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Abstract. We prove a-posteriori error-estimates for reduced-order modeling of quasilinear parabolic PDEs with non-monotone nonlinearity. We consider the solution of a semi-discrete in space equation as reference, and therefore incorporate reduced basis-, empirical interpolation-, and time-discretization-errors in our consideration. Numerical experiments illustrate our results.

1. Introduction

In the present paper we are concerned with a-posteriori error estimation for model order reduction applied to a semi-discrete in space quasilinear parabolic partial differential equation (PDE) with non-monotone nonlinearity. The PDE appears for instance as state equation in the optimal control problems from [**7**, **26**], and is used in the modeling of heat conduction, when the thermal conductivity of the material under consideration is temperature dependend, cf. e.g. [**35**, **36**, **42**].

The numerical treatment of evolution equations and related problems is challenging. For instance, the discretization of associated optimal control problems leads to large-scale optimization problems that are highly expensive to solve. This is especially true for nonlinear equations. Therefore, model order reduction-techniques (MOR) play an important role in this context. Their aim is to replace the high-dimensional original model by a suitable model with less degrees of freedom, the so-called reduced-order model. A prominent method of MOR for parabolic PDEs is the so-called Proper Orthogonal Decomposition (POD) method, [**47**]. This approach uses so-called snapshots of the dynamical system to construct a low-dimensional subspace of e.g. a high-dimensional finite-element space. More generally speaking, projection of a high-dimensional dynamical or parametric system onto smaller dimensional spaces leads to so-called reduced basis methods (RB), see e.g. [**23**]. These subspaces need to be in some sense capable of expressing the original trajectory of the system sufficiently well. The question of estimating the model order reduction error arises naturally and has been subject to intensive research. We refer e.g. to [**38**] and the references therein for RB-methods in PDE-constrained optimization in general, and to [**39**] or the survey [**21**] for POD in particular.

Since there is a huge amount of literature about POD/RB-MOR, not just for uncontrolled equations but even in the context of PDE-constrained optimization,

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we have to restrict ourselves to an incomplete literature overview. POD-error-estimates have been obtained in the a-priori regime for linear parabolic equations in [**30**], and for certain nonlinear equations e.g. in [**10**, **31**, **43**]. For recently obtained RB-a-posteriori error-estimates for quasilinear equations with monotone nonlinearity related to magneto(quasi)statics, both elliptic, and parabolic, we refer to [**24**, **25**]. For a general nonlinear parabolic PDE including a-posteriori error estimation on the time-discrete level we exemplarily cite [**13**], and refer to its introduction for an overview over further literature. Moreover, let us mention some results related to optimal control: A-posteriori POD-errors for linear-quadratic optimal control problems have been derived in [**46**]. The technique has been extended to nonlinear problems in [**29**] and problems with mixed control-state constraints in [**20**]. POD-a-posteriori error-estimates for an optimal control problem with semilinear state equation with monotone nonlinearity has been discussed in [**39**]. Appropriate coupling between numerical optimization and MOR is an active area of research, see e.g. [**18**]. For approaches based on a-posteriori error estimation and trust-region type-algorithms, respectively, we refer to [**5**, **41**], and in particular [**38**, **39**]. A different method, so-called optimality system POD, has been proposed in [**32**]. A related aspect, the interplay between POD and discretization, is under consideration e.g. in [**15–17**]. Balancing of discretization- and POD-errors for an optimal control problem is addressed in [**19**]. Finally, we exemplarily cite [**1**] and [**28**] for RB/POD-MOR applied to robust and multiobjective optimization, respectively.

For quasilinear parabolic PDEs or related optimal control problems, even the analysis of the full-order model is less complete. We mention [**22**, **37**] for the analysis of the equation itself based on the functional analytic tool of nonautonomous maximal parabolic regularity [**4**], and refer to [**7**, **8**] for first- and second-order optimality conditions of the respective optimal control problems. For additional state constraints we refer to [**27**]. A quasilinear version of the so-called thermistor problem has been addressed in [**35**, **36**], and convergence of the SQP method applied to the model problem from [**7**] has been proven in [**26**]. For earlier literature on quasilinear parabolic optimal control problems, and optimal control of quasilinear elliptic PDEs with refer to the introductions of [**7**, **8**]. Finite element discretization error-estimates for the state equation from [**8**] are obtained in [**9**]. Having in mind a coupling of numerical optimization and MOR à la [**38**, **39**] as a long-term goal, we start in the present paper with deriving a-posteriori reduced basis errors for the corresponding state equation. Herein, the presence of a non-monotone nonlinearity is the main difference to earlier publications concerned with RB-a-posteriori error-estimates for nonlinear PDEs [**24**, **25**, **39**], and also poses the main difficulty in our analysis. Moreover, compared to [**13**, **24**, **38**] we include time-discretization errors in our estimates which prevents the choice of unnecessarily accurate reduced-models in practice. Nevertheless, our reference solution is semi-discrete (in space), i.e. we fix a spatial discretization and do not address errors arising from this.

The paper is organized as follows: We start by introducing the model problem and the underlying assumptions in Section 2. Further, we provide a short overview over the results obtained on this equation so far, and introduce its semi-discretization (in space), and the reduced-order counterpart thereof. Moreover, we provide a sketchy outlook how the results subsequently obtained might be applied in the context of PDE-constrained optimization. In Section 3 we prove RB-a-posteriori error-estimates for a reduced-order trajectory that is continuous-in-time and piecewise C^1-in-time. Depending on how much regularity of the semi-discrete (in space) solution we are willing to exploit, we present two different approaches to obtain a-posteriori error-estimates. In case that hyperreduction of the nonlinearity is done by empirical interpolation (EIM) we provide estimates that also include the additional EIM-error in Section 4. Finally, we illustrate our results by numerical experiments in Section 5.

Notation. Throughout the paper we follow the conventions of [**7**] and use standard notation for (Bochner-)Lebesgue, (Bochner-)Sobolev-, and Hölder-spaces. By subscript D we denote incorporation of certain homogeneous Dirichlet boundary conditions. If the underlying spatial domain becomes clear from the context we will omit it, i.e. we write H^1_D instead of $H^1_D(\Omega)$ for instance.

2. Model Problem, Assumptions, and RB-MOR

In this section we first introduce the continuous model equation with non-monotone nonlinearity, and recall some regularity results under appropriate assumptions. Second, we state the semi-discrete (in space) version and its RB-reduction. Finally, we briefly explain how the results obtained in this paper might be applied to an optimal control problem associated to the model equation.

2.1. Model Problem, Assumptions, and Regularity Results. Let us start with defining the setting for the following quasilinear parabolic PDE:

$$\text{(Eq)} \qquad \left.\begin{aligned} \partial_t u + \mathcal{A}(u)u &= f \qquad && \text{on } I \times \Omega, \\ u(0) &= u_0 && \text{on } \Omega, \end{aligned}\right\}$$

with $f \in L^s(I, W^{-1,p}_D)$ and $u_0 \in (W^{1,p}_D, W^{-1,p}_D)_{1/s',s}$. The nonlinearity $\mathcal{A}$ is defined by

$$\langle \mathcal{A}(u)\varphi, \psi\rangle_{H^{-1}_D, H^1_D} := \int_\Omega \xi(u)\mu\nabla\varphi\nabla\psi \,\mathrm{d}x\mathrm{d}t, \qquad \varphi, \psi \in L^2(I, H^1_D),$$

whenever $u\colon I \times \Omega \to \mathbb{R}$ is measurable.

