Institut für Numerische Simulation
Rheinische Friedrich-Wilhelms-Universität Bonn
Endenicher Allee 19b • 53115 Bonn • Germany
phone +49 228 73-69828 • fax +49 228 73-69847
www.ins.uni-bonn.de

F. Hoppe, I. Neitzel

Convergence of the SQP Method for Quasilinear Parabolic Optimal Control Problems

INS Preprint No. 1907

November 2019

Optimization and Engineering manuscript No.
(will be inserted by the editor)

Convergence of the SQP Method for Quasilinear Parabolic Optimal Control Problems

Fabian Hoppe · **Ira Neitzel**

Received: date / Accepted: date

Abstract Based on the theoretical framework recently proposed by Bonifacius and Neitzel (2018) we discuss the sequential quadratic programming (SQP) method for the numerical solution of an optimal control problem governed by a quasilinear parabolic partial differential equation. Following well-known techniques, convergence of the method in appropriate function spaces is proven under some common technical restrictions.

Particular attention is payed to how the second order sufficient conditions for the optimal control problem and the resulting L^2-local quadratic growth condition influence the notion of "locality" in the SQP method. Further, a new regularity result for the adjoint state, which is required during the convergence analysis, is proven. Numerical examples illustrate the theoretical results.

Keywords optimal control · quasilinear parabolic partial differential equation · sequential quadratic programming · convergence analysis

Mathematics Subject Classification (2010) 35K59 · 49K20 · 90C48 · 49N60 · 65K10 · 90C55 · 49M15 · 49M37

1 Overview

Optimal control problems governed by linear and semilinear parabolic partial differential equations (PDEs) have been subject to intense research for several years.

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Projektnummer 211504053 - SFB 1060.

Fabian Hoppe
Institut für Numerische Simulation, Universität Bonn, Endenicher Allee 19b, 53115 Bonn, Germany,
Tel.: +49 228 73-69744, E-mail: hoppe@ins.uni-bonn.de

Ira Neitzel
Institut für Numerische Simulation, Universität Bonn, Endenicher Allee 19b, 53115 Bonn, Germany,
Tel.: +49 228 73-69837, E-mail: neitzel@ins.uni-bonn.de

Existence- and regularity of their solutions is well understood, first order necessary and second order sufficient optimality conditions have been proven, and discretization errors for different types of discretization are available, see e.g. the pioneering work of Lions (1971) concerned with linear PDEs and Hinze et al. (2009), or Tröltzsch (2010) for a recent overview covering theoretical and numerical aspects of both linear and nonlinear problems.

Recently, optimal control of quasilinear parabolic equations was addressed by Bonifacius and Neitzel (2018), Casas and Chrysafinos (2018), and Meinlschmidt et al. (2017a,b), Meinlschmidt and Rehberg (2016). The functional analytic framework for the analysis of the state equation is provided by the concept of maximal parabolic regularity of nonautonomous operators, see e.g. the work of Amann (2004, 2003, 2005), Meinlschmidt and Rehberg (2016), Haller-Dintelmann and Rehberg (2009), or further references in Bonifacius and Neitzel (2018). The highly non-trivial existence and regularity theory for solutions of the underlying PDE poses the main difficulty in the theoretical analysis of such problems. For a discussion of previous literature concerning optimal control of quasilinear PDEs see the introduction of Bonifacius and Neitzel (2018) and Casas and Chrysafinos (2018), respectively. In particular, optimal control of quasilinear elliptic equations has been considered by Casas and Tröltzsch (2009, 2011, 2012); Casas and Dhamo (2011) and Yousept (2013); de Los Reyes and Dhamo (2016); Nicaise and Tröltzsch (2017). Several physical models lead to quasilinear PDEs (e.g. temperature-dependent thermal conductivity), which motivates the analysis of this challenging class of problems from the applied point of view, see e.g. the socalled thermistor problem (Meinlschmidt et al. 2017a,b).

For the efficient numerical solution of nonlinear optimal control problems sequential quadratic programming (SQP) methods form a prominent class of state of the art algorithms: The nonlinear optimization problem is approximated by a sequence of linear quadratic subproblems that can be solved e.g. by application of the well-understood primal dual active set strategy. The analysis of such SQP methods for nonlinear optimal control problems has been addressed by several researchers, see e.g. Tröltzsch (1999), Goldberg and Tröltzsch (1998) for semilinear parabolic equations, Hintermüller and Hinze (2006), Hinze and Kunisch (2001), Wachsmuth (2007) for optimal control of time-dependent Navier-Stokes equation, Griesse et al. (2010, 2008) for semilinear elliptic problems with mixed constraints, and Heinkenschloss and Tröltzsch (1999) for optimal control of a phase field equation. For an overview concerning the origins of SQP methods in the context of PDE-constrained optimization we also refer to the introduction of Goldberg and Tröltzsch (1998). As further second order methods for the solution of nonlinear optimal control problems we mention the semismooth Newton method and versions of the primal dual active set strategy, respectively, see e.g. Hinze and Kunisch (2001), Hintermüller et al. (2007), Ito and Kunisch (2004).

In the present paper, we focus on the numerical solution of quasilinear parabolic optimal control problems by the SQP method. To our best knowledge, a corresponding convergence analysis in function space has not been carried out in the existing literature. The most closely related existing publications are those by Ulbrich and Ziems (Ulbrich and Ziems 2017; Ziems 2013; Ziems and Ulbrich 2011) and chapter 8 in the thesis of Feldhordt (2017), respectively. Ulbrich and Ziems consider trust-region and trust-region SQP methods for optimal control of general nonlinear PDE. The main difference to our result is that they inlcude discretization in their work and prove convergence of adaptive multilevel algorithms whereas we stick to the function space setting. In return, we are able to prove locally superlinear convergence around local minima fulfilling certain second order conditions avoiding the two norm gap (Ioffe 1979; Casas and Tröltzsch 2012), whereas Ulbrich and Ziems establish global convergence to a point fulfilling first order optimality conditions, but without explicit rate. Feldhordt (2017) considers optimal control of the socalled chemotaxis system and proves convergence of the SQP method assuming a rather strong second order sufficient condition. This corresponds to our interim result in Section 6.1, whereas our main focus during the rest of the paper is on the interplay of weaker second order conditions and the notation of “locality” in the SQP method. The second order sufficient conditions we refer to in the present paper are due to Bonifacius and Neitzel (2018). For the topic of second order conditions in PDE-constrained optimization in general we refer to Goldberg and Tröltzsch (1989), Bonnans (1998), or the recent survey by Casas and Tröltzsch (2015) and the references therein.

Many of our arguments in the present paper are similiar to those known from earlier publications. However, we believe that our consideration is of interest for three main reasons:

1. First, we demonstrate that the results on optimal control of quasilinear parabolic PDE obtained by Bonifacius and Neitzel (2018) allow to derive convergence of the SQP method. In particular, existence and regularity theory of quasilinear parabolic PDE is much more involved than the corresponding treatment of semilinear PDE. This makes the choice of the correct function spaces more complicated than in previous work on SQP methods and we believe that it is not clear a priori that –in the end– the arguments from the existing literature apply to the present model problem as well.
2. We show a new regularity result for the adjoint state in Section 7. The proof relies on maximal parabolic regularity arguments and is based on the work of Bonifacius and Neitzel (2018) and Haller-Dintelmann and Rehberg (2009). The result is crucial for our further analysis, because the improved regularity allows us to estimate the second derivative of the nonlinearity of the state equation in an appropriate way.
3. Finally, most proofs concerning convergence of the SQP method have been published before the introduction of a framework for second order sufficient conditions without two norm gap by Casas and Tröltzsch (2012). As shown by Bonifacius and Neitzel (2018) our model problem fits into this framework and hence it is natural to revisit convergence theory of the SQP method under the new aspect of absence of the two norm gap: If quadratic growth of the reduced objective functional holds L^2-locally (instead of L^∞-locally) around the optimal control, is it possible to replace L^∞-neighbourhoods from previous convergence proofs for the SQP method by L^2-neighbourhoods? – For our model problem, we give an answer to this question in Section 6.3, which is our main result.

The rest of this paper is organized as follows and keeps the main structure of previous results concerning the analysis of SQP methods, cf. in particular the work of Tröltzsch (1999), Wachsmuth (2007) and Goldberg and Tröltzsch (1998):

In Sections 2 and 3 we briefly recall the assumptions and the model problem as well as its first order optimality conditions from Bonifacius and Neitzel (2018). The idea of the SQP method is outlined together with appropriate second order sufficient conditions. To prepare the analysis of the convergence properties of the SQP method, we provide some auxiliary results that are specifically related to our quasilinear parabolic model problem in Section 4. The proof of a new regularity result for the adjoint state is postponed to Section 7. After that, we follow the standard argument to prove convergence of the SQP method in Sections 5 and 6. We utilize the connection to the Josephy-Newton method for a generalized equation originating from the first order optimality conditions. Convergence of this Newton method is proven in Section 5 and the interpretation of the iterates as the solutions of certain quadratic optimization problems is topic of Section 6. Assuming strong second order sufficient conditions we formulate our first main result in Section 6.1. The remaining two theoretical Sections 6.2 and 6.3 of the paper are devoted to the analysis of the SQP method under weaker second order assumptions. In particular we are able to replace the L^∞-neighbourhoods in the results of Tröltzsch (1999) and Wachsmuth (2007) by L^2-neighbourhoods in our final result in Section 6.3. Finally, we give short numerical examples that illustrate our theoretical findings in Section 8.

Notation For a Lipschitz domain Ω and $\theta \in (0,1]$, $k \in \mathbb{N}$, $p \in [1,\infty]$ we denote by $L^p = L^p(\Omega)$, $H^{\theta,p} = H^{\theta,p}(\Omega)$ and $W^{k,p} = W^{k,p}(\Omega)$ the usual Lebesgue-, Bessel-potential- and Sobolev-spaces, respectively. For the two latter families of spaces a subscript D denotes incorporation of previously defined homogeneous Dirichlet boundary conditions. With $H_D^{-\theta,p'}$ and $W_D^{-1,p'}$ we refer to the topological dual spaces of $H_D^{\theta,p}$ and $W_D^{1,p}$, where $\langle\cdot,\cdot\rangle$ stands for the duality pairing and –in case of Hilbert spaces– the scalar product. Norms $\|\cdot\|$ are indexed by the space they refer to. For some integrability exponent $r \in [0,\infty]$, we define the conjugate exponent r' by $1/r + 1/r' = 1$. Spaces of countinuously differentiable resp. Hölder continuous functions are denoted as usual by $\mathscr{C}^\alpha$.

The open and closed balls of radius $r > 0$ around x_0 in a normed space X are denoted by

$$\mathbb{B}_r^X(x_0) := \{x \in X\colon \|x - x_0\|_X < r\} \quad \text{and} \quad \overline{\mathbb{B}_r^X(x_0)} := \{x \in X\colon \|x - x_0\|_X \leq r\}.$$

With $(X,Y)_{r,s}$ or $[X,Y]_r$ we refer to real or complex interpolation spaces of two normed spaces X,Y, rescpectively. Given $I \subset \mathbb{R}$, a Banach space X, and a function $\phi\colon I \to X$, we denote by $\mathrm{tr}_t \phi$, $t \in I$, the trace $\phi(t) \in X$, if such a pointwise evaluation is welldefined.

The notation "... $\lesssim$ ..." will be used in order to express that "... $\leq C \cdot$ ..." holds with a generic constant $C > 0$, whose dependencies are not relevant for the present context. We use the double arrows "$\rightrightarrows$" to indicate set-valued maps.

2 Model Problem and Assumptions

We use notations and conventions of Bonifacius and Neitzel (2018) with some minor changes. We rename controls, states, adjoint states from q, u, z to u, y, p.

2.1 The Model Problem

Our model problem is the same as the one in Example 2.5 of Bonifacius and Neitzel (2018), and reads as follows:

$$\begin{cases} \min\limits_{y,u} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(I\times\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Lambda)}, \\ \text{subject to} \quad u \in U_{ad} \quad \text{and} \quad \begin{cases} \partial_t y + \mathscr{A}(y)y = Bu \\ y(0) = y_0. \end{cases} \end{cases} \tag{OCP}$$

Here, the quasilinear part $\mathscr{A}$ of the state equation is defined by

$$\mathscr{A}(y)\,\cdot\, := -\mathrm{div}(\xi(y)\mu\nabla\,\cdot\,),$$

The control operator B, Λ, and the admissible set U_{ad} will be specified in the following section.

2.2 Assumptions

We rely on the following assumptions that we repeat from Bonifacius and Neitzel (2018) with minor changes, cf. the following remark.

Assumption 1 $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a bounded domain with boundary $\partial\Omega$. $\Gamma_N \subset \partial\Omega$ is relatively open and denotes the Neumann boundary part whereas $\Gamma_D = \partial\Omega \setminus \Gamma_N$ denotes the Dirichlet boundary part equipped with homogeneous Dirichlet boundary conditions. We assume that $\Omega \cup \Gamma_N$ is Gröger regular (Bonifacius and Neitzel 2018, Definition A.1) such that every chart map in the Definition of Gröger regularity can be chosen volume preserving. The time interval $I = (0,T)$ with $T > 0$ is fixed.

Assumption 2 The function $\xi\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable with ξ'' being Lipschitz continuous on bounded subsets of $\mathbb{R}$. Let $\mu\colon \Omega \to \mathbb{R}^{d\times d}$, $\mu = \mu^T$, be measurable and uniformly bounded and coercive in the following sense:

$$0 < \mu_\bullet := \inf_{x\in\Omega} \inf_{z\in\mathbb{R}^d\setminus\{0\}} \frac{z^T\mu(x)z}{z^Tz}, \qquad \mu^\bullet := \sup_{x\in\Omega} \sup_{1\le i,j\le d} |\mu_{i,j}(x)| < \infty$$

We assume a coercivity condition $0 < \xi_\bullet \le \xi \le \xi^\bullet$ for ξ as well. With this we define as above

$$\langle \mathscr{A}(y)\varphi, \psi\rangle_{L^2(I,W_D^{1,2})} := \int_I\int_\Omega \xi(y)\mu\nabla\varphi\nabla\psi dxdt, \qquad \varphi,\psi \in L^2(I,W_D^{1,2}).$$

Assumption 3 We assume that there is $p \in (d,4)$ such that

$$-\operatorname{div}(\mu\nabla\cdot)+1\colon W_D^{1,p} \to W_D^{-1,p}$$

is a topological isomorphism and fix this choice of p. In the following we denote by $\mathscr{D}$ the domain of the unbounded operator $-\operatorname{div}(\mu\nabla\cdot)+1$ in the Bessel potential space $H_D^{-\zeta,p}$.

Assumption 4 Let $\zeta \in (0,1)$ and $s>1$ be fixed such that

$$\max\left\{1-\frac{1}{p},\frac{d}{p}\right\}<\zeta \qquad \text{and} \qquad \max\left\{\frac{2}{\zeta-d/p},\frac{2}{1-\zeta}\right\}<s$$

holds. The desired state $y_d \in L^\infty(I,L^{p/2})$, the initial condition for the state equation $y_0 \in (H_D^{-\zeta,p},\mathscr{D})_{1-1/s,s}$ and the regularization parameter $\gamma>0$ are fixed.

We introduce the measure space (Λ,ρ) by $\Lambda=\{\bullet\}^m\times I$ equippend with measure ρ being the product of the counting measure on the $m-$element set $\{\bullet\}^m$ with the Lebesgue measure on I. Within the control space $U:=L^s(\Lambda,\rho)=L^s(I,\mathbb{R}^m)$ the set of admissible controls is given by

$$U_{ad}:=\{u\in U\colon u_a\le u\le u_b \qquad \rho\text{-a.e. on }\Lambda\}$$

with fixed control bounds $u_a,u_b\in L^\infty(\Lambda)$. Finally, for fixed control basis functions $b_1,\dots,b_m\in H_D^{-\zeta,p}$ we define the bounded linear control operator by

$$B\colon U\to L^s(I,H_D^{-\zeta,p}), \qquad (Bu)(t):=\sum_{i=1}^m u_i(t)b_i.$$

Remark 1 The choice of control space and operator ("purely timedependent controls") corresponds to Example 2.5 of Bonifacius and Neitzel (2018), where the control space is chosen as $L^\infty(\Lambda)$ instead of $L^s(\Lambda)$. We will make use of measuring controls in L^s instead of L^∞ when applying the "interpolation trick", see the remark concluding Section 6.1. The reason for choosing purely timedependent controls – apart from practical motivation, see e.g. de Los Reyes et al. (2008)– is outlined in the remark at the end of Section 5.1. The symmetry property $\mu=\mu^T$ as well as the slightly higher spatial integrability of the desired state y_d ($L^{p/2}$ instead of L^2) are required to derive improved regularity for the adjoint state in Section 7.

