligned}
\bar{s}_k^p &= \sum_{\tilde{\Delta}\in\tilde{T}_k^d} (\int_{\tilde{\Delta}} |\tilde{f}-\tilde{P}\tilde{f}|\,dy)^p \\
&\leq C\left(\sum_{j=0}^{k}\sum_{\mathbf{i}\in\mathbb{Z}^d} 2^{(k-j)d}2^{(j-k-kd)p}|c_{j,\mathbf{i}}|^p + \sum_{j=k+1}^{\infty}\sum_{\mathbf{i}\in\mathbb{Z}^d} 2^{-jdp}|c_{j,\mathbf{i}}|^p\right) \\
&= C\left(2^{k(-1+d(1/p-1))p}\sum_{j=0}^{k} 2^{j(-d/p+1)p}c_j^p + \sum_{j=k+1}^{\infty} 2^{-jdp}c_j^p\right).
\end{aligned}$$

Since $1-d(1/p-1)<0$ for our parameter range $p>d/(d+1)$ implies

$$2^{k(-1+d(1/p-1))p}2^{j(-d/p+1)p} = 2^{-(k-j)(1-d(1/p-1))p}2^{-jdp} \leq 2^{-jdp},$$

this proves (43).

To estimate s_{k+1}^p by the right-hand side in (43), instead of the terms $\|\tilde{f}-\tilde{P}\tilde{f}\|^p_{L_1(\tilde{\Delta})}$ with $\tilde{\Delta}\in\tilde{T}_k^d$, we must now consider the terms

$$\begin{aligned}
2^{-kdp}|av_{\Delta'}(\tilde{f})-av_{\Delta''}(\tilde{f})|^p &= 2^{dp}\left|\int_{\Delta'}\tilde{f}\,dy - \int_{\Delta''}\tilde{f}\,dx\right|^p \\
&\leq 2^{dp}\sum_{j=0}^{\infty}\sum_{\mathbf{i}\in\mathbb{Z}^d:\,\mathrm{supp}(a_{j,\mathbf{i}})\cap(\Delta'\cup\Delta'')\neq\emptyset} |c_{j,\mathbf{i}}|^p\left|\int_{\Delta'}a_{j,\mathbf{i}}\,dy - \int_{\Delta''}a_{j,\mathbf{i}}\,dx\right|^p
\end{aligned}$$

for neighboring dyadic cubes Δ',Δ'' in $\tilde{T}_{k+1}$. But the estimates for the quantities

$$\left|\int_{\Delta'}a_{j,\mathbf{i}}\,dy - \int_{\Delta''}a_{j,\mathbf{i}}\,dx\right|^p$$

in the two cases $j>k$ and $j\leq k$ look completely the same as the estimates for $\|a_{j,\mathbf{i}} - \tilde{P}a_{j,\mathbf{i}}\|_{L_1(\tilde{\Delta})}$. The remaining steps can be repeated without change.

Together with (42), we have shown the uniform boundedness of the partial sum operators $\tilde{P}$ in $B^s_{p,q}(\mathbb{R}^d)$. This finishes the argument.

References

[1] R.A. DeVore, V.A. Popov, Interpolation of Besov spaces, Trans. Amer. Math. Soc. 305:1, 1988, 397–414.

[2] D.E. Edmunds, H. Triebel, Function spaces, entropy numbers, and differential operators, Cambridge Tracts in Mathematics v. 120, Cambr. Univ. Press, 1996.

[3] G. Garrigos, A. Seeger, and T. Ullrich, The Haar system as a Schauder basis in spaces of Hardy-Sobolev type, J. Four. Anal. Appl., to appear, arXiv:1609.08225.

[4] P. Oswald, On inequalities for spline approximation and spline systems in the space L^p $(0 < p < 1)$, in Approximation and Function Spaces, Proc. Int. Conf. Gdansk 1979 (Z. Ciesielski, ed.), PWN Warszawa/North-Holland Amsterdam, pp. 531–552, 1981.

[5] P. Oswald, Approximation by splines in the L^p metric, $0 < p < 1$, Math. Nachrichten, 94, 1980, 69–96 (in Russian).

[6] C. Schneider, On dilation operators in Besov spaces, Rev. Mat. Complut. 22:1, 2009, 111–128.

[7] H. Triebel, On Haar bases in Besov spaces, Serdica 4, 1978, 330–343.

[8] H. Triebel, Theory of Function Spaces III, Monogr. Math. 100, Birkhäuser, Basel, 2006.

[9] H. Triebel, Function Spaces and Wavelets on Domains, EMS Tracts in Mathematics 7, Europ. Math. Soc., Zurich, 2008.

[10] H. Triebel, Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration, EMS Tracts in Mathematics 11, Europ. Math. Soc., Zurich, 2010.

[11] W. Yuan, W. Sickel, D. Yang, Regularity properties of the orthonormal tensor product Haar system with respect to Besov-type spaces, Preprint, 2018.