Before going into the detailed assumptions, we would like to comment briefly on the non-monotone structure of the nonlinearity in (Eq). The main difficulty as well as the main novelty of this paper arise from this fact. Recall that a nonlinear operator $\mathcal{N}\colon X \to X^*$ on a Banach space X is called monotone if

$$\langle \mathcal{N}(x) - \mathcal{N}(y), x - y\rangle_{X^*,X} \geq 0 \qquad \forall x, y \in X,$$

and strongly monotone if there exists a constant $c > 0$ such that

$$\langle \mathcal{N}(x) - \mathcal{N}(y), x - y\rangle_{X^*,X} \geq c\|x - y\|^2_X \qquad \forall x, y \in X,$$

cf. e.g. [**40**, **48**] for this notion and its application in the theory of nonlinear PDEs. It has turned out that exploitation of strong monotonicity of the nonlinear terms is also an important step in the derivation of RB-a-posteriori error-estimates for semilinear parabolic [**39**], quasilinear elliptic [**25**], and quasilinear parabolic [**24**] PDEs, respectively. Note that the quasilinear nonlinearities in [**24**, **25**] refer to problems from magneto(quasi)statics and depend on the gradient of the solution. The nonlinear operator $H^1_D \to H^{-1}_D$ under consideration in the present paper, however, is given by the map $u \mapsto \mathcal{A}(u)u$, and hence it depends on the solution and not on its gradient. The counterexample [**14**, Example 8.18] shows that this essentially changes the structure of the nonlinearity: It cannot be expected to be monotone, and therefore it is specifically different to those considered in [**24**, **25**, **39**]. In fact, the main difficulty in the derivation of RB-a-posteriori error-estimates will be to find a workaround for the missing strong monotonicity of our nonlinearity.

For the rest of this paper, we rely on the following minimal assumptions:

Assumption 2.1.
(1) $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, is a bounded domain with boundary $\partial\Omega$. $\Gamma_N \subset \partial\Omega$ is relatively open and denotes the Neumann boundary part, whereas $\Gamma_D = \partial\Omega \setminus \Gamma_N$ is the part of the boundary equipped with homogeneous Dirichlet conditions. By subscript D we denote that a space of functions on Ω incorporates such homogeneous Dirichlet boundary conditions on Γ_D.
(2) For some $T > 0$ we define the time interval $I = [0, T]$.
(3) The function ξ: $\mathbb{R} \to \mathbb{R}$ is differentiable with bounded derivative. Let μ: $\Omega \to \mathbb{R}^{d\times d}$, $\mu = \mu^T$, be measurable, uniformly bounded, and coercive in the following sense:

$$0 < \mu_\bullet := \inf_{x\in\Omega} \inf_{z\in\mathbb{R}^d\setminus\{0\}} \frac{z^T\mu(x)z}{z^Tz}, \qquad \mu^\bullet := \sup_{x\in\Omega} \sup_{1\leq i,j\leq d} |\mu_{i,j}(x)| < \infty$$

We assume a coercivity condition $0 < \xi_\bullet \leq \xi \leq \xi^\bullet$ for ξ as well. With this we define as above

$$\langle \mathcal{A}(u)\varphi, \psi\rangle_{L^2(I,W_D^{1,2})} := \int_I \int_\Omega \xi(u)\mu\nabla\varphi\nabla\psi \mathrm{d}x\mathrm{d}t, \qquad \varphi, \psi \in L^2(I, W_D^{1,2}),$$

whenever u is a measurable function on Ω.

Since we restrict ourselves to semi-discrete (in space) solutions in the present paper, these assumptions suffice for our purpose. We refer to the introduction of Subsection 3.2, where we point out that the regularity assumptions on the semi-discrete reference solution will automatically be ensured by space-discretization.

The discussion of (Eq) on the continuous in space and time level, however, would require additional assumptions that we omit for the reason of clarity. We only recall two regularity results from the literature that might be seen as motivation for exploiting the respective regularity of the semi-discrete in space solution lateron: A detailed analysis of the equation on $W^{-1,p}$-spaces can be found in [**37**, section 5], and regularity on the slightly more regular Bessel potential-spaces $H^{-\zeta,p}$ has been addressed in [**7**, section 3.2]. In particular, $C^\alpha(I, W_D^{1,p})$-regularity of the solution with some $\alpha > 0$ and $p \in (d, 4)$ is obtained under fairly general regularity assumptions that admit certain constellations of non-smooth domains and coefficients, mixed boundary conditions, and distributional right-hand sides f. An equation similar to the one in the present paper, but in a slightly more regular setting, has been considered in [**8**, Theorem 2.3]. In particular, $C^{1,1}$-smooth domains and coefficients, homogeneous Dirichlet boundary conditions, and integrable right-hand sides are required. In return the authors discuss a possibly unbounded nonlinearity, include a semilinear term, and obtain W^2-regularity results that enable the derivation of finite element error-estimates [**9**]. Applying their setting to (Eq) yields $C^{0,\alpha}(I, W^{1,\infty})$-regularity of the solutions with some $\alpha > 0$, cf. also [**27**, Corollary 5.4].

2.2. Semi-discretization in space and RB-MOR.

We now introduce the semi-discrete (in space) counterpart of (Eq). Its solution will serve as the so-called truth-solution, i.e. the reference solution to which the a-posteriori error-estimates will refer to. In particular, note that this means that we do not address spatial discretization errors in this paper. Moreover, we also introduce the RB-reduced counterpart of the semi-discrete (in space) equation.

Let V_h be a H^1_D-conforming finite element space on Ω and I_h: $H^1_D \to V_h$ be an appropriate interpolation operator. We may introduce the semi-discrete (in space) counterpart of (Eq) as follows: Find $u_h \in W^{1,2}(I, V_h^*) \cap L^2(I, V_h)$ such that

$$\left.\begin{aligned} \langle \partial_t u_h(t), \varphi_h \rangle_{H_D^{-1}, H_D^1} + \langle \mathcal{A}(u_h(t)) u_h(t), \varphi_h \rangle_{H_D^{-1}, H_D^1} = \langle f(t), \varphi_h \rangle_{H_D^{-1}, H_D^1} \\ \forall t \in I, \varphi_h \in V_h, \\ u_h(0) = I_h u_0. \end{aligned}\right\} \tag{Eq$_h$}$$

Due to finite-dimensionality of V_h, (Eq$_h$) results in a system of ordinary differential equations (ODEs) for the coefficients of u_h w.r.t. some basis of V_h. This allows to discuss existence of solutions to (Eq$_h$). Hereby, we restrict ourselves to piecewise continuous (w.r.t. time) right-hand sides for simplicity. With regard to possible applications in PDE-constrained optimization, where f will be given by appropriate space- and time-discretization of a control function, this is not a severe restriction at all.

Proposition 2.2. *Assume that V_h is a subspace of $C(\overline{\Omega})$ and that the right-hand side fulfills $f \in C_{\text{pcw}}([0,T], H_D^{-1})$. Then, there exists a unique solution of (Eq$_h$) on the whole time interval $[0,T]$ with regularity*

$$u_h \in C(I, V_h) \cap C^1_{\text{pcw}}(I, V_h).$$

Here and in the following we denote by subscript "pcw" that the stated regularity holds piecewise w.r.t. a partition of the interval I. For the rest of the paper we will refer to u_h as the semi-discrete (in space) solution to (Eq), or, shorter, the truth-solution.