3 Optimality Conditions and SQP Method

We follow Goldberg and Tröltzsch (1998), Tröltzsch (1999), Wachsmuth (2007). From Bonifacius and Neitzel (2018), Section 4.1, recall the following notation:

$$\begin{aligned}\mathscr{A}'(y)v &:= -\operatorname{div}\left(\xi(y)v\mu\nabla y\right),\\ \mathscr{A}''(y)[v_1,v_2] &:= -\operatorname{div}\left(\xi'(y)(v_1\mu\nabla v_2+v_2\mu\nabla v_1)+\xi''(y)v_1v_2\mu\nabla y\right)\end{aligned}$$

for $v,v_1,v_2\in W^{1,r}(I,W_D^{-1,p})\cap L^r(I,W_D^{1,p})$ and $r\in(1,\infty)$. The divergence operators have to be understood in weak form, of course.

3.1 First Order Necessary Optimality Conditions

In Bonifacius and Neitzel (2018), Lemma 4.1, the existence of a global solution to (**OCP**) is established. Further, any local solution to (**OCP**) fulfills the following system of equations, cf. Bonifacius and Neitzel (2018, Lemmas 4.6-4.8):

$$\begin{aligned} \partial_t y + \mathscr{A}(y)y &= Bu, \\ y(0) &= y_0 \end{aligned} \tag{SE}$$

$$\begin{aligned} -\partial_t p + \mathscr{A}(y)^* p + \mathscr{A}'(y)^* p &= y - y_d, \\ p(T) &= 0 \end{aligned} \tag{AE}$$

$$(\gamma u + B^* p, v - u)_{L^2(\Lambda)} \geq 0 \text{ for all } v \in U_{ad}, \tag{FON}$$

This optimality system consists of the state equation (SE), the adjoint equation (AE), and the variational inequality (FON). The underlying function spaces are introduced in the next section. For reasons of shortness we will sometimes write the state equation as

$$e(y,u) := (\partial_t y + \mathscr{A}(y)y - Bu, \quad \mathrm{tr}_0 y - y_0) = 0 \tag{1}$$

with the $\mathscr{C}^2$-map

$$e\colon (W^{1,s}(I, W_D^{-1,p}) \cap L^s(I, W_D^{1,p})) \times L^s(\Lambda) \to L^s(I, W_D^{-1,p}) \times (W_D^{-1,p}, W_D^{1,p})_{1-1/s,s}.$$

By $L_{OCP}(y,u,p) := J(y,u) - \langle p, e_1(y,u) \rangle$ we denote the Lagrangian of (**OCP**).

3.2 Generalized Equation and SQP Method

We reformulate the optimality system as the generalized equation

$$0 \in F(y,u,p) + N(y,u,p) \tag{GE}$$

with the maps

$$F(y,u,p) := \begin{pmatrix} \partial_t y + \mathscr{A}(y)y - Bu \\ \mathrm{tr}_0 y - y_0 \\ -\partial_t p + \mathscr{A}(y)^* p + \mathscr{A}'(y)^* p - (y - y_d) \\ \mathrm{tr}_T p \\ \gamma u + B^* p \end{pmatrix}$$

$$\text{and} \qquad N(y,u,p) := \left(\{0\}, \{0\}, \{0\}, \{0\}, N_{U_{ad}}(u)\right)^T,$$

where $N_{U_{ad}}(u)$ denotes the normal cone of the closed convex set U_{ad} at the point $u \in L^s(\Lambda)$, i.e. $N_{U_{ad}}(u) = \left\{ v \in L^s(\Lambda) \colon (v, w-u)_{L^2(\Lambda)} \leq 0 \text{ for all } w \in U_{ad} \right\}$. To make the definition of F and N precise, F is understood as map $F\colon X_s \to Z_s$ with

$$X_s := \left(W^{1,s}(I, W_D^{-1,p}) \cap L^s(I, W_D^{1,p}) \right) \times L^s(\Lambda) \times \left(W^{1,s}(I, W_D^{-1,p'}) \cap L^s(I, W_D^{1,p'}) \right)$$

and

$$Z_s := L^s(I, W_D^{-1,p}) \times (W_D^{-1,p}, W_D^{1,p})_{1-1/s,s} \times L^s(I, W_D^{-1,p'}) \\ \times (W_D^{-1,p'}, W_D^{1,p'})_{1-1/s,s} \times L^s(\Lambda).$$

Accordingly, N is understood as set valued map $X_s \rightrightarrows Z_s$. We equip X_s and Z_s with the canonical norms

$$\|(y,u,p)\|_{X_s} := \|y\|_{W^{1,s}(I,W_D^{-1,p})\cap L^s(I,W_D^{1,p})} + \|u\|_{L^s(\Lambda)} + \|p\|_{W^{1,s}(I,W_D^{-1,p'})\cap L^s(I,W_D^{1,p'})},$$
$$\|(f,y_0,g,p_T,r)\|_{Z_s} := \|f\|_{L^s(I,W_D^{-1,p})} + \|y_0\|_{(W_D^{-1,p},W_D^{1,p})_{1-1/s,s}} + \|g\|_{L^s(I,W_D^{-1,p'})} \\ + \|p_T\|_{(W_D^{-1,p'},W_D^{1,p'})_{1-1/s,s}} + \|r\|_{L^s(\Lambda)}.$$

Having chosen these spaces, the following result holds:

Lemma 1 *F: $X_s \to Z_s$ is continuously Fréchet differentiable and N: $X_s \rightrightarrows Z_s$ has closed graph.*

Proof Differentiability has been used implicitely by Bonifacius and Neitzel (2018, Lemma 4.5) where the differentiability of the control to state map is shown by the implicit function theorem. The closed graph property is standard. □

Sometimes we will need the following subspaces X_∞ and Z_∞ of X_s, Z_s:

$$X_\infty := \left(W^{1,s}(I, W_D^{-1,p}) \cap L^s(I, W_D^{1,p})\right) \times L^\infty(\Lambda) \\ \times \left(W^{1,s}(I, W_D^{-1,p'}) \cap L^s(I, W_D^{1,p'})\right),$$
$$Z_\infty := L^s(I, W_D^{-1,p}) \times (W_D^{-1,p}, W_D^{1,p})_{1-1/s,s} \times L^s(I, W_D^{-1,p'}) \\ \times (W_D^{-1,p'}, W_D^{1,p'})_{1-1/s,s} \times L^\infty(\Lambda),$$

equipped with the canonical norms similiarly as above. Note that changing from X_s, Z_s to X_∞, Z_∞ means nothing more than replacing the $L^s(\Lambda)$-factors by $L^\infty(\Lambda)$-factors, i.e. considering controls in the L^∞- instead of the L^s-norm. The same result as before holds:

Lemma 2 *F: $X_\infty \to Z_\infty$ is continuously Fréchet differentiable and N: $X_\infty \rightrightarrows Z_\infty$ has closed graph.*

Due to Lemma 1 we can formulate the ansatz of the SQP method in its abstract form as the Josephy-Newton method for generalized equations, see Josephy (1979); Dontchev (1996); Alt (1990), or Hinze et al. (2009, chapter 2): Given an iterate $(y_k, u_k, p_k) \in X_s$, solve

$$0 \in F(y_k, u_k, p_k) + F'(y_k, u_k, p_k)(y - y_k, u - u_k, p - p_k) + N(y, u, p) \quad (2)$$

to obtain the new iterate $(y_{k+1}, u_{k+1}, p_{k+1}) \in X_s$. Writing down the full system of equations for (2) we find:

$$\begin{aligned} \partial_t y + \mathscr{A}(y_k)y + \mathscr{A}'(y_k)y &= Bu + \mathscr{A}'(y_k)y_k \\ \mathrm{tr}_0 y &= y_0 \end{aligned} \tag{3}$$

$$\begin{aligned} -\partial_t p + \mathscr{A}(y_k)^* p + \mathscr{A}'(y_k)^* p &= y - y_d - \mathscr{A}''(y_k)[y - y_k, \cdot]^* p_k \\ \mathrm{tr}_T p &= 0 \end{aligned} \tag{4}$$

$$0 \in \gamma u + B^* p + N_{U_{ad}}(u). \tag{5}$$

Obviously, the current u-iterate u_k has canceled out, which implies that the next iterate (y, u, p) depends on y_k and p_k but not on u_k. This is due to the structure of our model problem. Note that the first two equations (3) are equivalent to the linearized state equation

$$0 = e(y_k, u_k) + e_y(y_k, u_k)(y - y_k) + e_u(y_k, u_k)(u - u_k). \tag{6}$$

A standard computation shows that

$$\frac{1}{2} L''_{OCP}(y_k, u_k, p_k)[(y - y_k, u - u_k)]^2 + J'(y_k, u_k)(y - y_k, u - u_k) \tag{7}$$

is equal (up to addition of constants) to the expression

$$J_k(y, u) := \frac{1}{2}\|y - y_d\|^2 + \frac{\gamma}{2}\|u\|^2 - \frac{1}{2}\langle p_k, \mathscr{A}''(y_k)[y - y_k, y - y_k]\rangle, \tag{8}$$

that finally fulfills: The system of equations (3),(4),(5) is the formal optimality system of the following optimal control problem:

$$\begin{cases} \min\limits_{y,u} J_k(y, u) \\ \text{subject to} \quad u \in U_{ad} \quad \text{and equation (3).} \end{cases} \tag{\textbf{QP}}$$

This is the classical formulation of the SQP method as sequence of quadratic problems to solve. Note that these computations were completely formal in the sense that we do not know whether (**QP**) is convex or not. Hence, we cannot say whether there is a unique minimizer or whether the optimality system (3),(4),(5) is a sufficient characterization for this minimizer. This issue will be addressed in the following section utilizing the assumption of second order sufficient conditions.

3.3 Second Order Sufficient Conditions and SQP

Depending on second order sufficient conditions (SSCs) for (**OCP**) based on those derived in Bonifacius and Neitzel (2018) we have to restrict the admissible set for (**QP**) to ensure convexity.

Assumption 5 From now on let $\bar{u} \in U_{ad}$ be a fixed L^2-local minimizer for (**OCP**), i.e. there is $r > 0$ such that

$$u \in U_{ad} \text{ and } \|u-\bar{u}\|_{L^2(I,\mathbb{R}^m)} < r \qquad \Longrightarrow \qquad j(u) \geq j(\bar{u}).$$

Let $\bar{y}$ and $\bar{p}$ the state and adjoint state associated with $\bar{u}$. For $\sigma \geq 0$ we define the σ-active set of $\bar{u}$ as

$$A^\sigma(\bar{u}) := \{x \in \Lambda \colon |\gamma\bar{u} + B^*\bar{p}|(x) > \sigma\}$$

and the corresponding subspace

$$C^\sigma(\bar{u}) := \{v \in L^2(\Lambda) \colon v = 0 \text{ on } A^\sigma(\bar{u})\}$$

of directions vanishing on $A^\sigma(\bar{u})$. We assume that the following second order sufficient condition for (**OCP**) is satisfied at $\bar{u}$: There exists $\delta > 0$ such that

$$\begin{cases} L''_{OCP}(\bar{y},\bar{u},\bar{p})[(y,u)]^2 \geq \delta\|u\|^2_{L^2(\Lambda)} \\ \text{for all} \quad (y,u) \in W^{1,2}(I,W_D^{-1,p}) \cap L^2(I,W_D^{1,p}) \times L^2(\Lambda) \quad \text{s.t.} \\ \qquad u \in C^\sigma(\bar{u}), \\ \qquad e_y(\bar{y},\bar{u})y + e_u(\bar{y},\bar{u})u = 0. \end{cases} \tag{SSC-σ}$$

Condition (SSC-σ) is stronger than the second order sufficient condition derived by Bonifacius and Neitzel (2018, Theorem 4.14) which has smallest possible gap to the corresponding necessary condition. However we conclude from the cited result:

Theorem 1 *Let Assumption 5 hold with $\sigma \geq 0$. Then there are $\varepsilon, \eta > 0$ such that the quadratic growth condition*

$$j(u) \geq j(\bar{u}) + \eta\|u-\bar{u}\|^2_{L^2(\Lambda)}$$

holds for all $u \in U_{ad} \cap \overline{\mathbb{B}^{L^2}_\varepsilon(\bar{u})}$.

We also mention the work of Casas and Chrysafinos (2018) in which second order optimality conditions analogous to those of Bonifacius and Neitzel (2018), but for a slightly different setting w.r.t. the domain, the boundary conditions and the boundedness properties of the nonlinearity, were derived. Casas and Chrysafinos deal with $C^{1,1}$-smooth domains, homogeneous Dirichlet boundary conditions and locally Lipschitz continuous coefficients for the state equation, which enables them to consider W^2-regularity of the states.

Remark 2 Second order sufficient conditions related to strongly active sets turned out to be suitable assumptions for the analysis of SQP methods, see e.g. Tröltzsch (1999); Goldberg and Tröltzsch (1998), Wachsmuth (2007), which work with the same assumption as we do. That we do not work with the SSCs formulated by Bonifacius and Neitzel (2018) directly has two reasons: First, we require the coercivity condition in (SSC-σ) to hold on a vector space instead of just a cone in the proof of the L^2-stability result in Section 5.1. Second, in Section 6.2 we will make use of the fact that strongly active sets behave well under small perturbations for $\sigma > 0$.

Remark 3 Strongest possible second order conditions, i.e. coercivity of L''_{OCP} on the whole space $L^2(\Lambda)$ will be refered to by $\sigma = \infty$. In this case it holds $C^\infty(\bar{u}) = L^2(\Lambda)$ and $U^\infty_{ad} = U_{ad}$. See e.g. Griesse et al. (2010, 2008), Feldhordt (2017) or Heinkenschloss and Tröltzsch (1999) for such an assumption in the context of SQP methods. In Section 6.1 we state our main theorem for this special case.

For our analysis we will heavily rely on $L^\infty(I, W^{1,p'})$-regularity of the adjoint state $\bar{p}$ associated with the optimal control $\bar{u}$, cf. the remarks in Section 4.2. For better readability we postpone the proof of the corresponding regularity theorem to Section 7 and state here only

Lemma 3 *It holds* $\bar{p} \in L^\infty(I, W^{1,p'}_D)$.

Proof Set $r = s$, $y = \bar{y}$, $w = \bar{y} - y_d$ and $w_T = 0$ in Theorem 10 of Section 7. Due to $y_d \in L^\infty(I, L^{p/2})$ and $L^{p/2} \hookrightarrow H^{-\zeta,p}$ all requirements are fulfilled. It follows $\bar{p} \in W^{1,s}(I, H^{-\zeta,p}_D) \cap L^s(I, \mathscr{D})$, i.e. even $\bar{p} \in L^\infty(I, W^{1,p}_D)$ by Theorem 9 (b). □

Finally, we introduce the modified admissible set as

$$U^\sigma_{ad} := U_{ad} \cap (\bar{u} + C^\sigma(\bar{u})) = \{u \in U_{ad} \colon u = \bar{u} \text{ on } A^\sigma(\bar{u})\}$$

and define the corresponding restricted quadratic problem as follows:

$$\begin{cases} \min\limits_{y,u} J_k(y,u) \\ \text{subject to} \quad u \in U^\sigma_{ad} \quad \text{and equation (3)} \end{cases} \tag{$\mathbf{QP}$-σ}$$

Using the relation of J_k to the second derivative of the Lagrangian of (**OCP**) (see (7) and (8)) it is clear that (**QP**-σ) is a linear quadratic and under Assumption 5 strictly coercive and therefore strictly convex optimal control problem, at least for $(y_k, u_k, p_k) = (\bar{y}, \bar{u}, \bar{p})$. This will be crucial for the convergence analysis of the SQP method.

4 Auxiliary Results

Before going into the details of the convergence analysis for the SQP method we collect some auxiliary results in the following section.

4.1 A Property of the Control Operator

Recall from Assumption 4 the definition of the control operator that refers to the case of purely timedependent controls. Obviously, B is continuous from $L^2(\Lambda)$ to $L^2(I, W^{-1,p}_D)$ and therefore its adjoint B^* is defined on $L^2(I, W^{1,p'}_D)$ with values in $L^2(\Lambda)$. To derive the L^∞-stability result from the L^2-stability result in Section 5.1, we need to perform a bootstrapping argument that requires us to know how B^* behaves restricted to a space of more regular functions.

To simplify notation, let B: $L^s(I,\mathbb{R}) \to L^s(I,H^{-\zeta,p})$ be defined by $u \mapsto u \cdot b_1$ with only a single fixed control function $b_1 \in H_D^{-\zeta,p}$. Of course, this yields

$$(B^*v)(t) = \langle b_1, v(t)\rangle_{W_D^{-1,p},W_D^{1,p'}} \qquad \text{for every } v \in L^2(I,W_D^{1,p'}).$$

It is obvious that B maps $L^r(\Lambda)$ into $L^r(I,H_D^{-\zeta,p})$ for $r \in [2,\infty]$. To obtain $B^*v \in L^q(\Lambda)$, we have to ensure that $v \in L^q(I,H_D^{\zeta,p'})$ holds. We need the following Lemma:

Lemma 4 *It holds*

$$(W_D^{-1,q}, W_D^{1,q})_{\theta,1} \hookrightarrow H_D^{2\theta-1,q}$$

for $0 < \theta < 1$ *and* $q \in (1,\infty)$ *as long as* $2\theta - 1 \notin \{1/q, -1/q'\}$.