Proof. It suffices to argue for $f \in C(I, H_D^{-1})$. For the case of piecewise continuous f we argue similarly by glueing together the solutions obtained on each subinterval on which f is continuous. Let $(\varphi_i)_{i=1,\dots,N_h}$, $N_h = \dim V_h$, be a basis for V_h. Writing $u_h(t) = \sum_{i=1}^{N_h} \mathbf{x}_i(t)\varphi_i$, the coefficient vector $\mathbf{x}(t) \in \mathbb{R}^{N_h}$ fulfills

$$\mathbf{M}\partial_t \mathbf{x}(t) + \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t) = \mathbf{f}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{1}$$

with

$$\begin{aligned} \mathbf{M} &:= \left(\langle \varphi_i, \varphi_j \rangle_{L^2}\right)_{i,j}, \\ \mathbf{A}(\mathbf{z}) &:= \left(\int_\Omega \xi \left(\sum_{n=1}^{N_h} \mathbf{z}_n \varphi_n \right) \mu \nabla \varphi_i \nabla \varphi_j \, dx \right)_{i,j}, \\ \mathbf{f}(t) &:= \left(\langle f(t), \varphi_i \rangle_{H_D^{-1}, H_D^1} \right)_i, \end{aligned}$$

and $\mathbf{x}_0$ being the coefficient vector of $I_h u_0$. Since the basis functions are continuous and due to Lipschitz-continuity of ξ, it is clear that $\mathbf{A}(\cdot)$ is Lipschitz-continuous as map $\mathbb{R}^{N_h} \to \mathbb{R}^{N_h \times N_h}$. Further, since f was assumed to be continuous with respect to time, also $\mathbf{f}$ is continuous with respect to time. Now, the Picard-Lindelöf Theorem implies that there exists a unique C^1-solution $\mathbf{x}$: $[0,T] \to \mathbb{R}^{N_h}$ of (1). Hereby, global-in-time existence is ensured by uniform boundedness of $\|\mathbf{A}(\cdot)\|_{\mathbb{R}^{N_h \times N_h}}$, that follows from boundedness of ξ, and Gronwall's Lemma. □ □

Note that the embedding $V_h \hookrightarrow C(\overline{\Omega})$ is crucial in the previous argument: It ensures Lipschitz continuity of $\mathbf{A}(\cdot)$ and therefore existence of a solution to (1) via Picard-Lindelöf. If V_h is a classical Lagrange finite element space on a polygonal (polyhedral) domain Ω equipped with a triangular (tetrahedral) mesh, this assumption is obviously fulfilled. However, we would like to point out that except for the assumptions from Proposition 2.2 we do not rely on further details of spatial discretization. In general, V_h will have a rather large dimension which makes the numerical solution of a time-discretized counterpart of (Eq_h) expensive. This is in particular a problem in so-called many-query scenarios, i.e. in a context that requires to solve (Eq) for many different right-hand sides f. A possible way out of this problem is the so-called reduced basis-approach: The dynamical system of equation (Eq_h) is projected onto a smaller subspace of V_h that hopefully allows to express the characteristics of the system. In other words: We replace the finite element space V_h by a much smaller n-dimensional subspace $V_h^n \subset V_h$ that is related to the physical properties of the system. Such a reduced space might be obtained by the well-known POD approach [**30**, **47**] for instance. Although our arguments do not rely on the particular choice of V_h^n and therefore also cover general RB-methods, we clearly have in mind V_h^n's obtained by POD and also restrict our numerical experiments in Section 5 to this case. Having at hand a reduced ansatz-space $V_h^n \subset V_h$ and a suitable projection P_n: $V_h \to V_h^n$, we introduce the reduced-order counterpart of (Eq_h) as follows: Find $u_h^n \in W^{1,2}(I,(V_h^n)^*) \cap L^2(I, V_h^n)$ such that

$$(\mathrm{Eq}_h\text{-}\mathrm{RB}_n) \quad \left.\begin{aligned} \langle \partial_t u_h^n(t), \varphi_h^n\rangle_{H_D^{-1},H_D^1} + \langle \mathcal{A}(u_h^n(t))u_h^n(t), \varphi_h^n\rangle_{H_D^{-1},H_D^1} &= \langle f(t), \varphi_h^n\rangle_{H_D^{-1},H_D^1} \\ &\qquad \forall t \in I, \varphi_h^n \in V_h^n, \\ u_h^n(0) &= P_n I_h u_0. \end{aligned}\right\}$$

A reader familiar with ROM-techniques may already have noticed that the nonlinear term in (1) does not allow for efficient evaluation within the reduced-order model. We can overcome this issue by so-called hyperreduction techniques, e.g. the empirical interpolation method (EIM), which will be addressed in Section 4.

2.3. Outlook towards Optimal Control. In [**7**, Example 2.5] and [**26**] the following optimal control problem governed by (Eq) has been addressed:

$$(\mathrm{OCP}) \qquad \left.\begin{aligned} &\min_{u,q} J(u,q) := \frac{1}{2}\|u-u_d\|^2_{L^2(I\times\Omega)} + \frac{\gamma}{2}\|q\|^2_{L^2(I,\mathbb{R}^k)} \\ &\text{s.t.} \qquad q \in Q_{ad} := \left\{q \in L^2(I,\mathbb{R}^k)\colon q_a \le q \le q_b \text{ a.e. on } I\right\}, \\ &\qquad\qquad \text{and (Eq) with right-hand side } f = \sum_{i=1}^k q_i b_i. \end{aligned}\right\}$$

Hereby, $u_d \in L^2(I\times\Omega)$ denotes the desired state, $\gamma > 0$ is a Tikhonov-parameter, $b_i \in H_D^{-\zeta,p} \subset H_D^1$, $i = 1,\dots,k$, are some fixed spatial control functions, and $q_a, q_b \in L^\infty(I,\mathbb{R}^k)$ define box-constraints for the control. Fixing a space-discretization for (Eq) as described earlier in this section results in a semi-discrete (in space) counterpart (OCP_h) of (OCP), which we may consider again as reference object. In numerical algorithms for the solution of (OCP_h), we may have to evaluate the semi-discrete reduced functional $j(q) := J(u_h(q), q)$ where $u_h(q)$ denotes the solution of

(Eq_h) associated with several control functions q. Since repeated evaluation of j_h is costly, RB-MOR can be applied to (Eq_h). Therefore, and due to additional time-discretization, we only have the possibility to compute an approximate solution $u_h^{n,m} = u_h^{n,m}(q)$ instead of $u_h(q)$. A short computation shows that the resulting error in the reduced functional can be estimated as follows:

$$|J(u_h^{n,m}, q) - J(u_h, q)| \leq \left[\frac{1}{2}\|u_h^{n,m} - u_h\|_{L^2(I,L^2)} + \|u_h^{n,m} - u_d\|_{L^2(I,L^2)}\right] \|u_h^{n,m} - u_h\|_{L^2(I,L^2)}.$$

Consequently, we immediately obtain an a-posteriori error for the reduced functional of (OCP_h) if we have a $L^2(I, L^2)$-estimate for solutions of (Eq_h) at hand. This may be regarded as motivation for the results in the present paper. Note that the above estimate for the functional error differs from the estimates obtained in [**38**, Theorems 4 and 9] because we do not utilize adjoint information. Deriving a-posteriori reduced modeling errors for the adjoint equation of (OCP_h), and consequently for the gradient of the reduced functional, is beyond the scope of the present paper. We refer to [**38, 39**] for such estimates in case of different model problems.

3. A-posteriori RB-error-estimates

In this section we state and prove our first main results: A-posteriori error-estimates for (Eq_h) including both reduced-order and time-discretization errors; the technique of the proofs requires us to restrict ourselves to continuous-in-time trajectories. Moreover, for the reason of clarity we exclude hyperreduction for the nonlinearity at this point, and address this issue in the following section.

We roughly follow the ansatz of [**39**], where a semilinear equation with monotone nonlinearity has been discussed. To overcome the difficulties arising from the fact that our nonlinearity is not monotone we present two different approaches: The first approach relies on exploiting $L^\infty(I, W^{1,\infty})$-regularity of the truth-solution u_h and allows to obtain explicit estimates of "classical" structure in terms of the error in the initial condition and the V_h^*-residual of the discrete solution under consideration. As a semi-discrete in space solution, u_h obviously exhibits the required regularity for any fixed (spatial) discretization level. However, since the error-estimates will depend on the value of the $L^\infty(I, W^{1,\infty})$-norm of u_h it is desirable to have uniform bounds for this norm for all sufficiently fine spatial discretization levels. We believe that we can only expect such a uniform bound if the continuous in space and time solution of (Eq) exhibits $L^\infty(I, W^{1,\infty})$-regularity, which is guaranteed in the setting of [**8**]. However, in the less regular setting of [**7**] we cannot expect such a result. Therefore, the second approach is motivated by the intention to exploit less regularity of the truth-solution, more precisely: $L^\infty(I, W^{1,p})$-regularity for some $p > d$. For continuous in space and time solutions of (Eq) this regularity is guaranteed in the setting of [**7**]. The price to pay for exploiting less regularity of u_h is that we do not obtain an explicit formula for the error-estimate; instead the evaluation of the estimate requires the solution of a certain ODE. We start by fixing the following notation and assumptions:

Assumption 3.1. (1) Assume that $V_h \subset H_D^1 \cap C(\overline{\Omega})$ is an N_h-dimensional conforming finite element space, and V_h^n a n-dimensional subspace of V_h.

By $u_h \in C(I, V_h) \cap C^1_{\mathrm{pcw}}(I, V_h)$ we denote the truth-solution, i.e. the unique solution to (Eq_h).

(2) Moreover, let $u_h^n \in C(I, V_h^n) \cap C^1_{\mathrm{pcw}}(I, V_h^n)$ be arbitrary. By $e_h^n := u_h^n - u_h$ we denote the error with respect to the truth-solution.

We have in mind the following situation: u_h^n is the solution of a time-discrete counterpart of $(\mathrm{Eq}_h\text{-RB}_n)$, and we want to estimate how good u_h^n approximates the truth-solution u_h. Note that in order to ensure that u_h^n meets the regularity requirements of Assumption 3.1 we have to choose a time-discretization for $(\mathrm{Eq}_h\text{-RB}_n)$ that results in piecewise C^1-solutions, e.g. the Crank-Nicolson scheme in its CG1-DG0 Petrov-Galerkin form. Time-discrete solutions of $(\mathrm{Eq}_h\text{-RB}_n)$ obtained by Discontinuous Galerkin time-discretization, e.g. backward Euler, do not fulfill Assumption 3.1. Since discontinuous time-discretization might be of particular interest in the context of PDE-constrained optimization we outline an approach to overcome this restriction in Remark 4.4.

3.1. Some preliminary calculations. In this subsection we follow the residual-based ansatz of [**39**] as far as possible without modification, i.e. up to the point where strong monotonicity of the nonlinearity would be required. From that point on we develop two different approaches that will be discussed in the following subsections.

First, we introduce the residual of u_h^n by

$$r_h^n(t) := \partial_t u_h^n(t) + \mathcal{A}(u_h^n(t))u_h^n(t) - f(t) \in (V_h^n)^* \hookrightarrow H_D^{-1}, \quad t \in I. \tag{2}$$

A short computation utilizing (Eq_h) shows that

$$\langle r_h^n(t), \varphi_h \rangle_{H_D^{-1}, H_D^1} = \langle \partial_t e_h^n(t), \varphi_h \rangle_{H_D^{-1}, H_D^1} + \langle \mathcal{A}(u_h^n)u_h^n - \mathcal{A}(u_h)u_h, \varphi_h \rangle_{H_D^{-1}, H_D^1} \tag{3}$$

holds for all $\varphi_h \in V_h$. We consider V_h as a vector space canonically equipped with the H_D^1-norm. Therefore, its dual V_h^* is canonically equipped with the following norm:

$$\|\ell_h\|_{V_h^*} := \sup_{0 \neq \psi_h \in V_h} \frac{\langle \ell_h, \psi_h \rangle_{H_D^{-1}, H_D^1}}{\|\psi_h\|_{H_D^1}} = \sup_{0 \neq \psi_h \in V_h} \frac{\ell_h(\psi_h)}{\|\psi_h\|_{H_D^1}}. \tag{4}$$

Note that this norm is not equal to the H_D^{-1}-norm, because we only test with elements ψ_h from V_h in (4). For later use we state the following observation:

Lemma 3.2. *Let Assumption 3.1 hold. Then the function $[0,T] \to \mathbb{R}$, $t \mapsto \|r_h^n(t)\|^2_{V_h^*}$ is piecewise continuous.*

Proof. This follows from the definition of r_h^n and the fact that u_h^n is piecewise C^1. □ □

Plugging in $\varphi_h = e_h^n(t)$ for every fixed t in (3), and using the classical integration by parts formula from [**40**, Remark 7.5] we obtain

$$\frac{d}{dt}\frac{1}{2}\|e_h^n(t)\|^2_{L^2} + \langle \mathcal{A}(u_h^n)u_h^n - \mathcal{A}(u_h)u_h, e_h^n(t) \rangle_{H_D^{-1}, H_D^1} = \langle r_h^n(t), e_h^n(t) \rangle_{H_D^{-1}, H_D^1}. \tag{5}$$

Note that the second summand on the left-hand side of (5) causes problems in our case: If the nonlinearity $u \mapsto \mathcal{A}(u)u$ was strongly monotone, we could proceed as done in [**39**] for a semilinear term and estimate as follows:

$$\langle \mathcal{A}(u_h^n)u_h^n - \mathcal{A}(u_h)u_h, e_h^n(t) \rangle_{H_D^{-1}, H_D^1} \geq c|e_h^n(t)|^2_{H_D^1}.$$

However, as pointed out in Subsection 2.1 such an estimate cannot be expected to hold true. We cannot even bound the term under consideration from below by zero. Therefore, we have to proceed in a different way and split the problematic term into a coercive part and a remainder as follows:

$$
\begin{aligned}
&\langle \mathcal{A}(u_h^n)u_h^n - \mathcal{A}(u_h)u_h, u_h^n - u_h\rangle_{H_D^{-1},H_D^1} \\
&\qquad = \int_\Omega (\xi(u_h^n)\mu\nabla u_h^n - \xi(u_h)\mu\nabla u_h)\nabla(u_h^n - u_h)\mathrm{d}x \\
&\qquad = \int_\Omega \xi(u_h^n)\mu\nabla e_h^n\nabla e_h^n\mathrm{d}x + \int_\Omega (\xi(u_h^n) - \xi(u_h))\mu\nabla u_h\nabla e_h^n\mathrm{d}x \\
&\qquad \geq \xi_\bullet\mu_\bullet|e_h^n|_{H_D^1}^2 + \int_\Omega (\xi(u_h^n) - \xi(u_h))\mu\nabla u_h\nabla e_h^n\mathrm{d}x.
\end{aligned}
$$

Plugging this in into (5) yields

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\|e_h^n(t)\|_{L^2}^2 + \xi_\bullet\mu_\bullet|e_h^n(t)|_{H_D^1} \\
&\qquad \leq \langle r_h^n(t), e_h^n(t)\rangle_{H_D^{-1},H_D^1} - \int_\Omega (\xi(u_h^n(t)) - \xi(u_h(t)))\mu\nabla u_h(t)\nabla e_h^n(t)\mathrm{d}x,
\end{aligned}
\tag{6}
$$

i.e. except for the remainder term that we have shifted to the right-hand-side we have preserved a similar structure as in [**39**]. Formula (6) will serve as the common basis for our two different approaches in the following subsections. The main challenge in both cases is to estimate the second summand on the right-hand side in (6) in such a way that Gronwall's Lemma or a similar comparison principle can be applied to the resulting inequality.

3.2. A-posteriori estimates – Approach I. The approach of this subsection is closer to [**39**] than the second one, and relies on $L^\infty(I, W^{1,\infty})$-regularity of the truth-solution.