Proof This is a direct consequence of Griepentrog et al. (2002, Theorem 3.5). □

Now, set $\theta := (\zeta+1)/2$. For $r \in (1,\infty)$ there are two possibilities: If $\theta < 1-1/r$, then it holds for $0 \le \rho < 1-1/r-\theta$

$$W^{1,r}(I,W_D^{-1,p'}) \cap L^r(I,W_D^{1,p'}) \hookrightarrow \mathscr{C}^\rho(I,(W_D^{-1,p'},W_D^{1,p'})_{\theta,1}) \hookrightarrow \mathscr{C}^\rho(I,H_D^{\zeta,p'}),$$

i.e. B^* is continous from $W^{1,r}(I,W_D^{-1,p'}) \cap L^r(I,W_D^{1,p'})$ to $L^\infty(\Lambda)$. Otherwise, if $\theta > 1-1/r$, we obtain $q \ge 1$ such that $1/q > \theta - (1-1/r) > 0$ and

$$W^{1,r}(I,W_D^{-1,p'}) \cap L^r(I,W_D^{1,p'}) \hookrightarrow L^q(I,(W_D^{-1,p'},W_D^{1,p'})_{\theta,1}) \hookrightarrow L^q(I,H_D^{\zeta,p'}),$$

which means that B^* maps $W^{1,r}(I,W_D^{-1,p'}) \cap L^r(I,W_D^{1,p'})$ to $L^q(\Lambda)$. For the two embeddings we refer e.g. to Amann (2003, formula (1.2)). We will come back to this in Section 5.1: Given an estimate on the control in L^r, we have estimates for linearized state and adjoint state in $W^{1,r}(I,W_D^{-1,p}) \cap L^r(I,W_D^{1,p})$ and $W^{1,r}(I,W_D^{-1,p'}) \cap L^r(I,W_D^{1,p'})$ respectively. Application of B^* either yields an estimates for the control in L^q with some $q > r$ or in L^∞ if r already was large enough.

4.2 Some Properties of $\mathscr{A}''$

Recall the definition of $\mathscr{A}''$ from the beginning of Section 3. For the proof of the L^2- and L^∞-stability results in Section 5.1 we need the following

Lemma 5 *It holds*

$$\|\mathscr{A}''(y)[v,\cdot]^*p\|_{L^r(I,W_D^{-1,p'})} \le C(\xi,\mu,y)\|p\|_{L^\infty(I,W_D^{1,p'})}\|y\|_{L^\infty(I,W_D^{1,p})}\|v\|_{L^r(I,W_D^{1,p})}.$$

The constant C *can be chosen uniformly with respect to* y *for* y*'s coming from a bounded subset of* $W^{1,s}(I,W_D^{-1,p}) \cap L^s(I,W_D^{1,p})$.

Proof Estimate $\langle \mathscr{A}''(y)[v,\cdot]^*p, w\rangle = \langle \mathscr{A}''(y)[v,w],p\rangle$ for an arbitrary testfunction $w \in L^{r'}(I,W_D^{1,p})$ utilizing Hölders inequality. □

In Lemma 5 we bounded the norm of $\mathscr{A}''(\bar{y})[v,\cdot]^*\bar{p}$ in the space $L^r(I,W_D^{-1,p'})$ against the norm of v in the space $W^{1,r}(I,W_D^{1,p}) \cap L^r(I,W_D^{1,p})$ for each $r \in [2,s]$ by estimating $\langle \mathscr{A}''(y)[v,w],p\rangle$ with arguments $v \in L^r(I,W_D^{1,p})$ resp. $w \in L^{r'}(I,W_D^{1,p})$. This generality will be necessary in the bootstrapping argument in the proof of the L^∞-stability, which was already mentioned in the previous Section 4.1. As explained in the remark after Lemma 6, this requires bounds for y in $L^\infty(I,W_D^{1,p})$ and p in $L^\infty(I,W^{1,p'})$. However, in the next section we will require an estimate of $\langle \mathscr{A}''(y)[v,w],p\rangle$ directly (and not of $\mathscr{A}(y)''[v,\cdot]^*p$) which allows us to use the arguments v and w from the space $W^{1,2}(I,W_D^{-1,p}) \cap L^2(I,W_D^{1,p})$ in Lemma 6. In that case we can exploit more regularity of v,w, which allows to relax the assumptions on y and p.

Lemma 6 *It holds*

$$|\langle \mathscr{A}''(y)[v,w],p\rangle| \le C(\xi,\mu,y)\|y\|_{L^s(I,W_D^{1,p})}\|p\|_{L^s(I,W_D^{1,p'})} \cdot \|v\|_{W^{1,2}(I,W_D^{1,p})\cap L^2(I,W_D^{1,p})}\|w\|_{W^{1,2}(I,W_D^{1,p})\cap L^2(I,W_D^{1,p})}.$$

The constant C can be chosen uniformly with respect to y for y's coming from a bounded subset of $W^{1,s}(I,W_D^{-1,p}) \cap L^s(I,W_D^{1,p})$.

Proof The proof works similiar as for Lemma 5, but now we try to exploit more regularity of v and w. Using embeddings due to Amann (2003, formula (1.2)) and Griepentrog et al. (2002, Theorem 3.5) we find

$$W^{1,2}(I,W_D^{-1,p}) \cap L^2(I,W_D^{1,p}) \hookrightarrow L^q(I,L^\infty),$$

with some $q \in (2,\infty)$ satisfying

$$\frac{2}{q}+\frac{2}{s} \le 1 \qquad \text{and} \qquad \frac{1}{q}+\frac{1}{2}+\frac{1}{s} \le 1. \tag{9}$$

Now, an application of Hölders inequality (the temporal integrability exponents match due to (9)) yields the desired result. The uniform choice of the constant with respect to y follows from the boundedness of ξ and its derivatives on bounded sets of $\mathbb{R}$ and the compactness of the embedding $W^{1,s}(I,W_D^{-1,p}) \cap L^s(I,W_D^{1,p}) \hookrightarrow \mathscr{C}^0(\overline{I\times\Omega})$. □

Remark 4 The difference in the regularities assumed for y and p in the two Lemmas is essential: Lemma 5 will be applied in Section 5.1 only for $y=\bar{y}$ and $p=\bar{p}$, i.e. the required regularity is guaranteed by Lemma 3 for $\bar{p}$ and Theorem 9 (1), (2b) for $\bar{y}$, respectively. In Section 4.3 we will have to apply Lemma 6 for $y=y_k$, $p=p_k$ with y_k,p_k being iterates of the SQP method, i.e. y_k and p_k are solutions of the linearized state and adjoint equation. Hence, the regularity requirements for Lemma 6 are met, but not immediately those of Lemma 5.

Remark 5 (Necessity of higher regularity for the adjoint state) Note that Lemma 5 cannot be improved: The limiting factor is the summand

$$\int_{I\times\Omega} \xi'(y)w\nabla p\nabla v,$$

which has to be estimated for $v \in W^{1,r}(I, W_D^{-1,p}) \cap L^r(I, W_D^{1,p})$ and $w \in L^{r'}(I, W_D^{1,p})$, $r \in [2,s]$. The function w has temporal integrability r' and spatial integrability ∞, whereas ∇v has temporal integrability r and spatial integrability p, which is the best we can expect from the assumptions each. This implies that we require $p \in L^\infty(I, W_D^{1,p'})$ in order to be able to estimate the above integral.

4.3 Derivatives associated to (**QP**)

In this section we provide results on the first and second derivatives of the reduced objective functionals associated to the quadratic subproblems (**QP**). We will apply them in Section 6.3 briefly before obtaining our main result.

Recall the definition of the space X_s from Section 3.2 and denote by $j_k \colon L^2(\Lambda) \to \mathbb{R}$ the reduced functional associated with the linear quadratic optimal control problem (**QP**) at $(y_k, u_k, p_k) \in X_s$. In particular note that j_k'' is constant, because j_k is a quadratic functional, which makes us write j_k'' instead of $j_k''(v)$ for some v, because $v \mapsto j_k''(v)[\cdot,\cdot]$ is constant and hence independent of such v.

Proposition 1 *Let Assumptions 1-4 and 5 be satisfied. Then, it holds uniformly in* $u \in L^2(\Lambda)$

$$|(j_k'' - j''(\bar{u}))u^2| \lesssim \Big(\|y_k - \bar{y}\|_{W^{1,s}(I, W_D^{-1,p}) \cap L^s(I, W_D^{1,p})} + \|p_k - \bar{p}\|_{W^{1,s}(I, W_D^{-1,p'}) \cap L^s(I, W_D^{1,p'})} \Big) \|u\|_{L^2}^2$$

as $y_k \to \bar{y}$, $p_k \to \bar{p}$ *in the above norms.*

Proof Recall by (7) that $j_k'' \cdot u^2 = L_{OCP}''(y_k, u_k, p_k)(y, u)^2$ with

$$e_y(y_k, u_k)y + e_u(y_k, u_k)u = 0, \tag{10}$$

holds. We expand this as

$$\begin{aligned} L_{OCP}''(y_k, u_k, p_k)(y,u)^2 = &\underbrace{L_{OCP}''(\bar{y}, \bar{u}, \bar{p})(\tilde{y}, u)^2}_{=:(I)} \\ &- \underbrace{\big(L_{OCP}''(\bar{y}, \bar{u}, \bar{p})(\tilde{y}, u)^2 - L_{OCP}''(\bar{y}, \bar{u}, \bar{p})(y, u)^2\big)}_{=:(II)} \\ &- \underbrace{\big(L_{OCP}''(\bar{y}, \bar{u}, \bar{p}) - L_{OCP}''(y_k, u_k, p_k)\big)(y,u)^2}_{=:(III)} \end{aligned} \tag{11}$$

with $\tilde{y} \in W^{1,2}(I, W_D^{-1,p}) \cap L^2(I, W_D^{1,p})$ defined by

$$e_y(\bar{y}, \bar{u})\tilde{y} + e_u(\bar{y}, \bar{u})u = 0. \tag{12}$$

From the definition of the Lagrangian we know $(I) = j''(\bar{u})u^2$. Hence it remains to show that the contribution of (II) and (III) gets uniformly small as claimed above. By definition we have

$$\begin{aligned}(II) &= \underbrace{\|\tilde{y}\|^2 - \|y\|^2}_{=:(IIa)} - \underbrace{\langle \bar{p}, \mathscr{A}''(\bar{y})\tilde{y}^2 - \mathscr{A}''(\bar{y})y^2\rangle}_{=:(IIb)},\\ (III) &= \langle p_k, \mathscr{A}''(y_k)y^2\rangle - \langle \bar{p}, \mathscr{A}''(\bar{y})y^2\rangle\\ &= \underbrace{\langle p_k - \bar{p}, \mathscr{A}''(y_k)y^2\rangle}_{=:(IIIa)} + \underbrace{\langle \bar{p}, (\mathscr{A}''(y_k) - \mathscr{A}''(\bar{y}))y^2\rangle}_{=:(IIIb)},\end{aligned}$$

wherein the summands

$$(IIa) = \langle \tilde{y}+y, \tilde{y}-y\rangle \qquad \text{and} \qquad (IIb) = \langle \bar{p}, \mathscr{A}''(\bar{y})[\tilde{y}+y, \tilde{y}-y]\rangle \tag{13}$$

can be estimated using the boundedness of the solution operator of the linearized state equation (Bonifacius and Neitzel 2018, Proposition 4.4) and applying Lemma 6 and a similiar argument as in the proof of Lemma 6. In particular recall Remark 4. In the same way one can treat (III) as well. □

For the gradient of j_k we find:

Proposition 2 *If $(y_k, u_k, p_k) \to (\bar{y}, \bar{u}, \bar{p})$ in X_s, $v_k \to \bar{u}$ in L^s, it holds*

$$\nabla j_k(v_k) \to \nabla j(\bar{u}), \qquad \textit{strongly in } L^2(\Lambda).$$

Proof We split

$$\nabla j_k(v_k) - \nabla j(\bar{u}) = \underbrace{\nabla j_k(v_k) - \nabla j(v_k)}_{=:(A)} + \underbrace{\nabla j(v_k) - \nabla j(\bar{u})}_{=:(B)}$$

and estimate both summands. For some $v \in U_{ad}$, e.g. $v = v_k$, introducing the following quantities will be helpfull:

$y(v)$	state associated to v w.r.t. (**OCP**),
$p(v)$	adjoint state associated to v w.r.t. (**OCP**),
$y_k(v)$	state associated to v w.r.t. (**QP**)
$p_k(v)$	adjoint state associated to v w.r.t. (**QP**).

Regarding (B) we know from Bonifacius and Neitzel (2018, Proposition 4.9) that

$$\|\nabla j(v_k) - \nabla j(\bar{u})\|_{L^2(\Lambda)} \le \gamma \|v_k - \bar{u}\|_{L^2} + \|B^*(p(v_k) - p(\bar{u}))\|_{L^2} \to 0 \quad \text{ as } v_k \to \bar{u} \text{ in } L^s,$$

holds, because the adjoint states $p(v_k)$ converge in $L^s(I, W_D^{1,p'})$ to $\bar{p}$. To estimate (A) first note that the states $y_k(v_k)$ of the quadratic problem converge to $\bar{y} = y(\bar{u})$ in $W^{1,2}(I, W_D^{-1,p}) \cap L^2(I, W_D^{1,p})$. This is shown using the convergence of the solution operators of the linearized state equation (Bonifacius and Neitzel 2018, Proposition 4.9). Utilizing similiar techniques as before the desired result follows after some straight forward computations. We omit the details. □

5 Generalized Newton Method on U_{ad}^{σ}

Following the standard arguments, see e.g. Tröltzsch (2000, 1999), Goldberg and Tröltzsch (1998), Alt et al. (2010), Griesse et al. (2010, 2008), Wachsmuth (2007) and Hintermüller and Hinze (2006), we show that the Newton-Josephy method applied to a modified version of the generalized equation (GE), see Section 3.2, converges. Our own contribution here is to verify that –under the correct choice of spaces and with help of suitable auxiliary results that have been achieved in the previous section– existing arguments apply to the quasilinear case as well.

We consider the generalized equation with modified admissible set, i.e. we replace (GE) by

$$0 \in F(y,u,p) + N^{\sigma}(y,u,p), \tag{GE-σ}$$

where U_{ad} is replaced by U_{ad}^{σ} in the definition of the normal cone map N, i.e.

$$N^{\sigma}(y,u,p) := \Big(\{0\}\,,\,\{0\}\,,\,\{0\}\,,\,\{0\}\,,\,N_{U_{ad}^{\sigma}}(u)\Big)^T,$$

where $N_{U_{ad}^{\sigma}}(u)$ denotes the normal cone of U_{ad}^{σ} at u. The map $F\colon X_s \to Z_s$ as well as the spaces X_s, Z_s, see Section 3.2 for the definitions, do not change.

To prove convergence of the generalized Newton method strong regularity in the sense of Robinson has to be shown at an optimal point $(\bar{y},\bar{u},\bar{p}) \in X_s$, i.e. for every perturbation $d \in Z_s$ sufficiently close to 0 the generalized equation

$$d \in F(\bar{y},\bar{u},\bar{p}) + F'(\bar{y},\bar{u},\bar{p})(y-\bar{y},u-\bar{u},p-\bar{p}) + N^{\sigma}(y,u,p) \tag{GE-σ-D}$$

needs to have a unique solution that depends Lipschitz continuous on $d \in Z_s$. For the definition of strong regularity we refer e.g. to Robinson (1980), Hinze et al. (2009, Definition 2.5).

Translating back this generalized equation for (y,u,p) into an optimal control problem yields

$$\begin{cases} \min\limits_{y,u} \dfrac{1}{2}\|y-y_d\|^2 + \dfrac{\gamma}{2}\|u\|^2 - \dfrac{1}{2}\langle \bar{p}, \mathscr{A}''(\bar{y})[y-\bar{y}]^2\rangle \\ \qquad\qquad + \langle d_T, \mathrm{tr}_T y\rangle - \langle d_u, u\rangle + \langle d_p, y\rangle \\ \text{subject to} \quad u \in U_{ad}^{\sigma} \\ \qquad \text{and} \quad \begin{pmatrix} d_y \\ d_0 \end{pmatrix} = e_y(\bar{y},\bar{u})(y-\bar{y}) + e_u(\bar{y},\bar{u})(u-\bar{u}) \end{cases} \tag{QP-σ-D}$$

for a given perturbation vector $d = (d_y, d_0, d_p, d_T, d_u) \in Z_s$ with components coming from the corresponding spaces. Note that (GE-σ-D) is indeed the first order necessary and (due to convexity) sufficient optimality condition for (**QP-σ-D**), because (**QP-σ-D**) is convex since only linear perturbation terms have been added to the convex objective function from (**QP-σ**). The perturbation in the corresponding affine linear state equation is only a constant and does not destroy convexity as well.

5.1 Stability of the Quadratic Problems (**QP-σ**)

We fix $d_0 = 0$ and $d_T = 0$, i.e. we assume that initial and final conditions are met exactly during the application of the SQP method, which is reasonable from the numerical point of view.