Theorem 3.3. *Let Assumptions 2.1 and 3.1 hold, and let $c_{\mathrm{Lip}} > 0$ be such that*

$$|u_h(t)|_{W^{1,\infty}} \leq c_{\mathrm{Lip}} \qquad \forall t \in I.$$

Moreover, let $\varepsilon, \eta > 0$ be chosen such that

$$\eta + \varepsilon|\xi'|_\infty\mu^\bullet c_{\mathrm{Lip}} = \xi_\bullet\mu_\bullet$$

and define $\beta := 2\left(\frac{1}{2\varepsilon}|\xi'|_\infty\mu^\bullet c_{\mathrm{Lip}} + \xi_\bullet\mu_\bullet\right)$. Then, the following a-posteriori error-estimates for u_h^n hold:

$$\|e_h^n(t)\|_{L^2}^2 \leq e^{\beta t}\|u_h^n(0) - u_h(0)\|_{L^2}^2 + \eta^{-1}\int_0^t e^{\beta(t-s)}\|r_h^n(s)\|_{V_h^*}^2\mathrm{d}s, \tag{7}$$

$$
\begin{aligned}
\|e_h^n\|_{L^2(I,L^2)}^2 \leq{}& \beta^{-1}\left(e^{\beta T} - 1\right)\|u_h^n(0) - u_h(0)\|_{L^2}^2 \\
&+ \eta^{-1}\beta^{-1}\int_0^T (e^{\beta(T-t)} - 1)\|r_h^n(t)\|_{V_h^*}^2\mathrm{d}t.
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\|e_h^n\|_{L^2(I,H_D^1)}^2 \leq{}& \xi_\bullet^{-1}\mu_\bullet^{-1}e^{\beta T}\|u_h^n(0) - u_h(0)\|_{L^2}^2 \\
&+ \xi_\bullet^{-1}\mu_\bullet^{-1}\eta^{-1}\int_0^T e^{\beta(T-t)}\|r_h^n(t)\|_{V_h^*}^2\mathrm{d}t.
\end{aligned}
\tag{9}
$$

Proof. We proceed with the argument from the previous subsection. Starting with the estimate (6) we bound the remaining term of the nonlinearity in the following way:

$$\left|\int_\Omega (\xi(u_h^n(t)) - \xi(u_h(t)))\mu\nabla u_h(t)\nabla e_h^n(t)\mathrm{d}x\right| \le |\xi'|_\infty \mu^\bullet c_{\mathrm{Lip}} \|e_h^n\|_{L^2}\|e_h^n\|_{H_D^1}. \tag{10}$$

Using $W^{1,\infty}$-regularity for u_h we can estimate one of the e_h^n-factors in the L^2-norm, which would not be possible assuming only $W^{1,p}$-regularity for u_h with some finite p. With help of Young's inequality we arrive at

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|e_h^n(t)\|_{L^2}^2 + \xi_\bullet\mu_\bullet\|e_h^n(t)\|_{H_D^1}^2 \le \xi_\bullet\mu_\bullet\|e_h^n(t)\|_{L^2}^2 + \langle r_h^n(t), e_h^n(t)\rangle_{H_D^{-1},H_D^1}&\\ + |\xi'|_\infty\mu^\bullet c_{\mathrm{Lip}}\left(\frac{1}{2\varepsilon}\|e_h^n\|_{L^2} + \frac{\varepsilon}{2}\|e_h^n\|_{H_D^1}\right)&\end{aligned}$$

with some $\varepsilon > 0$. Another application of Young's inequality yields

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|e_h^n(t)\|_{L^2}^2 + \xi_\bullet\mu_\bullet\|e_h^n(t)\|_{H_D^1}^2 \le \left(\frac{1}{2\varepsilon}|\xi'|_\infty\mu^\bullet c_{\mathrm{Lip}} + \xi_\bullet\mu_\bullet\right)\|e_h^n(t)\|_{L^2}^2 + \frac{1}{2\eta}\|r_h^n(t)\|_{V_h^*}^2&\\ + \left(\frac{\eta}{2} + \frac{\varepsilon}{2}|\xi'|_\infty\mu^\bullet c_{\mathrm{Lip}}\right)\|e_h^n\|_{H_D^1}^2&\end{aligned}$$

with $\eta > 0$. Now, choose η, ε as in the statement of the theorem and obtain

$$\frac{d}{dt}\frac{1}{2}\|e_h^n(t)\|_{L^2}^2 + \frac{1}{2}\xi_\bullet\mu_\bullet\|e_h^n(t)\|_{H_D^1}^2 \le \beta\cdot\frac{1}{2}\|e_h^n(t)\|_{L^2}^2 + \frac{1}{2\eta}\|r_h^n(t)\|_{V_h^*}^2, \tag{11}$$

where we will use from now on the abbreviation $\beta = 2\left(\frac{1}{2\varepsilon}|\xi'|_\infty\mu^\bullet c_{\mathrm{Lip}} + \xi_\bullet\mu_\bullet\right)$ to enhance readability. With help of Gronwall's Lemma [**12**, Corollary 2] we obtain an a-posteriori estimate for the $L^\infty(I, L^2)$-error from this:

$$\|e_h^n(t)\|_{L^2}^2 \le \|P_n I_h u_0 - I_h u_0\|_{L^2}^2 e^{\beta t} + \eta^{-1}\int_0^t \|r_h^n(s)\|_{V_h^*}^2 e^{\beta(t-s)}\mathrm{d}s.$$

The second summand thereof is integrated using integration by parts, i.e.

$$\int_0^T e^{\beta t}\left(\int_0^t e^{-\beta s}\|r_h^n(s)\|_{V_h^*}^2\mathrm{d}s\right)\mathrm{d}t = \beta^{-1}\int_0^T (e^{\beta(T-t)} - 1)\|r_h^n(t)\|_{V_h^*}^2\mathrm{d}t,$$

and together with the first summand we obtain the $L^2(I, L^2)$-estimate (21). As in [**39**] the $L^2(I, H^1)$-estimate (22) is obtained from (11) by integrating with respect to time over $I = [0, T]$ and using (21). □ □

In order to exploit less regularity of u_h in (10) one might be tempted to estimate in $\xi(u_h^n(t)) - \xi(u_h(t))$ in L^q with some $q > 2$ such that $H_D^1 \hookrightarrow L^q$. However, we would not be able to shift the resulting $\|e_h^n(t)\|_{H_D^1}^2$-term to the right-hand side of (11) via Young's inequality. Hence, application of Gronwall's Lemma would not be possible.

3.3. A-posteriori estimates – Approach II.

We derive a-posteriori error-estimates that rely on $L^\infty(I, W_D^{1,p})$-regularity for the truth-solution u_h with some $p > d$, only. We start with the following auxiliary result:

Lemma 3.4. *Let Assumptions 2.1 and 3.1 hold, and let $\varepsilon, \eta > 0$ satisfy*

$$\xi_\bullet\mu_\bullet = \eta + \varepsilon\cdot\mu^\bullet(2\xi^\bullet)^{1-\frac{2}{q}}|\xi'|_\infty^{\frac{2}{q}} c_p.$$

The error-function $t \mapsto \|e_h^n(t)\|_{L^2}^2$ satisfies the differential inequality

$$\varphi'(t) \le \alpha\varphi(t) + \beta\varphi(t)^r + \gamma(t), \qquad t \in [0,T], \tag{12}$$

with the constants $\alpha = 2\xi_\bullet\mu_\bullet$, $\beta = \varepsilon^{-1}\mu^\bullet(2\xi^\bullet)^{1-\frac{2}{q}}|\xi'|_\infty^{\frac{2}{q}}c_p$, the function $\gamma = \gamma(t) = \eta^{-1}\|r_h^n(t)\|_{V_h^*}^2$, and the exponent $r = \frac{2}{q} = 1 - \frac{2}{p} \in (0,1)$.

We would like to mention the following interesting analogy: Boundedness of ξ is essential in our argument below, as it already was for the discussion of the equation on the continuous level, cf. [**7**, **37**].

Proof. We pick up the argument from Subsection 3.1. We start by estimating the second summand on the right-hand side of (6) as follows:

$$\left|\int_\Omega (\xi(u_h^n(t)) - \xi(u_h(t)))\mu\nabla u_h(t)\nabla e_h^n(t)\mathrm{d}x\right| \\ \le \mu^\bullet\|\xi(u_h^n(t)) - \xi(u_h(t))\|_{L^q}|u_h(t)|_{W^{1,p}}|e_h^n(t)|_{H^1}$$

with $p^{-1} + q^{-1} + \frac{1}{2} = 1$, i.e. $q = \frac{2p}{p-2}$. Note that we define the $W^{1,p}$-semi-norm as follows:

$$|\varphi|_{W^{1,p}}^p := \int_\Omega |\nabla\varphi|_2^p dx = \int_\Omega \left(\sum_{i=1}^d \left(\frac{\partial\varphi}{\partial x_i}\right)^2\right)^{p/2} \mathrm{d}x.$$

Next, we apply the well-known Riesz-Thorin interpolation-inequality

$$\|f\|_{L^q} \le \|f\|_{L^\infty}^{1-\frac{2}{q}}\|f\|_{L^2}^{\frac{2}{q}}, \qquad f \in L^\infty, q \in (2,\infty),$$

to the Lipschitz estimate $\|\xi(u_h^n(t)) - \xi(u_h(t))\|_{L^2} \le |\xi'|_\infty\|u_h^n(t) - u_h(t)\|_{L^2}$ and arrive at

$$\left|\int_\Omega (\xi(u_h^n(t)) - \xi(u_h(t)))\mu\nabla u_h(t)\nabla e_h^n(t)\mathrm{d}x\right| \\ \le \mu^\bullet(2\xi^\bullet)^{1-\frac{2}{q}}|\xi'|_\infty^{\frac{2}{q}}|u_h(t)|_{W^{1,p}}\|e_h^n(t)\|_{L^2}^{\frac{2}{q}}|e_h^n(t)|_{H^1}. \tag{13}$$

Note that this is the point where uniform boundedness of the nonlinearity ξ enters. As before, we will estimate the product of the last two factors by Young's inequality and move the H^1-semi-norm term to the left-hand side of (6) in order to obtain an ODE for the L^2-error. This is why we are not able to convert the $\|e_h^n(t)\|_{L^2}^{2/q}$-term to an $\|e_h^n(t)\|_{L^2}^2$-term by application of Young's inequality, because we need to generate an $|e_h^n(t)|_{H^1}^2$-term from the second factor, such that this term can be canceled by the left-hand side of (6).

Plugging (13) into (6) and using Young's inequality twice we obtain the following:

$$\frac{d}{dt}\frac{1}{2}\|e_h^n(t)\|_{L^2}^2 + \xi_\bullet\mu_\bullet\|e_h^n(t)\|_{H_D^1} \le \frac{\eta}{2}\|e_h^n(t)\|_{H_D^1}^2 + \frac{1}{2\eta}\|r_h^n(t)\|_{V_h^*}^2 + \xi_\bullet\mu_\bullet\|e_h^n(t)\|_{L^2}^2 \\ + \mu^\bullet(2\xi^\bullet)^{1-\frac{2}{q}}|\xi'|_\infty^{\frac{2}{q}}|u_h(t)|_{W^{1,p}} \cdot \left(\frac{\varepsilon}{2}\|e_h^n(t)\|_{H^1}^2 + \frac{1}{2\varepsilon}\|e_h^n(t)\|_{L^2}^{\frac{4}{q}}\right).$$

Here, $\varepsilon, \eta > 0$ are the parameters appearing in Young's inequality. Choosing them as in the statement of the lemma yields

$$\begin{aligned}(14)\quad &\frac{d}{dt}\|e_h^n(t)\|_{L^2}^2 + \xi_\bullet\mu_\bullet\|e_h^n(t)\|_{H^1}^2 \\ &\le \eta^{-1}\|r_h^n(t)\|_{V_h^*}^2 + 2\xi_\bullet\mu_\bullet\|e_h^n(t)\|_{L^2}^2 + \varepsilon^{-1}\mu^\bullet(2\xi^\bullet)^{1-\frac{2}{q}}|\xi'|_\infty^{\frac{2}{q}}c_p\|e_h^n(t)\|_{L^2}^{4/q},\end{aligned}$$

from which the claim follows. □ □

Let us briefly comment on the rather challenging structure of (12): First, note that $r \in (0,1)$, i.e. the right-hand side in (12) does only depend Lipschitz-continuous on $\varphi(t)$, if $\varphi(t)$ stays uniformly away from zero, which can be ensured only for $\varphi(0) > 0$. Moreover, the Lipschitz-constant on sets bounded uniformly away from zero increases, if r gets smaller. The latter, however, is the case if $p > d$ gets smaller, i.e. if we exploit less regularity of the truth-solution. In other words: The smaller the initial error, and the less regularity of the truth-solution we use, the more ill-posed (12) becomes.

Theorem 3.5. *Let Assumptions 2.1 and 3.1 hold, and let $p > d$ and $c_p > 0$ such that*

$$|u_h(t)|_{W^{1,p}} \le c_p \qquad \forall t \in I.$$

Moreover, we assume that the initial error does not vanish, i.e. $\|e_h^n(0)\|_{L^2} > 0$. Let $\varepsilon, \eta > 0$ be chosen such that

$$\xi_\bullet\mu_\bullet = \eta + \varepsilon \cdot \mu^\bullet(2\xi^\bullet)^{1-\frac{2}{q}}|\xi'|_\infty^{\frac{2}{q}}c_p$$

holds. Given the constants $\alpha = 2\xi_\bullet\mu_\bullet$, $\beta = \varepsilon^{-1}\mu^\bullet(2\xi^\bullet)^{1-\frac{2}{q}}|\xi'|_\infty^{\frac{2}{q}}c_p$, and $r = 1-\frac{2}{p}$, let $\varphi\colon [0,T] \to [0,\infty)$ be the solution to

$$\begin{aligned}\varphi'(t) &= \alpha\varphi(t) + \beta\varphi(t)^r + \eta^{-1}\|r_h^n(t)\|_{V_h^*}^2, \qquad t \in I,\\ \varphi(0) &= \|u_h^n(0) - u_h(0)\|_{L^2}^2.\end{aligned}$$

Then the following a-posteriori error-estimates hold true:

$$(15)\qquad \|e_h^n(t)\|_{L^2}^2 \le \varphi(t), \quad \forall t \in I, \qquad \|e_h^n\|_{L^2(I,L^2)}^2 \le \int_0^T \varphi(s)\mathrm{d}s,$$

$$\begin{aligned}(16)\qquad \|e_h^n\|_{L^2(I,H_D^1)}^2 \le \frac{1}{\xi_\bullet\mu_\bullet}\Big(&\|u_h^n(0) - u_h(0)\|_{L^2}^2 + \eta^{-1}\|r_h^n\|_{L^2(I,V_h^*)}^2 \\ &\alpha\int_0^T \varphi(s)ds + \beta\int_0^T \varphi(s)^{2/q}\mathrm{d}s\Big).\end{aligned}$$

Proof. In order to apply [**12**, Theorem 44] to Lemma 3.4 we have to verify that $f(t,z) = \alpha z + \beta z^r + \eta^{-1}\|r_h^n(t)\|_{V_h^*}^2$ satisfies the required assumption, i.e. that given $\varphi_0 > 0$ there is $\varepsilon > 0$ such that the ODE

$$\varphi'(t) = f(t,\varphi(t)), \qquad \varphi(0) = \varphi_0 + \delta,$$

has a solution on the time interval $I = [0,T]$ as long as $\delta \in [0,\varepsilon]$. Existence of a local solution on some time interval $[0,T_{\max})$ with $T_{\max} \in (0,T]$ is clear due to Peano's existence-theorem, since $t \mapsto \|r_h^n(t)\|_{V_h^*}^2$ is piecewise continuous on $[0,T]$ (see Lemma 3.2) and $u \mapsto \alpha u + \beta u^r$ even admits a continuous extension $\mathbb{R} \to \mathbb{R}$, $u \mapsto \alpha u + \beta \mathrm{sign}(u)|u|^r$. Further, due to $\alpha > 0, \beta > 0$, and $\|r_h^n(t)\|_{V_h^*}^2 \ge 0$ it is clear that φ is monotone increasing for $\varphi_0 > 0$. The theory of ODEs shows that either $T_{\max} = T$ or $\varphi(t) \to \infty$ as $t \to T_{\max}$. In the latter case there has to be $t_0 \in (0,T)$ such that $\varphi(t) > 1$ for $t \geq t_0$. For those $t \geq t_0$ it holds due to $r \in (0,1)$ that

$$\varphi'(t) = \alpha\varphi(t) + \beta\varphi(t)^r + \eta^{-1}\|r_h^n(t)\|_{V_h^*}^2 \leq (\alpha+\beta)\varphi(t) + \eta^{-1}\|r_h^n(t)\|_{V_h^*}^2.$$

By Gronwall's Lemma [**12**, Corollary 2] we conclude that $\varphi(t)$ stays bounded on $[0,T]$, which contradicts the assumption $T_{\max} < T$. Therefore, all solutions φ have to exist on the whole time interval $[0,T]$. Thus, we have shown the estimate for the $L^\infty(I,L^2)$-error, from which we immediately obtain the $L^2(I,L^2)$-error by integration. Following again [**39**] we integrate (12) to obtain (16). □ □

An explicit comparison principle for (12) à la [**12**, Corollary 2] would allow to obtain also explicit formulas in the estimates of Theorem 3.5. Unfortunately, we only found such results in the literature for the special cases $\gamma \equiv 0$ or $\alpha = 0$ [**12**, Theorems 21 and 23], that are not of interest in the present context.