Proposition 3 *Let Assumptions 1-4 and 5 hold. Denote with $(y^i, u^i, p^i) \in X_s$, $i = 1, 2$, the solution of* (**QP-σ-D**) *for arbitrary perturbation vectors $d^i \in Z_s$. Then it holds*

$$\|u^2 - u^1\|_{L^2}^2 \lesssim \|d_u^2 - d_u^1\|_{L^2}^2 + \|d_y^2 - d_y^1\|_{L^2(I,W^{-1,p})}^2 + \|d_p^2 - d_p^1\|_{L^2(I,W^{-1,p'})}^2.$$

The hidden constant depends on the data of (**OCP**) *and $(\bar{y}, \bar{u}, \bar{p})$, but not on d^i.*

To enhance clarity we state the KKT-system of the perturbed problems, that can easily be derived from (GE-σ-D) using (2) and (3)-(5), before starting the proof:

$$\begin{cases} \partial_t y^i + \mathscr{A}(\bar{y})y^i + \mathscr{A}'(\bar{y})y^i = Bu^i + \mathscr{A}'(\bar{y})\bar{y} + d_y^i \\ y^i(0) = y_0 \\ -\partial_t p^i + \mathscr{A}(\bar{y})^* p^i + \mathscr{A}'(\bar{y})^* p^i = y^i - y_d - \mathscr{A}''(\bar{y})[y^i - \bar{y}, \cdot]^* \bar{p} + d_p^i \\ p^i(T) = 0 \\ d_u^i \in \gamma u^i + B^* p^i + N_{U_{ad}^\sigma}(u^i). \end{cases} \tag{14}$$

In the following we use the short notation $\Delta_y := y^2 - y^1$, $\Delta_u := u^2 - u^1$, $\Delta_p := p^2 - p^1$ (and similiarly for d_y, d_u, d_p). From (14) we derive:

$$\partial_t \Delta_y + \mathscr{A}(\bar{y})\Delta_y + \mathscr{A}'(\bar{y})\Delta_y = B\Delta_u + \Delta_{d_y}, \tag{15}$$

$$-\partial_t \Delta_p + \mathscr{A}(\bar{y})^* \Delta_p + \mathscr{A}'(\bar{y})^* \Delta_p = \Delta_y - \mathscr{A}''(\bar{y})[\Delta_y, \cdot]^* \bar{p} + \Delta_{d_p}, \tag{16}$$

with vanishing initial and final condition, respectively: $\Delta_y(0) = 0$ and $\Delta_p(T) = 0$.

Proof The proof relies on the linear quadratic structure of (**QP-σ-D**) and regularity results for the linearized state equation resp. the adjoint equation.

Hence it works completely analogous to Goldberg and Tröltzsch (1998) and we omit the details and only mention the required regularity results (Bonifacius and Neitzel 2018, Propositions 4.4 resp. 4.7) and that terms containing $\mathscr{A}''$ are estimated with help of Lemma 5. □

This shows L^2-stability of the quadratic problems (**QP-σ**) with respect to perturbations measured in corresponding norms. Utilizing a standard bootstrapping argument as e.g. in Tröltzsch (2000) we can show the corresponding L^s- resp. L^∞-stability result:

Theorem 2 *Let Assumptions 1-4 and 5 hold. Then, for the* (y^i, u^i, p^i), $i = 1,2$, *from the previous Proposition we have*

$$\|u^2 - u^1\|_{L^s} \lesssim \|d_u^2 - d_u^1\|_{L^s} + \|d_y^2 - d_y^1\|_{L^s(I,W^{-1,p})} + \|d_p^2 - d_p^1\|_{L^s(I,W^{-1,p'})},$$
$$\|u^2 - u^1\|_{L^\infty} \lesssim \|d_u^2 - d_u^1\|_{L^\infty} + \|d_y^2 - d_y^1\|_{L^s(I,W^{-1,p})} + \|d_p^2 - d_p^1\|_{L^s(I,W^{-1,p'})}$$

and

$$\|(y^1, u^1, p^1) - (y^2, u^2, p^2)\|_{X_s} \lesssim \|d^1 - d^2\|_{Z_s},$$
$$\|(y^1, u^1, p^1) - (y^2, u^2, p^2)\|_{X_\infty} \lesssim \|d^1 - d^2\|_{Z_\infty}.$$

In particular, the generalized equation (GE-σ) *is strongly regular at its solution* $(\bar{y}, \bar{u}, \bar{p})$ *with respect to the spaces* X_s, Z_s *and* X_∞, Z_∞.

Proof Again, the proof follows the techniques from Goldberg and Tröltzsch (1998); Tröltzsch (2000). From the projection formula $u^i = \mathrm{Proj}_{U^\sigma_{ad}}\left(-\frac{1}{\gamma}(B^* p^i - d_u^i)\right)$, $i = 1,2$, we infer that

$$|\Delta_u| \leq \frac{1}{\gamma}\left(|B^*\Delta_p| + |\Delta_{d_u}|\right)$$

holds pointwise on Λ. Thus, we can bound Δ_u in the $L^q(\Lambda)$-norm, if we can bound $B^*\Delta_p$ and Δ_{d_u} in the $L^q(\Lambda)$-norm. We apply a bootstrapping argument that relies on the property of B^* from Section 4.1: Assume that we already know

$$\|\Delta_u\|_{L^r} \lesssim \|\Delta_{d_u}\|_{L^r} + \|\Delta_{d_y}\|_{L^r(I,W^{-1,p})} + \|\Delta_{d_p}\|_{L^r(I,W^{-1,p'})}$$

for some $r \in [2,s)$. Using the regularity theory of the linearized state resp. adjoint equation for (15) resp. (16) we conclude

$$\|\Delta_p\|_{L^r(I,W_D^{-1,p'})} \lesssim \|\Delta_{d_u}\|_{L^r} + \|\Delta_{d_y}\|_{L^r(I,W^{-1,p})} + \|\Delta_{d_p}\|_{L^r(I,W^{-1,p'})}.$$

At this point we need the full strength of Lemma 5 to estimate the $\mathscr{A}''$-terms for different $r \in [2,s]$. Note that $\bar{p} \in L^\infty(I, W^{1,p'})$ holds due to Lemma 3. Our discussion of B^* from Section 4.1 shows that either

$$(\zeta + 1)/2 < 1 - 1/r, \text{ which implies } \|B^*\Delta_p\|_{L^\infty} \lesssim \|\Delta_p\|_{L^r(I,W^{-1,p'})}$$

or

$$(\zeta + 1)/2 > 1 - 1/r, \text{ which implies } \|B^*\Delta_p\|_{L^q} \lesssim \|\Delta_p\|_{L^r(I,W^{-1,p'})}$$

with some q fulfilling $1/q > 1/r + (\zeta - 1)/2$ holds. In the first case it follows

$$\|\Delta_u\|_{L^\infty} \lesssim \|\Delta_{d_u}\|_{L^\infty} + \|\Delta_{d_y}\|_{L^s(I,W_D^{-1,p})} + \|\Delta_{d_p}\|_{L^s(I,W_D^{-1,p'})}$$

and we are done. In the second case we have

$$\|\Delta_u\|_{L^q} \lesssim \|\Delta_{d_u}\|_{L^q} + \|\Delta_{d_y}\|_{L^q(I,W^{-1,p})} + \|\Delta_{d_p}\|_{L^q(I,W^{-1,p'})}$$

and we repeat the procedure with $r = q$ as long as the first holds, which is clearly the case due to Assumption 4 if $r = s$ is reached. Note that $(\zeta - 1)/2 < 0$ is fixed and that we can avoid q being equal to the exceptional cases of Lemma 4 due to the strict inequality that allows small perturbations. □

Remark 6 In addition to the case of purely timedependent control Bonifacius and Neitzel (2018) discuss the case of distributed control, i.e. $U = L^s(I \times \Omega)$ in Assumption 4 and B is the embedding $L^s(I \times \Omega) \hookrightarrow L^s(I, H_D^{-\zeta,p})$.

The main difficulty when generalizing our results to the setting of distributed control lies in keeping the arguments for Proposition 3 and Theorem 2 working. In that case, B^* is the imbedding $L^{s'}(I, W^{1,p'}) \hookrightarrow L^{s'}(I \times \Omega)$ and a similiar discussion as in Section 4.1 has to be done. Sufficiently good estimates for Δ_p could be obtained using the regularity theorem from Section 7, whereas the corresponding estimates for Δ_y would require an analogous analysis of the linearized state equation on $H^{-\zeta,p}$-spaces, which is beyond scope and focus of this paper.

5.2 Convergence of the Generalized Newton Method

Invoking a general result on the convergence of generalized Newton methods, e.g. Hinze et al. (2009), Theorem 2.19, our previous results allow to derive the following

Theorem 3 *Let assumptions 1-4 and 5 hold.*

1. *Then there is a radius $r_{Newton} > 0$ such that for any tripple $(y_0, u_0, p_0) \in X_s$ fulfilling*
$$(y_0, u_0, p_0) \in \mathbb{B}^{X_s}_{r_{Newton}}((\bar{y}, \bar{u}, \bar{p}))$$
the sequence of iterates generated by the Newton-Josephy method for equation (GE-σ) *with (y_0, u_0, p_0) as start is welldefined, stays in the ball $\mathbb{B}^{X_s}_{r_{Newton}}((\bar{y}, \bar{u}, \bar{p}))$ and converges q-superlinearly to $(\bar{y}, \bar{u}, \bar{p})$ in X_s.*
2. *The same result as in (1) holds with X_∞ instead of X_s.*

Proof The proof is standard, see e.g. Tröltzsch (1999), Goldberg and Tröltzsch (1998), Wachsmuth (2007), Hintermüller and Hinze (2006), Griesse et al. (2008, 2010).

6 Convergence of the SQP Method

The welldefinedness of the iterates in Theorem 3 is so far only ensured by some generalized implicit function theorem and the strong regularity of (GE-σ) at $(\bar{y}, \bar{u}, \bar{p})$. Convexity of the quadratic subproblems (**QP-**σ) is so far only known in the case $(y_k, u_k, p_k) = (\bar{y}, \bar{u}, \bar{p})$, i.e. the relation of possible minimizers of (**QP-**σ) and solutions of (GE-σ) is unclear at the moment.

Therefore, this final section is devoted to the interpretation of the Newton iterates as solutions of some linear quadratic optimal control problems. In a first step (Section 6.1) we consider the quadratic problems restricted to U^σ_{ad}, i.e. the set of those controls from U_{ad} that coincide with the optimal control $\bar{u}$ on the σ-active set of $\bar{u}$. It is not possible to avoid such rather technical restrictions completely, cf. the example given by Goldberg and Tröltzsch (1998, Section 6) or the last remark of Tröltzsch (1999), but they can be slightly relaxed: The quadratic subproblems have to be restricted to $U_{ad} \cap \mathbb{B}^{L^2}_\rho(\bar{u})$ with some radius $\rho > 0$, as shown in Sections 6.2 and 6.3. That this restriction can be done in terms of L^2-balls around $\bar{u}$ (instead of L^∞-balls as in previous results) is –to our best knowledge– a new result that we obtain by careful application of the SSCs.

6.1 SQP Method on U_{ad}^{σ}

In this section we relate the iterates of the Newton method to solutions of (**QP-**σ), see Section 3.3 for the definition of U_{ad}^{σ} and **QP-**σ. To do so we will show that the formal optimality conditions for (**QP-**σ) encoded in the Newton equations for (GE-σ) are indeed sufficient optimality conditions for (**QP-**σ). Following again the work of Tröltzsch (1999), Goldberg and Tröltzsch (1998), and Wachsmuth (2007) this is done by showing strict convexity for (**QP-**σ) for (y_k, u_k, p_k) sufficiently close to $(\bar{y}, \bar{u}, \bar{p})$. We prove convergence of the SQP method under the technical restriction to replace U_{ad} by U_{ad}^{σ}. Assuming strongest possible SSCs, i.e. $U_{ad} = U_{ad}^{\sigma}$, this yields our first main result.

Recall the definition of the space X_s from Section 3.2. The following result corresponds to Lemma 6.2, Corollary 6.3 by Tröltzsch (1999).

Proposition 4 *Let Assumptions 1-4 and 5 be satisfied. Then, the linear quadratic SQP problem* (**QP-**σ) *is a strictly convex optimization problem as long as* (y_k, u_k, p_k) *is sufficiently close to* $(\bar{y}, \bar{u}, \bar{p})$ *in* X_s.

Proof The optimization problems (**QP-**σ) are of linear quadratic type. To show strict convexity it suffices to show coercivity, but the latter is an immediate consequence of the second order sufficient condition (SSC-σ) and the uniform estimate from Proposition 1. □

Now we are ready to show locally superlinear convergence of the SQP method with quadratic problems on U_{ad}^{σ}:

Theorem 4 *Let the assumptions of Theorem 3 be fulfilled.*

1. *There is a radius* $r_{SQP\text{-}\sigma} > 0$ *such that for any start tripple* $(y_0, u_0, p_0) \in X_s$ *fulfilling*
$$(y_0, u_0, p_0) \in \mathbb{B}^{X_s}_{r_{SQP-\sigma}}((\bar{y}, \bar{u}, \bar{p}))$$
the sequences of iterates generated by the generalized Newton method applied to (GE-σ) *resp. generated by the SQP method with quadratic subproblems* (**QP-**σ) *are both welldefined, coincide, stay in the ball* $\mathbb{B}^{X_s}_{r_{SQP\ \sigma}}((\bar{y}, \bar{u}, \bar{p}))$ *and converge superlinearly to* $(\bar{y}, \bar{u}, \bar{p})$ *in* X_s.
2. *The statement analogous to (1) with* X_s *replaced by* X_∞ *is true, too.*
3. *There is a radius* $\tilde{r}_{SQP\text{-}\sigma} > 0$ *such that the SQP method with quadratic subproblems* (**QP-**σ) *and initial iterate* (y_0, u_0, p_0) *with*
$$\|y_0 - \bar{y}\|_{W^{1,s}(I, W_D^{-1,p}) \cap L^s(I, W_D^{1,p})} + \|p_0 - \bar{p}\|_{W^{1,s}(I, W_D^{-1,p'}) \cap L^s(I, W_D^{1,p'})} \le \tilde{r}_{SQP\text{-}\sigma}$$
converges superlinearly in X_s *and* X_∞ *to* $(\bar{y}, \bar{u}, \bar{p})$. *In particular we can choose*
$$u_0 \in U_{ad}, \qquad \|u_0 - \bar{u}\|_{L^2(\Lambda)} \quad \textit{sufficiently small},$$
$$y_0, p_0 \ \textit{state and adjoint state associated to } u_0.$$

Proof For (1) and (2) the proof works analogous to that of Theorem 6.4 in Tröltzsch (1999). For (3) note that (**QP-**σ) is actually independent of the current control iterate u_k, cf. also the remark after (5), which shows the first statement in (3). By interpolation, cf. Remark 7, it holds $\|u_0-\bar{u}\|_{L^s} \leq C\|u_0-\bar{u}\|_{L^2}^{2/s}$ for all $u_0 \in U_{ad}$ with a constant $C>0$ depending only on u_a and u_b. From this we conclude by continuity

$$\|(y_0,u_0,p_0)-(\bar{y},\bar{u},\bar{p})\|_{X_s} \lesssim \|u_0-\bar{u}\|_{L^2}^{2/s},$$

which shows the second statement of (3). □

Assuming strongest possible second order sufficient conditions, i.e. coercivity of the second derivative of the Lagrangian on the whole space instead of only on a subspace, we are able to state our first main result. Note that it is possible to formulate all "closeness" required for convergence of the SQP method with respect to L^2-norms.

Theorem 5 *Let the Assumptions 1-4 be fulfilled and let the second order sufficient condition* (SSC-σ) *from Assumption 5 hold on the whole space* $L^2(\Lambda)$*. Then the SQP method for* (**OCP**) *started in* $(y_0,u_0,p_0) \in X_s$*,*

$$u_0 \in U_{ad}, \qquad \|u_0-\bar{u}\|_{L^2(\Lambda)} \quad \textit{sufficiently small},$$
$$y_0, p_0 \textit{ state and adjoint state associated to } u_0,$$

converges superlinearly in X_s *and* X_∞ *to* $(\bar{y},\bar{u},\bar{p})$.

Proof Use Theorem 4 (3) together with $U_{ad}^{\sigma}=U_{ad}$. □

Remark 7 (The "interpolation trick") Using the equivalence of the topologies generated by the L^2- and the L^s-norm on the L^∞-bounded set U_{ad}, respectively, might look rather trivial, but indeed this observation is a key argument for many proofs concerning second order conditions without two norm gap, see e.g. Casas and Tröltzsch (2012, Proposition 3.4) or Bonifacius and Neitzel (2018, Theorem 4.14). Here, we made use of this "interpolation trick" in Theorem 4 (3) and 5 to tighten the unsatisfying gap between the quadratic growth condition for j implied by (SSC-σ) – this growth condition holds L^2-locally – and the L^s-local convergence of the SQP method.

For the rest of Section 6 we will be concerned with relaxing this rather abstract and technical condition towards a more natural restriction.