4. A-posteriori RB- and EIM-error-estimates

It is a well-known issue in RB-methods that the evaluation of nonlinear terms such as $\xi(u)$ requires access to the full number of degrees of freedom. Since the reasoning behind MOR is to avoid such computations within the full model, alternatives have to be found. In order to allow for an efficient offline-online splitting, the evaluation of nonlinearities in the reduced-order model for (Eq) needs to be done by methods of hyperreduction, e.g. the Empirical Interpolation Method (EIM, [**6**]). In this section we describe a very basic version of the latter technique applied to our model problem, and show how the additional errors can be incorporated in the a-posteriori error-estimates of Theorems 3.3 and 3.5 using the same technique as in [**24, 25**].

4.1. Empirical Interpolation of $\mathcal{A}$. First, we introduce EIM as far as required for our purpose and as concise as possible. In order to present the main idea as clearly as possible, we stick to the continuous setting and omit space-discretization; the generalization to finite element spaces with a nodal basis is straightforward. Given so-called snapshots $y_1, ..., y_N \in C(\Omega)$, and a tolerance $\mathrm{tol}_{\mathrm{EIM}} > 0$, determine via a Greedy procedure (for details, see e.g. [**6, 47**]) some functions $\Xi_1, ..., \Xi_m \in C(\Omega)$, and interpolation points $x_1, ..., x_m \in \Omega$ such that

$$\xi(y_\ell(x_j)) = \sum_{k=1}^m c_{\ell,k}\Xi_k(x_j), \qquad \ell = 1, ..., N, \quad j = 1, ..., m.$$

implies $\left\|\xi(y_\ell) - \sum_{k=1}^m c_{\ell,k}\Xi_k\right\|_{L^\infty} \leq \mathrm{tol}_{\mathrm{EIM}}$. For some $w \in C(\Omega)$ we define the EIM-approximation of $\xi(w)$ as

$$\xi_m^{\mathrm{EIM}}(w) = \sum_{k=1}^m c_k \Xi_k$$

where $c \in \mathbb{R}^m$ solves the $m \times m$-system $\xi(w(x_j)) = \sum_{k=1}^m c_k \Xi_k(x_j)$, $j = 1, ..., m$. With this we may introduce a RB-EIM-reduced counterpart of (Eq) as

$$
(\mathrm{Eq}_h\text{-}\mathrm{RB}_n\text{-}\mathrm{EIM}_m)\quad
\left.\begin{aligned}
\langle \partial_t u_h^{n,m}(t), \varphi_h^n \rangle_{H_D^{-1},H_D^1} + \langle \mathcal{A}_m^{\mathrm{EIM}}(u_h^{n,m}(t)) u_h^{n,m}(t), \varphi_h^n \rangle_{H_D^{-1},H_D^1} &= \langle f(t), \varphi_h^n \rangle_{H_D^{-1},H_D^1} \\
&\qquad \forall t \in I, \varphi_h^n \in V_h^n, \\
u_h^{n,m}(0) &= P_n I_h u_0,
\end{aligned}\right\}
$$

where $\mathcal{A}_m^{\mathrm{EIM}}$ denotes the EIM-reduced version of the nonlinear differential operator defined by

$$
\langle \mathcal{A}_m^{\mathrm{EIM}}(u)\varphi, \psi \rangle_{H_D^{-1},H_D^1} := \int_\Omega \xi_m^{\mathrm{EIM}}(u) \mu \nabla u \nabla \varphi \mathrm{d}x, \qquad \varphi, \psi \in H_D^1.
$$

Note that there is an efficient online evaluation of $\mathcal{A}_m^{\mathrm{EIM}}$, because the stiffness-matrices associated to the operators $-\nabla \cdot \Xi_k \mu \nabla$ can be precomputed in the offline-phase. Therefore, we only have to deal with m and n degrees of freedom, respectively, when dealing with $\mathcal{A}_m^{\mathrm{EIM}}$. In the following we will denote the EIM-error by

$$
\triangle_m^{\mathrm{EIM}}(u) := \|\xi(u) - \xi_m^{\mathrm{EIM}}(u)\|_{L^\infty}.
$$

For sophisticated algorithmic coupling of model order reduction and hyperreduction we refer to [**13, 44**] for instance. Other kinds of hyperreduction include e.g. discrete empirical interpolation (DEIM, [**11**]), or dynamic mode decomposition (DMD, [**2**]).

4.2. A-posteriori RB-error-estimates including the EIM-error. In this section we extend the results from Section 3 by incorporating also EIM-errors: It is clear that $\mathcal{A}(u_h^n)u_h^n$, and therefore r_h^n, cannot be computed efficiently during the online-phase due to the fact that the assembly of the stiffness-matrix for $\mathcal{A}(u_h^n)$ requires us to use the full number of degrees of freedom. Hence, the estimates of Theorems 3.3 and 3.5 cannot be evaluated efficiently in the online-phase. Consequently, evaluation of r_h^n has to be avoided. Instead, given an arbitrary $u_h^{n,m}$ fulfilling Assumption 3.1, we introduce the EIM-reduced residual $r_h^{n,m}$ of $u_h^{n,m}$ as

$$
r_h^{n,m}(t) := \partial_t u_h^{n,m}(t) + \mathcal{A}_m^{\mathrm{EIM}}(u_h^{n,m}(t)) u_h^{n,m}(t) - f(t) \in V_h^* \hookrightarrow H_D^{-1}, \qquad t \in I. \tag{17}
$$

It is obvious, that $r_h^{n,m}$ admits an efficient online evaluation. It remains to show how the error $u_h^{n,m} - u_h$ to the truth-solution can be estimated in terms of $r_h^{n,m}$ instead of r_h^n. Since all changes in the arguments already known from Section 3 are straightforward utilizing the estimates (18) and (19) below, we omit the details and only state the results. A short computation as in Section 3.1 shows that the RB-EIM-error $e_h^{n,m} := u_h^{n,m} - u_h$ fulfills

$$
\begin{aligned}
\langle r_h^{n,m}(t), \varphi_h \rangle_{H_D^{-1},H_D^1} = \langle \partial_t e_h^{n,m}(t), \varphi_h \rangle_{H_D^{-1},H_D^1} &+ \langle \mathcal{A}(u_h^{n,m}) u_h^{n,m} - \mathcal{A}(u_h) u_h, \varphi_h \rangle_{H_D^{-1},H_D^1} \\
&+ \langle \mathcal{A}_m^{\mathrm{EIM}}(u_h^{n,m}) u_h^{n,m} - \mathcal{A}(u_h^{n,m}) u_h^{n,m}, \varphi_h \rangle_{H_D^{-1},H_D^1}.
\end{aligned}
$$

As before it follows:

$$
\begin{aligned}
(18)\quad \frac{d}{dt}\frac{1}{2}\|e_h^{n,m}(t)\|_{L^2}^2 + \xi_\bullet\mu_\bullet|e_h^{n,m}(t)|_{H^1}^2 &\le \langle r_h^{n,m}(t), e_h^{n,m}(t)\rangle_{H_D^{-1},H_D^1} \\
&\quad - \int_\Omega (\xi(u_h^{n,m}) - \xi(u_h))\mu\nabla u_h\nabla e_h^{n,m}\mathrm{d}x \\
&\quad - \int_\Omega (\xi^{\mathrm{EIM}}(u_h^{n,m}) - \xi(u_h^{n,m}))\mu\nabla u_h^{n,m}\nabla e_h^{n,m}\mathrm{d}x.
\end{aligned}
$$