6.2 Generalized Newton Method on U_{ad} resp. $U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}$

To establish convergence of the SQP method restricted to $U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}$ we have to consider convergence of the Newton method for the associated generalized equation first. Our arguments follow in particular the presentation by Wachsmuth (2007), but similiar results are also due to Goldberg and Tröltzsch (1998) and Tröltzsch (1999). To replace L^∞-locality by L^2-locality in the statements of Proposition 5 is –to our best knowledge– a new result, which will serve as main step towards our final result in the last section.

In the following we consider the perturbed generalized equation

$$d \in F(\bar{y},\bar{u},\bar{p}) + F'(\bar{y},\bar{u},\bar{p})(y-\bar{y},u-\bar{u},p-\bar{p}) + N(y,u,p). \tag{GE-D}$$

Note that we now use the normal cone map N associated with the true set of admissible controls U_{ad} instead of the normal cone map N^σ associated with the modified admissible set U_{ad}^σ that was used for the definition of (GE-σ-D) in the previous sections. Furthermore, note that (GE-D) can be understood as generalized equation in the spaces X_s, Z_s resp. X_∞, Z_∞ both. For the definition of these spaces see Section 3.2. As before, the generalized equation (GE-D) is the formal optimality system of the following perturbed optimal control problem:

$$\begin{cases} \min\limits_{y,u} \frac{1}{2}\|y-y_d\|^2 + \frac{\gamma}{2}\|u\|^2 - \frac{1}{2}\langle \bar{p}, \mathscr{A}''(\bar{y})[y-\bar{y}]^2\rangle - \langle d_u, u\rangle + \langle d_p, y\rangle \\ \text{subject to} \quad u \in U_{ad} \\ \quad\text{and} \quad \begin{pmatrix} d_y \\ 0 \end{pmatrix} = e_y(\bar{y},\bar{u})(y-\bar{y}) + e_u(\bar{y},\bar{u})(u-\bar{u}) \end{cases} \tag{\textbf{QP-D}}$$

The reduced objective function for (**QP-D**) will be denoted by j_d. Note that we did not discuss properties of this optimization problem so far. Further, we introduce the following notation for the strongly active sets:

$$A_d^\sigma(u) := \{x \in \Lambda \colon |\nabla j_d(u)|(x) = |B^* p + \gamma u - d_u|(x) > \sigma\},$$
$$A^\sigma(u) := A_0^\sigma(u), \qquad \text{i.e. } d = 0 \text{ in the definition above.}$$

Here, p denotes the adjoint state associated with u with respect to (**QP-D**) with perturbation vector d, see (14). Note that $A_0^\sigma(\bar{u})$ coincides with the strongly active set for $\bar{u}$ defined in Assumption 5.

In Section 5 we observed that under Assumptions 1-4 and 5 the restricted optimal control problem (**QP-σ-D**), i.e. problem (**QP-D**) restricted to U_{ad}^σ, is strictly convex and admits a unique solution $(\bar{y}_d,\bar{u}_d,\bar{p}_d)$. This holds true for arbitrarily large perturbation vectors d. In particular, the map $d = (d_y,d_p,d_u) \mapsto (\bar{y}_d,\bar{u}_d,\bar{p}_d)$ was shown to be Lipschitz from Z_∞ to X_∞ in Theorem 2, say with modulus $L' > 0$. It follows that the mapping

$$\begin{aligned} &Z_\infty \to L^\infty(\Lambda), \\ &d \mapsto \gamma\bar{u}_d + B^*\bar{p}_d - d_u = \nabla j_d(\bar{u}_d) \end{aligned} \tag{17}$$

is Lipschitz as well, say with modulus $L > 0$.

Remark 8 Of course, even the map $Z_s \to X_s$, $d \mapsto (\bar{y}_d,\bar{u}_d,\bar{p}_d)$ is Lipschitz continuous as shown in Theorem 2, which implies that $d \mapsto \gamma\bar{u}_d + B^*\bar{p}_d - d_u$ is Lipschitz continuous from Z_s to $L^s(\Lambda)$. Unfortunately, we rely on L^∞-estimates in the following.

Assuming $\sigma > 0$ in (SSC-σ) we can draw some immediate conclusions from the Lipschitz continuity of (17) as done by Wachsmuth (2007, Corollaries 5.3 and 5.4):

Lemma 7 *Let Assumptions 1-4 and 5 with $\sigma > 0$ hold and suppose that $\|d\|_{Z_\infty} < \frac{\sigma}{2L}$.*

1. *It holds $A^\sigma(\bar{u}) \subset A_d^{\sigma/2}(\bar{u}_d)$ and the signs of $\nabla j_d(\bar{u}_d)$ and $\nabla j_0(\bar{u})$ coincide on $A^\sigma(\bar{u})$.*
2. *The solution $(\bar{y}_d, \bar{u}_d, \bar{p}_d)$ of* (**QP-σ-D**) *is a solution of* (GE-D) *as well, i.e. it holds*

$$\langle \gamma \bar{u}_d + B^* \bar{p}_d - d_u, u - \bar{u}_d \rangle_{L^2(\Lambda)} \geq 0, \qquad \forall u \in U_{ad}.$$

Proof Completely analogous to Wachsmuth (2007).

Lemma 7 shows that our solution of (**QP-σ-D**) that depends Z_∞-X_∞-Lipschitz on d is a solution of (GE-D) as well, if the perturbation d is small enough in Z_∞. To establish strong regularity of (GE) (with spaces X_∞, Z_∞) from this result we have to show that this solution is locally unique. This is done by proving that $(\bar{y}_d, \bar{u}_d, \bar{p}_d)$ is not only a global solution of (**QP-σ-D**) but even a local solution of (**QP-D**) fulfilling a quadratic growth condition on a ball around $(\bar{y}_d, \bar{u}_d, \bar{p}_d)$ with radius independent of d.

Proposition 5 *Let the Assumptions of Lemma 7 be satisfied.*

1. *Then there exist $0 < \tilde{\varepsilon} < \frac{\sigma}{2L}$ and $\tilde{\rho}, \eta > 0$, such that $(\bar{y}_d, \bar{u}_d, \bar{p}_d)$, i.e. the solution of* (**QP-σ-D**)*, is also a L^2-local solution of* (**QP-D**) *and satisfies the quadratic growth condition*

$$j_d(u) \geq j_d(\bar{u}_d) + \eta \|u - \bar{u}_d\|_{L^2}^2$$

for $\|u - \bar{u}_d\|_{L^2(\Lambda)} \leq \tilde{\rho}$, $u \in U_{ad}$, as long as $\|d\|_{Z_\infty} < \tilde{\varepsilon}$.
2. *There are $0 < \hat{\varepsilon} \leq \tilde{\varepsilon}$, $0 < \hat{\rho} \leq \tilde{\rho}$ such that $(\bar{y}_d, \bar{u}_d, \bar{p}_d)$ is the only stationary*[1] *point for* (**QP-D**) *in $\overline{\mathbb{B}_{\hat{\rho}}^{L^2}(\bar{u}_d)}$.*

The first statement of this Proposition corresponds to Theorem 5.5 (Wachsmuth 2007) with the L^∞-ball around $\bar{u}_d$ replaced by an L^2-ball. To establish quadratic growth L^∞-locally around $\bar{u}_d$, one could follow the direct proof of Theorem 5.17 (Tröltzsch 2010). Avoiding the two norm gap –which is our aim– can be done following ideas due to Casas and Tröltzsch (2012, Theorem 2.3), see also Tröltzsch and Wachsmuth (2006, Theorem 3.22), utilizing a proof by contradiction. We mention that similiar arguments were also used by Casas and Tröltzsch (2012) in the context of abstract finite element errors.

Note that for every single perturbation $d \in Z_\infty$, both properties in the proposition are directly implied by Theorem 2.3 resp. Corollary 2.6 from Casas and Tröltzsch (2012). The crucial point here is to guarantee that the radii of the respective balls can be chosen independently of the choice of d as long as $\|d\|_{Z_\infty}$ is small enough.

Proof For the proof of (1) we extended the technique presented by Casas and Tröltzsch (2012, Theorem 2.3) to our needs. First, note that due to the quadratic structure of (**QP-D**) it holds $j_d''(\bar{u}_d)[v_1, v_2] = j''(\bar{u})[v_1, v_2]$. In particular, j_d'' is independent of d.

[1] We call (y, u, p) stationary for (**QP-D**) if (y, u, p) fulfills the first order necessary conditions for (**QP-D**).

We are going to argue by contradiction and assume the contrary of our claim: There are sequences $(d_n)_n \subset Z_\infty$, $(h_n)_n \subset L^2(\Lambda)$ with

$$\|d_n\|_{Z_\infty} < \frac{1}{n}, \qquad \|h_n\|_{L^2} < \frac{1}{n} \text{ and } \bar{u}_{d_n} + h_n \in U_{ad}$$

such that

$$j_{d_n}(\bar{u}_{d_n} + h_n) - j_{d_n}(\bar{u}_{d_n}) < \frac{1}{n}\|h_n\|_{L^2}^2. \tag{18}$$

Define $v_n := \frac{h_n}{\|h_n\|_{L^2}}$ and $\rho_n := \|h_n\|_{L^2}$. It holds $d_n = (d_{y,n}, d_{p,n}, d_{u,n}) \to 0$ strongly in Z_∞, which implies $\bar{u}_{d_n} \to \bar{u}$ and $\nabla j_{d_n}(\bar{u}_{d_n}) \to \nabla j(\bar{u})$ strongly in $L^\infty(\Lambda)$. Due to $\|v_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$ we can w.l.o.g. assume that $v_n \rightharpoonup v_*$ weakly in $L^2(\Lambda)$ for some $v_* \in L^2(\Lambda)$.

Step 1: We prove $j'(\bar{u})v_* = 0$. We have

$$\begin{aligned} j'(\bar{u})v_* &= \langle \text{strong-}\lim_{n\to\infty} \nabla j_{d_n}(u_{d_n}), \text{weak-}\lim_{n\to\infty} v_n \rangle_{L^2} \\ &= \lim_{n\to\infty} \langle \nabla j_{d_n}(u_{d_n}), v_n \rangle_{L^2} \geq 0, \end{aligned} \tag{19}$$

because $\langle \nabla j_{d_n}(u_{d_n}), v_n \rangle_{L^2} = \frac{1}{\rho_n} \langle \nabla j_{d_n}(u_{d_n}), h_n \rangle_{L^2} \geq 0$ holds for every n due to $\bar{u}_{d_n} + h_n \in U_{ad}$ and Lemma 7 (2), for which we can assume w.l.o.g. that $\|d_n\|_{Z_\infty} < \frac{\sigma}{2L}$. Further, using the mean value theorem there are $\theta_n \in (0,1)$ such that

$$\frac{j_{d_n}(u_{d_n} + \rho_n v_n) - j_{d_n}(\bar{u}_{d_n})}{\rho_n} = \langle \nabla j_{d_n}(\bar{u}_{d_n} + \theta_n \rho_n v_n), v_n \rangle_{L^2}.$$

Due to the structure of (**QP-D**) –see e.g. (15), (16) and use regularity results as in the proof of Theorem 2– we know that $\nabla j_{d_n}(\bar{u}_{d_n} + \theta_n \rho_n v_n) \to \nabla j(\bar{u})$ strongly in $L^2(\Lambda)$, which implies

$$\frac{j_{d_n}(u_{d_n} + \rho_n v_n) - j_{d_n}(\bar{u}_{d_n})}{\rho_n} \to j'(\bar{u})v_* \qquad \text{as } n \to \infty. \tag{20}$$

On the other hand it holds by assumption (18):

$$\frac{j_{d_n}(u_{d_n} + \rho_n v_n) - j_{d_n}(\bar{u}_{d_n})}{\rho_n} < \frac{1}{\rho_n} \cdot \frac{1}{n} \|h_n\|_{L^2}^2 = \frac{\rho_n}{n} \to 0,$$

which together with (20) yields $j'(\bar{u})v_* \leq 0$ first and then together with (19):

$$j'(\bar{u})v_* = 0. \tag{21}$$

Step 2: We want to show $v_* = 0$ if $|\nabla j(\bar{u})| > 0$. To do so we show $v_* \geq 0$ if $\nabla j(\bar{u}) > 0$ and $v_* \leq 0$ if $\nabla j(\bar{u}) < 0$, which implies together with Step 1 the desired property: For $\sigma' > 0$ arbitrary define $A^{\sigma',a}(\bar{u}) := \{x \in \Lambda : \nabla j(\bar{u}) > \sigma'\}$. As in the proof of Lemma 7 we conclude that $\nabla j_{d_n}(\bar{u}_{d_n}) > 0$ on $A^{\sigma',a}(\bar{u})$ for all sufficiently large n, which implies $h_n, v_n \geq 0$ on $A^{\sigma',a}(\bar{u})$ for all such n. Because weak convergence preserves signs we conclude $v_* \geq 0$ on $A^{\sigma',a}(\bar{u})$. Since $\sigma' > 0$ was arbitrary it follows $v_* \geq 0$ whenever $\nabla j(\bar{u}) > 0$, as stated. The case $\nabla j(\bar{u}) < 0$ is shown in the same way.

Step 3: In Step 2 we have shown that $v_* \in C^0(\bar{u}) \subset C^\sigma(\bar{u})$ holds. For the definition of $C^0(\bar{u})$ and $C^\sigma(\bar{u})$ see Assumption 5. In the present step we will arrive at the final contradiction. First observe that by our assumption

$$\frac{\rho_n^2}{n} = \frac{1}{n}\|h_n\|_{L^2}^2 > j_{d_n}(\bar{u}_{d_n}+h_n) - j_{d_n}(\bar{u}_{d_n}) \overset{(\bigstar)}{=} j'_{d_n}(\bar{u}_{d_n})h_n + \frac{1}{2}j''(\bar{u})h_n \overset{(\blacksquare)}{\geq} \frac{\rho_n^2}{2} j''(\bar{u})v_n^2,$$

where we used the linear quadratic structure of (**QP-D**) at ($\bigstar$) and the first order optimality condition at ($\blacksquare$). It follows

$$j''(\bar{u})v_*^2 \leq \liminf_{n\to\infty} j''(\bar{u})v_n^2 \leq \liminf_{n\to\infty} \frac{2}{n} = 0, \tag{22}$$

where the first inequality comes from the weak lower semicontinuity of $j''(\bar{u})$, see Proposition 4.10 (Bonifacius and Neitzel 2018). Since $v_* \in C^\sigma(\bar{u})$ we can apply (SSC-σ) and conclude from (22) that $v_* = 0$. Using property (4.11) by Bonifacius and Neitzel (2018) at ($\blacktriangle$) we obtain

$$\gamma = \gamma \liminf_{n\to} \|v_n\|_{L^2}^2 \overset{(\blacktriangle)}{\leq} \liminf_{n\to\infty} j''(\bar{u})v_n^2 \overset{(22)}{=} 0,$$

which is the desired contradiction. □

The second part of the Proposition is shown similiarly adapting the proof of Corollary 2.6 by Casas and Tröltzsch (2012). We leave the details to the reader. □

Given a radius $\rho > 0$ we introduce another modification of the perturbed linear quadratic problem (**QP-D**)

$$\begin{cases} \min\limits_{y,u} \frac{1}{2}\|y-y_d\|^2 + \frac{\gamma}{2}\|u\|^2 - \frac{1}{2}\langle \bar{p}, \mathscr{A}''(\bar{y})[y-\bar{y}]^2\rangle - \langle d_u, u\rangle + \langle d_p, y\rangle \\ \text{subject to} \quad u \in U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})} \\ \text{and} \quad \begin{pmatrix} d_y \\ 0 \end{pmatrix} = e_y(\bar{y},\bar{u})(y-\bar{y}) + e_u(\bar{y},\bar{u})(u-\bar{u}) \end{cases} \tag{QP-D-ρ}$$

for which the following result holds:

Corollary 1 *Let the Assumptions of the Lemma 7 be satisfied.*

1. *There are $\varepsilon, \rho > 0$, such that the tripple $(\bar{y}_d, \bar{u}_d, \bar{p}_d)$, i.e. the unique solution of* (**QP-σ-D**)*, is also the unique solution of* (**QP-D-ρ**) *if $\|d\|_{Z_\infty} < \varepsilon$.*
2. *There are $\varepsilon, \tau > 0$, such that for $\|d\|_{Z_\infty} < \varepsilon$ the control $\bar{u}_d$ is the unique solution of* (GE-D) *that is contained in the set $U_{ad} \cap \overline{\mathbb{B}_\tau^{L^2}(\bar{u})}$.*

A result similiar to (2) –but with L^∞- instead of L^2-balls– was proven by Goldberg and Tröltzsch (1998, Theorem 5.4) using a different argument that relies on strongly active sets and continuity of (17).

Proof 1. Choose $\rho = \frac{2\tilde{\rho}}{3}$ and $\varepsilon < \min\left(\tilde{\varepsilon}, \frac{\tilde{\rho}}{3C}\right)$, where $C > 0$ is the Z_∞-L^2-Lipschitz constant for the map $d \mapsto \bar{u}_d$, cf. Theorem 2 for the Lipschitz continuity. Then, it holds in particular $\|d\|_{Z_\infty} < \tilde{\varepsilon}$, i.e. the previous Proposition applies, and

$$\bar{u}_d \in U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})} \subset U_{ad} \cap \overline{\mathbb{B}_{\tilde{\rho}}^{L^2}(\bar{u}_d)}$$

for all $\|d\|_{Z_\infty} < \varepsilon$. Since $\bar{u}_d$ is the unique minimizer of (**QP-D**) restricted to $U_{ad} \cap \overline{\mathbb{B}_{\tilde{\rho}}^{L^2}(\bar{u}_d)}$ by quadratic growth (Proposition 5 (1)) and this minimizer is contained in the smaller set $U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}$, we finally proved that $\bar{u}_d$ is the unique minimizer of (**QP-D**) restricted to $U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}$, i.e. the unique minimizer of (**QP-D-ρ**).

2. Similiarly as for (1). Now make use of Proposition 5 (2). □

We introduce another variation of (GE):

$$0 \in F(y,u,p) + N^\rho(y,u,p), \tag{GE-ρ}$$

with the set valued map $N^\rho(y,u,p) := \{\{0\},\{0\},\{0\},\{0\},N_{U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}}(u)\}^T$, where $N_{U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}}(u)$ denotes the normal cone of the closed convex set $U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}$ at some point u. The first part of the following result is similiar to Corollary 5.6 (Wachsmuth 2007), the second part to the observation on top of p. 240 by Goldberg and Tröltzsch (1998).

Theorem 6 *Let the Assumptions of Lemma 7 be fulfilled. It holds:*

1. *The generalized equation* (GE) *in the spaces X_∞, Z_∞ is strongly regular at $(\bar{y},\bar{u},\bar{p})$.*
2. *There is a $\rho > 0$ such that the generalized equation* (GE-ρ) *in the spaces X_∞, Z_∞ is strongly regular at $(\bar{y},\bar{u},\bar{p})$.*

Proof Both statements are consequences of Corollary 1 resp. Theorem 2. The first part is proven in the same way as in Wachsmuth (2007). We have to use that the L^∞-norm is stronger than the L^2-norm. For the second part note that for all u in the L^2-interior of the ball $\overline{\mathbb{B}_\rho^{L^2}(\bar{u})}$, i.e. in particular for all u sufficiently close to $\bar{u}$ in the L^∞-norm, the equality $N_{U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}}(u) = N_{U_{ad}}(u)$ holds, as already mentioned by Goldberg and Tröltzsch (1998). □

The following result is an immediate consequence of an abstract result (Hinze et al. 2009, Theorem 2.19) and Theorem 6. The closed graph property for the normal cone map N^ρ is standard.

Theorem 7 *Let Assumptions 1-4 and 5 hold. For any (y_0, p_0) sufficiently close to $(\bar{y},\bar{p})$ in the space*

$$W^{1,s}(I, W_D^{-1,p}) \cap L^s(I, W_D^{1,p}) \times W^{1,s}(I, W_D^{-1,p'}) \cap L^s(I, W_D^{1,p'})$$

it holds:

1. *The sequence of iterates generated by the Newton-Josephy method for* (GE) *with initial iterate* (y_0, u_0, p_0) *is welldefined and converges superlinearly in* X_∞ *to* $(\bar{y}, \bar{u}, \bar{p})$.
2. *The same holds true for the sequence of iterates generated by the Newton-Josephy method for* (GE-ρ) *with* ρ *from Theorem 6 (2).*

From Lemma 7 on we had to consider perturbations in Z_∞, i.e. we had to measure the control in $L^\infty(\Lambda)$. This is the reason why have to show strong regularity only in Z_∞, X_∞ and not in Z_s, X_s as well as we did before. That we impose no condition on u_0 is due to the fact that the Newton update equations for (GE) resp. (GE-ρ) are independent of the current u-iterate u_k, see the comment after equation (5).

6.3 SQP Method on $U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}$

Finally, we can investigate how the iterates of the generalized Newton method from Theorem 7 can be computed by solving linear quadratic optimal control problems restricted on $U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}$. For analogous results in the case of semilinear equations (but with L^∞- instead of L^2-balls) we refer to Tröltzsch (1999) and Goldberg and Tröltzsch (1998).

Lemma 8 *Let the Assumptions of Theorem 7 hold. Let* $(y_k, u_k, p_k) \in X_\infty$ *be a given tripple and consider the restricted quadratic subproblem* (**QP-**σ) *associated with this tripple. There exists an* X_∞*-neighbourhood* V_1 *of* $(\bar{y}, \bar{u}, \bar{p})$ *such that the map*

$$(y_k, u_k, p_k) \mapsto (y_{k+1}^\sigma, u_{k+1}^\sigma, p_{k+1}^\sigma)$$

is well-defined on V_1 *and Lipschitz continuous, where* $(y_{k+1}^\sigma, u_{k+1}^\sigma, p_{k+1}^\sigma)$ *denotes the unique solution of* (**QP-**σ).

Proof Existence and uniqueness of a solution to (**QP-**σ) is established in Proposition 4 for (y_k, u_k, p_k) sufficiently close to $(\bar{y}, \bar{u}, \bar{p})$. Define $\tilde{V}$ to be such a neighbourhood of $(\bar{y}, \bar{u}, \bar{p})$. To see Lipschitz continuity, note that $(y_{k+1}^\sigma, u_{k+1}^\sigma, p_{k+1}^\sigma)$ is solution of the parametrized generalized equation

$$\begin{aligned} 0 &\in G((y_k, u_k, p_k), (y, u, p)) + N^\sigma(y, u, p) \\ &:= F(y_k, u_k, p_k) + F'(y_k, u_k, p_k)(y - y_k, u - u_k, p - p_k) + N^\sigma(y, u, p) \end{aligned}$$

–with (y_k, u_k, p_k) being the parameter– and that

$$\begin{aligned} 0 &\in G((\bar{y}, \bar{u}, \bar{p}), (y, u, p)) + N^\sigma(y, u, p) \\ &= F(\bar{y}, \bar{u}, \bar{p}) + F'(\bar{y}, \bar{u}, \bar{p})(y - \bar{y}, u - \bar{u}, p - \bar{p}) + N^\sigma(y, u, p) \end{aligned}$$

is strongly regular at its solution $(\bar{y}, \bar{u}, \bar{p})$ according to Theorem 2. Further, G and G', i.e. F and F', depend continuously on (y_k, u_k, p_k), because $F\colon X_\infty \to Z_\infty$ is continuously differentiable (Lemma 2). Hence, Theorem 2.18 (Hinze et al. 2009) implies the desired Lipschitz continuity of $(y_k, u_k, p_k) \mapsto (y_{k+1}^\sigma, u_{k+1}^\sigma, p_{k+1}^\sigma)$ from X_∞ to X_∞ on a sufficiently small neighbourhood $\hat{V}$ of $(\bar{y}, \bar{u}, \bar{p})$. Now, $V_1 := \tilde{V} \cap \hat{V}$ yields the desired neighbourhood. $\square$

With the previous Lemma we have shown in particular that

$$\begin{aligned} X_\infty &\to L^\infty(\Lambda) \\ (y_k,u_k,p_k) &\mapsto \nabla j_k(u^\sigma_{k+1}) = \gamma u^\sigma_{k+1} + B^* p^\sigma_{k+1} \end{aligned} \tag{23}$$

is Lipschitz continuous on the X_∞-neighbourhood V_1 of $(\bar{y},\bar{u},\bar{p})$. With j_k we denoted the reduced functional of (**QP-**σ) and p^σ_{k+1} is the adjoint state (w.r.t. (**QP-**σ)) associated with the control u^σ_{k+1}, see equations (3), (4). The same argument as for Lemma 7 now shows

Lemma 9 *Let the Assumptions of Theorem 7 hold. There is an X_∞-neighbourhood V_2 of $(\bar{y},\bar{u},\bar{p})$ such that for all $(y_k,u_k,p_k) \in V_2$ the solution $(y^\sigma_{k+1},u^\sigma_{k+1},p^\sigma_{k+1})$ of* (**QP-**σ) *satisfies the first order necessary optimality conditions of* (**QP**)*.*

Proof State- and adjoint equation of (**QP**) and (**QP-**σ) coincide. We only have to show that $(y^\sigma_{k+1},u^\sigma_{k+1},p^\sigma_{k+1})$ fulfills the variational inequality of (**QP**) as well and this works completely analogous to Lemma 7 replacing (17) with (23). □

Now, we can show the following result that is similiar to Proposition 5:

Proposition 6 *Let the Assumptions of Theorem 7 hold. There is an X_∞-neighbourhood V_3 of $(\bar{y},\bar{u},\bar{p})$ and there are $\rho,\eta > 0$ such that for all tripples $(y_k,u_k,p_k) \in V_3$ the unique solution $(y_{k+1},u_{k+1},p_{k+1}) := (y^\sigma_{k+1},u^\sigma_{k+1},p^\sigma_{k+1})$ of* (**QP-**σ)

1. *is a L^2-local solution of* (**QP**) *satisfying the quadratic growth condition*

$$j_k(u) \ge j_k(u_{k+1}) + \eta \|u - u_{k+1}\|^2_{L^2}$$

for $u \in U_{ad}$ such that $\|u - u_{k+1}\|_{L^2(\Lambda)} \le \rho$.

2. *is the only stationary point for* (**QP**) *in $\overline{\mathbb{B}^{L^2}_\rho(u_{k+1})}$.*

Proof We proceed as in the proofs of Proposition 5 (1) and (2) and argue by contradiction. Instead of j_{d_n} and $\bar{u}_{d_n}$ we have to consider j_k and u_{k+1}. We only mention the essential ingredients that keep all the previous arguments working:

(i) For any sequence $(w_k) \subset U_{ad}$ such that $w_k \to \bar{u}$ in $L^2(\Lambda)$ it holds

$$\nabla j_k(w_k) \to \nabla j(\bar{u}), \qquad \text{strongly in } L^2(\Lambda).$$

This was shown in Proposition 2; use the "interpolation trick" (Remark 7) to obtain the required L^s-convergence $w_k \to \bar{u}$ from the given L^2-convergence.

(ii) If $u_k \to \bar{u}$ strongly in L^2 and $v_k \rightharpoonup v_*$ weakly in L^2 we have:

$$j''(\bar{u})v_*^2 \le \liminf_{k\to\infty} j_k''(u_k)v_k^2.$$

Using the boundedness of $(v_k)_k$, this is a consequence of Proposition 1 and the weak lower semicontinuity of j'', see Bonifacius and Neitzel (2018), (4.10):

$$\liminf_k j_k''(u_k)v_k^2 \ge \liminf_k \underbrace{\left(j_k''(u_k) - j''(u_k)\right) v_k^2}_{\to 0 \text{ uniformly in } v_k} + \liminf_k j''(u_k)v_k^2 \ge j''(\bar{u})v_*^2$$

(iii) If $v_* = 0$ in (2), then $\gamma \liminf_{k\to\infty} \|v_k\|_{L^2}^2 \le \liminf_{k\to\infty} j_k''(u_k)v_k^2$: This is shown by the same argument as above. □

Next, we obtain with the same argument as for Corollary 1:

Proposition 7 *Let the Assumptions of Theorem 7 hold.*

1. *There is an X_∞-neighbourhood V_4 of $(\bar{y},\bar{u},\bar{p})$ and a radius $\rho > 0$ such that for all $(y_k,u_k,p_k) \in V_4$ the next SQP iterate $(y_{k+1},u_{k+1},p_{k+1})$ given by the unique solution of* (**QP-σ**) *is also the unique solution of* (**QP**) *with admissible set $U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}$.*
2. *There is an X_∞-neighbourhood V_5 of $(\bar{y},\bar{u},\bar{p})$ and a radius $\rho > 0$, such that for all $(y_k,u_k,p_k) \in V_5$ the next SQP iterate $(y_{k+1},u_{k+1},p_{k+1})$ given by the unique solution of* (**QP-σ**) *is also the unique L^2-local solution of the global quadratic problem* (**QP**) *that is contained in $U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}$.*

For convenience of the reader we write down the quadratic problem which we will refer to in our final theorem:

$$\begin{cases} \min\limits_{y,u} J_k(y,u) := \dfrac{1}{2}\|y-y_d\|^2 + \dfrac{\gamma}{2}\|u\|^2 - \dfrac{1}{2}\langle p_k, \mathscr{A}''(y_k)[y-y_k]^2\rangle \\ \text{subject to} \quad u \in U_{ad} \cap \overline{\mathbb{B}_\rho^{L^2}(\bar{u})}, \\ \qquad \text{and} \quad \begin{cases} \partial_t y + \mathscr{A}(y_k)y + \mathscr{A}'(y_k)y = Bu + \mathscr{A}'(y_k)y_k \\ y(0) = y_0 \end{cases} \end{cases} \qquad (\mathbf{QP}(\rho, y_k, p_k))$$

Theorem 8 *Let Assumptions 1-4 and 5 with $\sigma > 0$ hold. Then there are radii $\rho > 0$, $r_{SQP} > 0$ such that for any initial guess*

$$(y_0,p_0) \in W^{1,s}(I,W_D^{-1,p}) \cap L^s(I,W_D^{1,p}) \times W^{1,s}(I,W_D^{-1,p'}) \cap L^s(I,W_D^{1,p'})$$

fulfilling

$$\|y_0-\bar{y}\|_{W^{1,s}(I,W_D^{-1,p})\cap L^s(I,W_D^{1,p})} + \|p_0-\bar{p}\|_{W^{1,s}(I,W_D^{-1,p'})\cap L^s(I,W_D^{1,p'})} \le r_{SQP}$$

the sequence of iterates generated by the sucessive solution of the SQP subproblems (**QP(ρ, y_k, p_k)**) *converges superlinearly in X_∞ to $(\bar{y},\bar{u},\bar{p})$.*

A possible choice of y_0, p_0 are state y_0 and adjoint state p_0 associated to some control $u_0 \in U_{ad}$ w.r.t. (**OCP**) *if $\|u_0-\bar{u}\|_{L^2}$ is chosen small enough.*

Proof Combine Proposition 7 with Theorem 7. □

This theorem is our main result. Note in particular that we tightened the gap between the L^2-local growth condition originating from the second order sufficient conditions and the "closeness"-conditions in the SQP method. The latter had been formulated with respect to L^∞ in the existing literature. Now, in Theorem 8 above all required "closeness" can be formulated with respect to the L^2-norm.

7 Regularity of the Adjoint State

In this section we prove the regularity required for the adjoint state in our analysis. In Bonifacius and Neitzel (2018, Proposition 4.7) it was shown that

$$\bar{p} \in W^{1,r}(I, W_D^{-1,p'}) \cap L^r(I, W_D^{1,p'}) \qquad \forall r \in [s', \infty),$$

whereas we need additional regularity $\bar{p} \in L^\infty(I, W^{1,p'})$ as explained in Remark 5. In fact, we will show even higher regularity for $\bar{p}$ in the theorem below than necessary.

To improve readability of our arguments, we start with a collection of results from Bonifacius and Neitzel (2018). As further reference for maximal parabolic regularity on $H^{-\zeta,p}$-spaces we mention the work of Haller-Dintelmann and Rehberg (2009). Some of the results cited below are originally due to them.

Theorem 9 *1. For every right hand side $f \in L^s(I, H_D^{-\zeta,p})$ there is a unique solution $y \in W^{1,s}(I, H_D^{-\zeta,p}) \cap L^s(I, \mathscr{D})$ to the nonlinear state equation*

$$\partial_t y + \mathscr{A}(y)y = f, \qquad y(0) = y_0. \tag{E}$$

2. *The following embeddings hold true:*
   (a) $\mathscr{D} \hookrightarrow W_D^{1,p} \hookrightarrow_c L^p \hookrightarrow H_D^{-\zeta,p}$
   (b) $W^{1,s}(I, H_D^{-\zeta,p}) \cap L^s(I, \mathscr{D}) \hookrightarrow_c \mathscr{C}^\alpha(I, W_D^{1,p})$ *for some* $\alpha > 0$.
3. *The linear map* $W_D^{1,p} \to \mathscr{L}(\mathscr{D}, H_D^{-\zeta,p})$, $\xi \mapsto -\mathrm{div}(\xi\mu\nabla\cdot)$ *is continuous.*
4. *Let y be a solution of* (E). *Then it holds:*
   (a) $\mathscr{A}(y)$ *has maximal parabolic regularity on* $L^r(I, H_D^{-\zeta,p})$ *for* $r \in (1, \infty)$.
   (b) $\mathscr{A}(y) + \mathscr{A}'(y)$ *has maximal parabolic regularity on* $L^r(I, W_D^{-1,p})$ *for every* $r \in (1, s]$.
   (c) $\mathscr{A}(y)^\bullet + \mathscr{A}'(y)^\bullet$ *(where* $^\bullet$ *indicates taking adjoints and reversing time) has maximal parabolic regularity on* $L^{r'}(I, W_D^{-1,p'})$ *for every* $r' \in [s', \infty)$.
   (d) $\mathscr{A}(y)^* + \mathscr{A}'(y)^*$ *has maximal parabolic regularity on* $L^{r'}(I, W_D^{-1,p'})$ *for every* $r' \in [s', \infty)$.
5. *For* $\tau \in (\frac{1+\zeta}{2}, 1)$ *it holds* $(H_D^{-\zeta,p}, \mathscr{D})_{\tau,1} \hookrightarrow W_D^{1,p}$.

Proof 1. Bonifacius and Neitzel (2018, Theorem 3.20 for regularity, Proposition 3.5 for existence)
2. Bonifacius and Neitzel (2018, (a) below Proposition 3.6, (b) Corollary 3.7)
3. Bonifacius and Neitzel (2018) Proposition 3.6(ii).
4. See Bonifacius and Neitzel (2018, Theorem 3.20 for (a), Proposition 4.4 (resp. text between formulas (4.4) and (4.5)) for (b), Proposition 4.7 for (c)).
   For (d): Bonifacius and Neitzel (2018, proof of Proposition 4.7) state that every autonomous operator $\mathscr{A}(y(t))^* + \mathscr{A}'(y(t))^*$ has maximal parabolic regularity on $W_D^{-1,p'}$. Since the map $t \mapsto \mathscr{A}(y(t))^* + \mathscr{A}'(y(t))^*$ is continuous from I to $\mathscr{L}(W_D^{1,p'}, W_D^{-1,p'})$ the nonautonomous operator inherits maximal parabolic regularity, see Amann (2004, Theorem 7.2)
5. Bonifacius and Neitzel (2018, Proposition 3.6(i)). □

Now, we fix $y \in W^{1,s}(I, H_D^{-\zeta,p}) \cap L^s(I, \mathscr{D})$. In particular, y can be a solution of (E) for some right hand side $f \in L^s(I, H_D^{-\zeta,p})$. It was shown, see Theorem 9 (4c) and Amann (2004, Proposition 3.1) resp. Amann (2003, formula (6.2)), that

$$\begin{aligned}\big(-\partial_t + \mathscr{A}(y)^* + \mathscr{A}'(y)^*, \mathrm{tr}_T\big) \colon W^{1,r}(I, W_D^{-1,p'}) \cap L^r(I, W_D^{1,p'}) \\ \to L^r(I, W_D^{-1,p'}) \times (W_D^{-1,p'}, W_D^{1,p'})_{1-1/r,r}\end{aligned} \tag{24}$$

is a topological isomorphism for $r \in [s', \infty)$. In fact, this also holds for every $r \in (1, \infty)$ due to continuity of $t \mapsto \mathscr{A}(y)^* + \mathscr{A}'(y)^*$ as map $I \to \mathscr{L}(W^{1,p'}, W^{-1,p'})$ by (Amann 2004, Proposition 7.1 and Theorem 7.1). The required continuity with respect to time is shown by Bonifacius and Neitzel (2018) in the proof of Proposition 4.7.

We want to obtain more regularity for the adjoint state and to do so we consider restrictions of the above isomorphism onto smaller spaces of more regular functions. First, note that a short computation shows $\mathscr{A}(y)^*|_{L^r(I, W_D^{1,p})} = \mathscr{A}(y)|_{L^r(I, W_D^{1,p})}$ and similiarly we can express $\mathscr{A}'(y)^*$ restricted to $L^r(I, W_D^{1,p})$ as first order differential operator $\mathscr{A}'(y)^*\varphi = \xi'(y)\mu\nabla y \nabla\varphi$. Standard Sobolev embeddings imply that under Assumption 4 it holds

$$L^{p/2} \hookrightarrow H_D^{-\zeta,p}. \tag{25}$$

We already know by Theorem 9 (4a) that $-\mathrm{div}(\xi(y)\mu\nabla\cdot)$ has maximal parabolic regularity on $L^r(I, H_D^{-\zeta,p})$ and that $t \mapsto -\mathrm{div}(\xi(y)\mu\nabla\cdot)$ is continuous as map $I \to \mathscr{L}(\mathscr{D}, H_D^{-\zeta,p})$, which follows from Theorem 9 (2a) and (3). As above, we infer from Amann (2004) that

$$\begin{aligned}(-\partial_t - \mathrm{div}(\xi(y)\mu\nabla\cdot), \mathrm{tr}_T) \colon W^{1,r}(I, H_D^{-\zeta,p}) \cap L^r(I, \mathscr{D}) \\ \to L^r(I, H_D^{-\zeta,p}) \times (H_D^{-\zeta,p}, \mathscr{D})_{1-1/r,r}\end{aligned}$$

is a topological isomorphism. Now, choose $\frac{1+\zeta}{2} < \theta < 1$ such that $\frac{1}{r} > 1 - \theta$. It follows by (Amann 2003, formula (1.2)) and Theorem 9 (2a), (5) that

$$W^{1,r}(I, H_D^{-\zeta,p}) \cap L^r(I, \mathscr{D}) \hookrightarrow_c L^r(I, (H_D^{-\zeta,p}, \mathscr{D})_{\theta,1}) \hookrightarrow L^r(I, W_D^{1,p})$$

holds. Hence, the operator

$$\begin{aligned}\mathscr{A}'(y)^* \colon W^{1,r}(I, H_D^{-\zeta,p}) \cap L^r(I, \mathscr{D}) \hookrightarrow_c L^r(I, W_D^{1,p}) \to L^r(I, L^{p/2}) \hookrightarrow L^r(I, H_D^{-\zeta,p}), \\ z \mapsto \xi'(y)\mu\nabla y \nabla z\end{aligned}$$

is compact as it can be expressed as composition of linear operators of which one is a compact embedding. We conclude that the sum

$$\begin{aligned}(-\partial_t - \mathrm{div}(\xi(y)\mu\nabla\cdot) + \xi'(y)\mu\nabla y\nabla\cdot, \mathrm{tr}_T) \colon W^{1,r}(I, H_D^{-\zeta,p}) \cap L^r(I, \mathscr{D}) \\ \to L^r(I, H_D^{-\zeta,p}) \times (H_D^{-\zeta,p}, \mathscr{D})_{1-1/r,r}\end{aligned}$$

is a Fredholm-operator of index 0 for every $r \in (1, \infty)$. Since it is the restriction of the isomorphism (24) above, its kernel is trivial and therefore we actually have an isomorphism. To sum this up we have shown the following regularity result:

Theorem 10 *Given $y \in W^{1,s}(I, H_D^{-\zeta,p}) \cap L^s(I, \mathscr{D})$ the map*

$$(-\partial_t - \operatorname{div}(\xi(y)\mu\nabla\cdot) + \xi'(y)\mu\nabla y\nabla\cdot, \operatorname{tr}_T) \colon W^{1,r}(I, H_D^{-\zeta,p}) \cap L^r(I, \mathscr{D}) \\ \to L^r(I, H_D^{-\zeta,p}) \times (H_D^{-\zeta,p}, \mathscr{D})_{1-1/r,r}$$

is a topological isomorphism for every $r \in (1,\infty)$, i.e. the adjoint equation

$$-\partial_t z - \operatorname{div}(\xi(y)\mu\nabla z) + \xi'(y)\mu\nabla y\nabla z = w, \\ z(T) = w_T$$

admits a unique solution $z \in W^{1,r}(I, H_D^{-\zeta,p}) \cap L^r(I, \mathscr{D})$ provided that $w \in L^r(I, H_D^{-\zeta,p})$ and $w_T \in (H_D^{-\zeta,p}, \mathscr{D})_{1-1/r,r}$.

Remark 9 Note that we did not need more assumptions than Bonifacius and Neitzel (2018) except for the slightly higher integrability of y_d. In the framework of maximal parabolic regularity on $W_D^{-1,p}$-spaces they discuss first order necessary and second order sufficient optimality conditions, but in order to deal with the adjoint equation in the maximal parabolic regularity context (Bonifacius and Neitzel 2018, Lemma 4.6, Proposition 4.7) they required states in $\mathscr{C}^\alpha(I, W_D^{1,p})$ which was achieved by consideration of the state equation on $H_D^{-\zeta,p}$ spaces. Since we aim at SQP methods having an adjoint equation with corresponding regularity theory is necessary anyway and therefore restriction to the $H_D^{-\zeta,p}$-setting is not superfluous.

Remark 10 Since μ was assumed to be symmetric we could identify $\mathscr{A}(y)^*$ with $\mathscr{A}(y)$ etc. directly. In fact, all arguments go through if we postulate the same assumptions for μ^T as already done for μ.

8 Numerical Examples

In this final section we present numerical examples in order to illustrate our theoretical results. To do so we have constructed socalled manufactured solution examples, i.e. an optimal control problem with analytically known solution, see Tröltzsch (2010, Section 2.9) for the construction of such examples. Further, we test with an example including boundary control, see Section 8.2

We implemented the SQP algorithm in `python` using an optimize-then-discretize approach and `FEniCS` (Alnæs et al. 2015; Logg et al. 2012) for the finite element discretization of the problem. The quadratic subproblems are solved by a semismooth Newton method, see e.g. Ulbrich (2011), Hintermüller and Hinze (2006). As observed in the existing literature the restriction of the quadratic subproblems to L^∞- or –in our case– L^2-balls is only required to prove convergence of the algorithm in function space. Fortunately, we can leave away this additional constraint in practice and solve the quadratic subproblems on U_{ad} without loosing convergence.

Initial guess for the SQP method is in all three examples $(y_0,u_0,p_0) := (0,0,0)$. To measure optimality of some iterate u we compute the L^2-norm of the residual of the projection formula

$$\mathrm{res}_{L^2}(u) := \left\| u - \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}B^* p(u)\right)\right\|_{L^2},$$

where the adjoint state $p(u)$ associated to u is computed using the implicit Euler scheme. The nonlinear equations appearing at each timestep during the solution of the state equation are solved by the built-in nonlinear solver of `FEniCS`. Convergence of the SQP-Algorithm is measured by the increments

$$\mathrm{incr}_k := \|y_{k+1} - y_k\|_{L^\infty} + \|u_{k+1} - u_k\|_{L^\infty} + \|p_{k+1} - p_k\|_{L^\infty}$$

Note that we do not compute the norm of the increments with respect to the norms appearing in Theorem 8 because we do not have the abstract exponents p,s at hand in a practical context.

8.1 Manufactured Solution Examples

8.1.1 Manufactured Solution Example in 1D

For $I=[0,1]$ and $\Omega=[0,1]$ we consider the problem

$$\begin{cases} \min\limits_{y,u} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(I\times\Omega)} + 10^{-3}\cdot\|u\|^2_{L^2([0,1])} \\ \text{subject to} \quad u \in \left\{ v\in L^2([0,1]) : -\frac{9}{10} \le v(x) \le \frac{\sqrt{2}}{2} \quad \text{a.e.}\right\}, \\ \text{and} \quad \begin{cases} \partial_t y - \mathrm{div}(\xi(y)\nabla y) = b\cdot u + f \quad \text{on } I\times\Omega, \\ y = 0 \quad \text{on } I\times\partial\Omega, \\ y(0) = \sin(\pi x_1), \end{cases} \end{cases} \tag{26}$$

and choose

$$\begin{aligned} \bar{y}(t,x) &= \cos(2\pi t)\sin(\pi x), \\ \bar{p}(t,x) &= \frac{1}{100}\sin(2\pi t)\sin(\pi x), \\ b(x) &= \mathbf{1}_{[1/3,2/3]}(x), \\ \xi(z) &= \frac{1}{2} + \frac{1}{1+\exp(-5z)}. \end{aligned}$$

With help of `Wolfram Mathematica` we compute the remaining quantities y_d, f, $\bar{u}$ such that the optimality system for (26) is fulfilled. In particular it holds

$$\bar{u}(t) = \min\left(\frac{\sqrt{2}}{2}, \max\left(-\frac{9}{10}, -\frac{10}{\pi}\sin(2\pi t)\right)\right).$$

Note that all our theoretical results remain true for a problem of type (26) since addition of the term f to the model problem (**OCP**) does not change its structural properties.

Discretization of spatial functions is done with piecewise linear finite elements on a equidistant partition of $\Omega = [0,1]$ in 100 subintervals. For time discretization we apply an implicit Euler discretization with 10000 timesteps, whereby the size of timesteps is chosen in order to roughly balance spatial and temporal discretization errors, cf. Casas and Chrysafinos (2019). Since there is no visible difference between the computed optimal control and the interpolated true solution, we do not show a plot. Superlinear convergence seems to be indicated in Figure 1 (lhs). Stagnation both of the projection residuals (Figure 1 (lhs)) resp. the errors to the interpolated KKT-tripple (Figure 2 (lhs)) can be explained by effects of discretization.

8.1.2 Manufactured Solution Example in 2D

For $I = [0,1]$ and $\Omega = [0,1]^2$ we consider a problem of the same structure as (26), but now with

$$\begin{aligned} y_0(x) &= \sin(\pi x_1)\sin(\pi x_2), \\ \bar{y}(t,x) &= \cos(2\pi t)\sin(\pi x_1)\sin(\pi x_2), \\ \bar{p}(t,x) &= \frac{1}{100}\sin(2\pi t)\sin(\pi x_1)\sin(\pi x_2), \\ b(x) &= \pi^2 \cdot \mathbf{1}_{[1/3,2/3]^2}(x) \end{aligned}$$

and the regularization parameter $\gamma = 2\cdot 10^{-3}$ in (26) replaced by $\gamma = 10^{-2}$. As before, the remaining quantities are computed utilizing `Wolfram Mathematica` and the optimal control is given by

$$\bar{u}(t) = \min\left(\frac{\sqrt{2}}{2}, \max\left(-\frac{9}{10}, -\sin(2\pi t)\right)\right).$$

Discretization of spatial functions is now done with piecewise linear finite elements on a triangular mesh generated by `mshr`, the mesh-generation tool of `FEniCS`, with maximum element diameter $h_{\max} \approx 2.55\cdot 10^{-2}$. For time discretization we apply an implicit Euler discretization with 1544 timesteps, whereby the size of timesteps $\tau \approx h_{\max}^2$ is chosen in order to roughly balance spatial and temporal discretization errors, cf. Casas and Chrysafinos (2019). It was not possible to choose a finer discretization while maintaining a reasonable amount of computational costs and hence convergence of the SQP method with respect to the true solution stagnates quite early compared to the 1D case, see Figure 2 (rhs). However, the behaviour of the increments resp. the projection residuals in Figure 1 (rhs) still seems to indicate superlinear convergence.

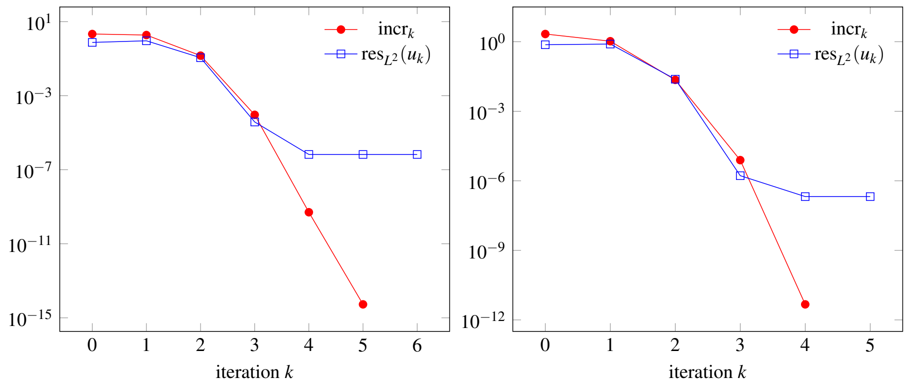

Fig. 1 Manufactured solution examples in 1D (lhs) resp. 2D (rhs): Convergence of the increments incr_k resp. projection residuals $\mathrm{res}_{L^2}(u_k)$ during the SQP iteration.

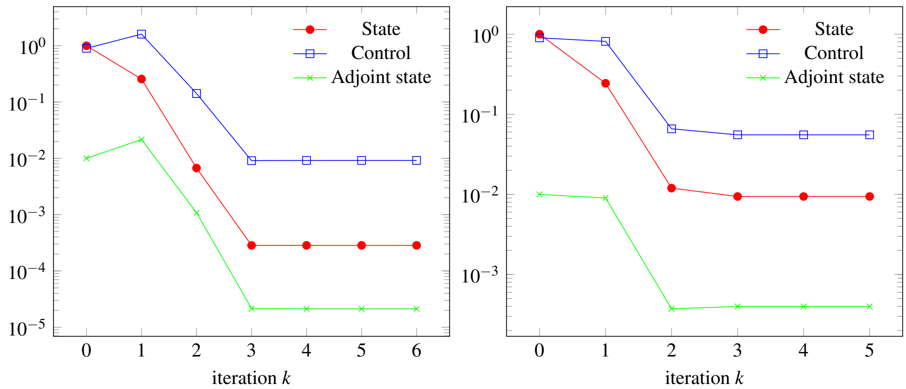

Fig. 2 Manufactured solution example in 1D (lhs) resp. 2D (rhs): Convergence of the L^∞-errors of state, control resp. adjoint state with respect to the interpolated true KKT-tripple.

8.2 Example including Boundary Control Functions

As a final example we consider for $I = [0,1]$ and $\Omega = [0,1]^2$ the following problem:

$$\begin{cases} \min\limits_{y,u} J(y,u) := \dfrac{1}{2}\|y-y_d\|^2_{L^2(I\times\Omega)} + \dfrac{10^{-4}}{2}\cdot\sum\limits_{i=1}^{4}\|u_i\|^2_{L^2([0,1])} \\ \text{subject to} \quad u \in \big\{v \in L^2([0,1],\mathbb{R}^4) : 0 \le v_1(t) \le 15, \\ \qquad\qquad\qquad\qquad -0.8 \le v_i \le 0 \text{ for } i = 2,3,4\big\}, \\ \quad\text{and} \quad \begin{cases} \partial_t y - \operatorname{div}(\xi(y)\nabla y) = u_1 \mathbf{1}_{K_2} \quad \text{on } I\times\Omega, \\ \xi(y)\partial_{n_\Omega} y = u_i \quad \text{on } I\times\Gamma_{N,i}, \quad i = 2,3,4, \\ \partial_{n_\Omega} y = 0 \quad \text{on } I\times\Gamma_{N,1}, \\ y = 0 \quad \text{on } I\times\partial\Gamma_D, \\ y(0) = 0. \end{cases} \end{cases} \tag{27}$$

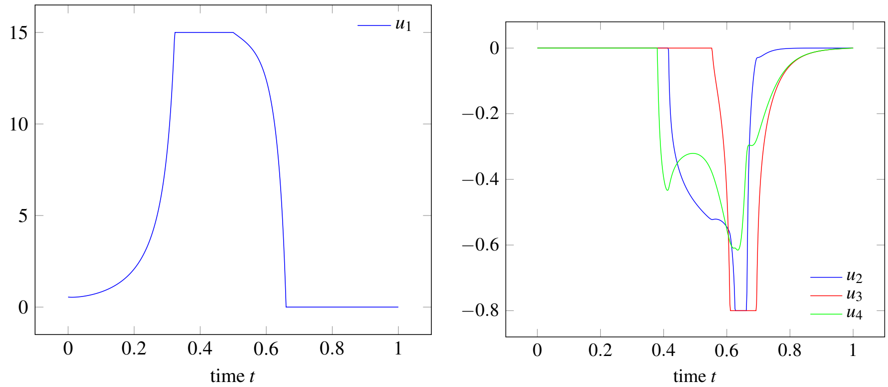

Fig. 3 Optimal controls computed with SQP for example (27): control u_1 (lhs) and controls u_2, u_3, u_4 (rhs).

The desired state is given by

$$y_d(t,x) := \mathbf{1}_{[1/3,2/3]}(t) \cdot \mathbf{1}_{K_1}(x),$$

where $K_1 := \mathbb{B}_{\frac{1}{4}}((0.6, 0.5))$ denotes the subdomain of Ω we want to "heat" during the times $t \in [1/3, 2/3]$ and $K_2 := \mathbb{B}_{\frac{1}{6}}((0.6, 0.5))$ is the subdomain of Ω at which we can apply a "heat source". At the boundary parts

$$\begin{aligned} \Gamma_{N,2} &:= \{x \in \partial\Omega : x_2 = 1\}, \\ \Gamma_{N,3} &:= \{x \in \partial\Omega : x_1 = 1, 0.5 \leq x_2 < 1\}, \\ \Gamma_{N,4} &:= \{x \in \partial\Omega : x_1 = 0, 0.5 \leq x_2 < 1\} \end{aligned}$$

we can "cool" via Neumann boundary control. $\Gamma_D := \{x \in \partial\Omega : x_2 = 0\}$ denotes the Dirichlet boundary equippend with homogeneous Dirichlet conditions. Natural boundary conditions hold on the remaining parts $\Gamma_{N,1}$ of the boundary. The nonlinearity is given by

$$\xi(y) := 0.9 + \frac{0.2}{1 + \exp(-20y)}.$$

Discretization is done with piecewise linear finite elements on a mesh generated by `mshr` with mesh size $h_{\max} \approx 4.24 \cdot 10^{-2}$ and 556 time steps. Figure 3 shows the optimal controls computed by the SQP method. Both the convergence of increments and residuals in Figure 4 seems to go along with our theoretical findings.

Acknowledgements The authors are grateful to Hannes Meinlschmidt (RICAM, Linz) for suggesting a more elegant and shorter proof for Theorem 10 than our original argument.

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Projektnummer 211504053 - SFB 1060.

References

Alnæs MS, Blechta J, Hake J, Johansson A, Kehlet B, Logg A, Richardson C, Ring J, Rognes ME, Wells GN (2015) The fenics project version 1.5. Archive of Numerical Software 3(100), DOI 10.11588/ans.2015.100.20553

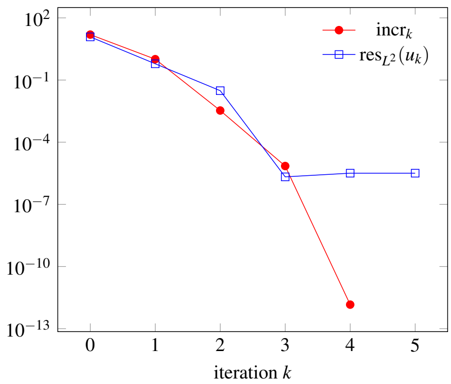

Fig. 4 Example (27): Convergence of the increments incr_k resp. projection residuals $\mathrm{res}_{L^2}(u_k)$ during the SQP iteration.

Alt W (1990) The Lagrange-Newton method for infinite-dimensional optimization problems. Numer Funct Anal Optim 11(3-4):201–224, DOI 10.1080/01630569008816371, URL https://doi.org/10.1080/01630569008816371

Alt W, Griesse R, Metla N, Rösch A (2010) Lipschitz stability for elliptic optimal control problems with mixed control-state constraints. Optimization 59(5-6):833–849, DOI 10.1080/02331930902863749, URL https://doi.org/10.1080/02331930902863749

Amann H (2003) Nonautonomous parabolic equations involving measures. Zap Nauchn Sem S-Peterburg Otdel Mat Inst Steklov (POMI) 306(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funktsii. 34):16–52, 229, DOI 10.1007/s10958-005-0376-8, URL https://doi.org/10.1007/s10958-005-0376-8

Amann H (2004) Maximal regularity for nonautonomous evolution equations. Adv Nonlinear Stud 4(4):417–430, DOI 10.1515/ans-2004-0404, URL https://doi.org/10.1515/ans-2004-0404

Amann H (2005) Quasilinear parabolic problems via maximal regularity. Adv Differential Equations 10(10):1081–1110

Bonifacius L, Neitzel I (2018) Second order optimality conditions for optimal control of quasilinear parabolic equations. Math Control Relat Fields 8(1):1–34, DOI 10.3934/mcrf.2018001, URL https://doi.org/10.3934/mcrf.2018001

Bonnans JF (1998) Second-order analysis for control constrained optimal control problems of semilinear elliptic systems. Appl Math Optim 38(3):303–325, DOI 10.1007/s002459900093, URL https://doi.org/10.1007/s002459900093

Casas E, Chrysafinos K (2018) Analysis and optimal control of some quasilinear parabolic equations. Mathematical Control and Related Fields 8(3-4):607–623

Casas E, Chrysafinos K (2019) Numerical analysis of quasilinear parabolic equations under low regularity assumptions. Numerische Mathematik DOI 10.1007/s00211-019-01071-5, URL https://doi.org/10.1007/s00211-019-01071-5

Casas E, Dhamo V (2011) Optimality conditions for a class of optimal boundary control problems with quasilinear elliptic equations. Control Cybernet 40(2):457–490

Casas E, Tröltzsch F (2009) First- and second-order optimality conditions for a class of optimal control problems with quasilinear elliptic equations. SIAM J Control Optim 48(2):688–718, DOI 10.1137/080720048, URL https://doi.org/10.1137/080720048

Casas E, Tröltzsch F (2011) Numerical analysis of some optimal control problems governed by a class of quasilinear elliptic equations. ESAIM Control Optim Calc Var 17(3):771–800, DOI 10.1051/cocv/2010025, URL https://doi.org/10.1051/cocv/2010025

Casas E, Tröltzsch F (2012) A general theorem on error estimates with application to a quasilinear elliptic optimal control problem. Comput Optim Appl 53(1):173–206, DOI 10.1007/s10589-011-9453-8, URL https://doi.org/10.1007/s10589-011-9453-8

Casas E, Tröltzsch F (2012) Second order analysis for optimal control problems: improving results expected from abstract theory. SIAM J Optim 22(1):261–279, DOI 10.1137/110840406, URL https://doi.org/10.1137/110840406

Casas E, Tröltzsch F (2015) Second order optimality conditions and their role in PDE control. Jahresber Dtsch Math-Ver 117(1):3–44, DOI 10.1365/s13291-014-0109-3, URL https://doi.org/10.1365/s13291-014-0109-3

Dontchev AL (1996) Local analysis of a Newton-type method based on partial linearization. In: The mathematics of numerical analysis (Park City, UT, 1995), Lectures in Appl. Math., vol 32, Amer. Math. Soc., Providence, RI, pp 295–306

Feldhordt H (2017) Boundary control of a chemotaxis system. Dissertation, Univeristät Duisburg-Essen, URL https://nbn-resolving.org/urn:nbn:de:hbz:464-20170809-110745-3

Goldberg H, Tröltzsch F (1989) Second order optimality conditions for a class of control problems governed by nonlinear integral equations with application to parabolic boundary control. Optimization 20(5):687–698, DOI 10.1080/02331938908843489, URL https://doi.org/10.1080/02331938908843489

Goldberg H, Tröltzsch F (1998) On a Lagrange-Newton method for a nonlinear parabolic boundary control problem. Optim Methods Softw 8(3-4):225–247, DOI 10.1080/10556789808805678, URL https://doi.org/10.1080/10556789808805678

Griepentrog JA, Gröger K, Kaiser HC, Rehberg J (2002) Interpolation for function spaces related to mixed boundary value problems. Math Nachr 241:110–120, DOI 10.1002/1522-2616(200210)244:1¡110::AID-MANA110¿3.0.CO;2-S, URL https://doi.org/10.1002/1522-2616(200210)244:1¡110::AID-MANA110¿3.0.CO;2-S

Griesse R, Metla N, Rösch A (2008) Convergence analysis of the SQP method for nonlinear mixed-constrained elliptic optimal control problems. ZAMM Z Angew Math Mech 88(10):776–792, DOI 10.1002/zamm.200800036, URL https://doi.org/10.1002/zamm.200800036

Griesse R, Metla N, Rösch A (2010) Local quadratic convergence of SQP for elliptic optimal control problems with mixed control-state constraints. Control Cybernet 39(3):717–738

Haller-Dintelmann R, Rehberg J (2009) Maximal parabolic regularity for divergence operators including mixed boundary conditions. J Differential Equations 247(5):1354–1396, DOI 10.1016/j.jde.2009.06.001, URL https://doi.org/10.1016/j.jde.2009.06.001

Heinkenschloss M, Tröltzsch F (1999) Analysis of the Lagrange-SQP-Newton method for the control of a phase field equation. Control Cybernet 28(2):177–211

Hintermüller M, Hinze M (2006) A SQP-semismooth Newton-type algorithm applied to control of the instationary Navier-Stokes system subject to control constraints. SIAM J Optim 16(4):1177–1200, DOI 10.1137/030601259, URL https://doi.org/10.1137/030601259

Hintermüller M, Volkwein S, Diwoky F (2007) Fast solution techniques in constrained optimal boundary control of the semilinear heat equation. In: Control of coupled partial differential equations, Internat. Ser. Numer. Math., vol 155, Birkhäuser, Basel, pp 119–147

Hinze M, Kunisch K (2001) Second order methods for optimal control of time-dependent fluid flow. SIAM J Control Optim 40(3):925–946, DOI 10.1137/S0363012999361810, URL https://doi.org/10.1137/S0363012999361810

Hinze M, Pinnau R, Ulbrich M, Ulbrich S (2009) Optimization with PDE constraints, Mathematical Modelling: Theory and Applications, vol 23. Springer, New York

Ioffe AD (1979) Necessary and sufficient conditions for a local minimum. III. Second order conditions and augmented duality. SIAM J Control Optim 17(2):266–288, DOI 10.1137/0317021, URL https://doi.org/10.1137/0317021

Ito K, Kunisch K (2004) The primal-dual active set method for nonlinear optimal control problems with bilateral constraints. SIAM J Control Optim 43(1):357–376, DOI 10.1137/S0363012902411015, URL https://doi.org/10.1137/S0363012902411015

Josephy N (1979) Newton's Method for Generalized Equations. Tech. rep.

Lions JL (1971) Optimal control of systems governed by partial differential equations. Translated from the French by S. K. Mitter. Die Grundlehren der mathematischen Wissenschaften, Band 170, Springer-Verlag, New York-Berlin

Logg A, Mardal KA, Wells GN, et al. (2012) Automated Solution of Differential Equations by the Finite Element Method. Springer, DOI 10.1007/978-3-642-23099-8

de Los Reyes JC, Dhamo V (2016) Error estimates for optimal control problems of a class of quasilinear equations arising in variable viscosity fluid flow. Numer Math 132(4):691–720, DOI 10.1007/s00211-015-0737-2, URL https://doi.org/10.1007/s00211-015-0737-2

de Los Reyes JC, Merino P, Rehberg J, Tröltzsch F (2008) Optimality conditions for state-constrained PDE control problems with time-dependent controls. Control Cybernet 37(1):5–38

Meinlschmidt H, Rehberg J (2016) Hölder-estimates for non-autonomous parabolic problems with rough data. Evol Equ Control Theory 5(1):147–184, DOI 10.3934/eect.2016.5.147, URL https://doi.org/10.3934/eect.2016.5.147

Meinlschmidt H, Meyer C, Rehberg J (2017a) Optimal control of the thermistor problem in three spatial dimensions, Part 1: Existence of optimal solutions. SIAM J Control Optim 55(5):2876–2904, DOI 10.1137/16M1072644, URL https://doi.org/10.1137/16M1072644

Meinlschmidt H, Meyer C, Rehberg J (2017b) Optimal control of the thermistor problem in three spatial dimensions, Part 2: Optimality conditions. SIAM J Control Optim 55(4):2368–2392, DOI 10.1137/16M1072656, URL https://doi.org/10.1137/16M1072656

Nicaise S, Tröltzsch F (2017) Optimal control of some quasilinear Maxwell equations of parabolic type. Discrete Contin Dyn Syst Ser S 10(6):1375–1391, DOI 10.3934/dcdss.2017073, URL https://doi.org/10.3934/dcdss.2017073

Robinson SM (1980) Strongly regular generalized equations. Math Oper Res 5(1):43–62, DOI 10.1287/moor.5.1.43, URL https://doi.org/10.1287/moor.5.1.43

Tröltzsch F (1999) On the Lagrange-Newton-SQP method for the optimal control of semilinear parabolic equations. SIAM J Control Optim 38(1):294–312, DOI 10.1137/S0363012998341423, URL https://doi.org/10.1137/S0363012998341423

Tröltzsch F (2000) Lipschitz stability of solutions of linear-quadratic parabolic control problems with respect to perturbations. Dynam Contin Discrete Impuls Systems 7(2):289–306

Tröltzsch F (2010) Optimal control of partial differential equations, Graduate Studies in Mathematics, vol 112. American Mathematical Society, Providence, RI, DOI 10.1090/gsm/112, URL https://doi.org/10.1090/gsm/112, theory, methods and applications, Translated from the 2005 German original by Jürgen Sprekels

Tröltzsch F, Wachsmuth D (2006) Second-order sufficient optimality conditions for the optimal control of Navier-Stokes equations. ESAIM Control Optim Calc Var 12(1):93–119, DOI 10.1051/cocv:2005029, URL https://doi.org/10.1051/cocv:2005029

Ulbrich M (2011) Semismooth Newton methods for variational inequalities and constrained optimization problems in function spaces, MOS-SIAM Series on Optimization, vol 11. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, DOI 10.1137/1.9781611970692, URL https://doi.org/10.1137/1.9781611970692

Ulbrich S, Ziems JC (2017) Adaptive multilevel trust-region methods for time-dependent PDE-constrained optimization. Port Math 74(1):37–67, DOI 10.4171/PM/1992, URL https://doi.org/10.4171/PM/1992

Wachsmuth D (2007) Analysis of the SQP-method for optimal control problems governed by the nonstationary Navier-Stokes equations based on L^p-theory. SIAM J Control Optim 46(3):1133–1153, DOI 10.1137/S0363012904443506, URL https://doi.org/10.1137/S0363012904443506

Yousept I (2013) Optimal control of quasilinear h(curl)-elliptic partial differential equations in magnetostatic field problems. SIAM J Control Optim 51(5):3624–3651, DOI 10.1137/120904299, URL https://doi.org/10.1137/120904299

Ziems JC (2013) Adaptive multilevel inexact SQP-methods for PDE-constrained optimization with control constraints. SIAM J Optim 23(2):1257–1283, DOI 10.1137/110848645, URL https://doi.org/10.1137/110848645

Ziems JC, Ulbrich S (2011) Adaptive multilevel inexact SQP methods for PDE-constrained optimization. SIAM J Optim 21(1):1–40, DOI 10.1137/080743160, URL https://doi.org/10.1137/080743160