The second summand on the right-hand side can be estimated as in Subsections 3.2 and 3.3. The third summand is estimated as follows:

(19)

$$
\begin{aligned}
\left|\int_\Omega (\xi^{\mathrm{EIM}}(u_h^{n,m}) - \xi(u_h^{n,m}))\mu\nabla u_h^{n,m}\nabla e_h^{n,m}\mathrm{d}x\right| &\le \Delta_m^{\mathrm{EIM}}(u_h^{n,m})\mu^\bullet|u_h^{n,m}|_{H^1}\|e_h^{n,m}\|_{H^1} \\
&\le \frac{1}{2\delta}\Delta_m^{\mathrm{EIM}}(u_h^{n,m})\mu^\bullet|u_h^{n,m}|_{H^1}^2 + \frac{1}{2}\delta\Delta_m^{\mathrm{EIM}}(u_h^{n,m})\mu^\bullet\|e_h^{n,m}\|_{H^1}^2,
\end{aligned}
$$

where $\delta > 0$ is the parameter in Young's inequality. With this, we are ready to state the modified versions of the two main results from Section 3, beginning with the modified version of Theorem 3.3:

Theorem 4.1. *Let Assumptions 2.1 and 3.1 hold, and let $c_{\mathrm{Lip}} > 0$ such that*

$$
|u_h(t)|_{W^{1,\infty}} \le c_{\mathrm{Lip}} \qquad \forall t \in I.
$$

Given $u_h^{n,m}$, choose $\varepsilon, \eta, \delta > 0$ such that

$$
\eta + \varepsilon|\xi'|_\infty\mu^\bullet c_{\mathrm{Lip}} + \delta\Delta_m^{\mathrm{EIM}}\mu^\bullet = \xi_\bullet\mu_\bullet,
$$

is satisfied with the EIM-error $\Delta_m^{\mathrm{EIM}} := \sup_{t\in I}\Delta_m^{\mathrm{EIM}}(u_h^{n,m}(t))$. Moreover, we introduce the constant $\beta := 2\left(\frac{1}{2\varepsilon}|\xi'|_\infty\mu^\bullet c_{\mathrm{Lip}} + \xi_\bullet\mu_\bullet\right)$. Then the following a-posteriori error-estimates for $u_h^{n,m}$ hold true:

(20)

$$
\begin{aligned}
\|e_h^{n,m}(t)\|_{L^2}^2 &\le e^{\beta t}\|u_h^{n,m}(0) - u_h(0)\|_{L^2}^2 \\
&\quad + \int_0^t e^{\beta(t-s)}\left(\eta^{-1}\|r_h^{n,m}(s)\|_{V_h^*}^2 + \delta^{-1}\Delta_m^{\mathrm{EIM}}\mu^\bullet|u_h^{n,m}(s)|_{H^1}^2\right)\mathrm{d}s,
\end{aligned}
$$

(21)

$$
\begin{aligned}
\|e_h^{n,m}\|_{L^2(I,L^2)}^2 &\le \beta^{-1}\left(e^{\beta T} - 1\right)\|u_h^{n,m}(0) - u_h(0)\|_{L^2}^2 \\
&\quad + \beta^{-1}\int_0^T (e^{\beta(T-t)} - 1)\left(\eta^{-1}\|r_h^{n,m}(t)\|_{V_h^*}^2 + \delta^{-1}\Delta_m^{\mathrm{EIM}}\mu^\bullet|u_h^{n,m}(t)|_{H^1}^2\right)\mathrm{d}t.
\end{aligned}
$$

(22)

$$
\begin{aligned}
\|e_h^{n,m}\|_{L^2(I,H_D^1)}^2 &\le \xi_\bullet^{-1}\mu_\bullet^{-1}e^{\beta T}\|u_h^{n,m}(0) - u_h(0)\|_{L^2}^2 \\
&\quad + \xi_\bullet^{-1}\mu_\bullet^{-1}\int_0^T e^{\beta(T-t)}\left(\eta^{-1}\|r_h^{n,m}(t)\|_{V_h^*}^2 + \delta^{-1}\Delta_m^{\mathrm{EIM}}\mu^\bullet|u_h^{n,m}(t)|_{H^1}^2\right)\mathrm{d}t.
\end{aligned}
$$

We also fix the following simplified estimates, that are less sharp but exhibit a favorable structure: They are weighted sums of the initial L^2-error, the L^2-V_h^*-norm of the residual, and the EIM-error. This allows to determine the optimal choice of the parameters $\varepsilon, \eta, \delta$ for these simpler estimates.

Corollary 4.2. *Under the assumptions of the previous theorem it holds:*

$$\|e_h^{n,m}(t)\|_{L^2}^2 \le e^{\beta t}\|u_h^{n,m}(0)-u_h(0)\|_{L^2}^2 + e^{\beta t}\eta^{-1}\int_0^t \|r_h^{n,m}(s)\|_{V_h^*}^2 \mathrm{d}s + \delta^{-1}e^{\beta t}\Delta_m^{\mathrm{EIM}}\mu^\bullet \int_0^t |u_h^{n,m}(s)|_{H^1}^2 \mathrm{d}s,$$

$$\|e_h^{n,m}\|_{L^2(I,L^2)}^2 \le \beta^{-1}\left(e^{\beta T}-1\right)\|u_h^{n,m}(0)-u_h(0)\|_{L^2}^2 + \beta^{-1}(e^{\beta T}-1)\left(\eta^{-1}\|r_h^{n,m}\|_{L^2(I,V_h^*)}^2 + \delta^{-1}\Delta_m^{\mathrm{EIM}}\mu^\bullet|u_h^{n,m}|_{L^2(I,H^1)}^2\right)$$

$$\|e_h^{n,m}\|_{L^2(I,H_D^1)}^2 \le e^{\beta T}\xi_\bullet^{-1}\mu_\bullet^{-1}\|u_h^{n,m}(0)-u_h(0)\|_{L^2}^2 + e^{\beta T}\xi_\bullet^{-1}\mu_\bullet^{-1}\left(\eta^{-1}\|r_h^{n,m}\|_{L^2(I,V_h^*)}^2 + \delta^{-1}\Delta_m^{\mathrm{EIM}}\mu^\bullet|u_h^{n,m}|_{L^2(I,H^1)}^2\right)$$

The same technique as in Section 3.3 yields the following result:

Theorem 4.3. *Let Assumptions 2.1 and 3.1 hold, and let $p > d$ and $c_p > 0$ such that*

$$|u_h(t)|_{W^{1,p}} \le c_p \qquad \forall t \in I.$$

Moreover, we assume that the initial error does not vanish, i.e. $\|e_h^{n,m}(0)\|_{L^2} > 0$. Choose $\varepsilon, \eta, \delta > 0$ such that

$$\xi_\bullet\mu_\bullet = \eta + \varepsilon\cdot\mu^\bullet(2\xi^\bullet)^{1-\frac{2}{q}}|\xi'|_\infty^{\frac{2}{q}}c_p + \delta\Delta_m^{\mathrm{EIM}}\mu^\bullet$$

is satisfied for the EIM-error $\Delta_m^{\mathrm{EIM}} = \sup_{t\in I}\Delta_M^{\mathrm{EIM}}(u_h^{n,m}(t))$. Given the constants $\alpha = 2\xi_\bullet\mu_\bullet$, $\beta = \varepsilon^{-1}\mu^\bullet(2\xi^\bullet)^{1-\frac{2}{q}}|\xi'|_\infty^{\frac{2}{q}}c_p$, and $r = 1-\frac{2}{p}$, let $\varphi\colon [0,T] \to [0,\infty)$ be the solution to

$$\varphi'(t) = \alpha\varphi(t) + \beta\varphi(t)^r + \eta^{-1}\|r_h^{n,m}(t)\|_{V_h^*}^2 + \delta^{-1}\Delta_m^{\mathrm{EIM}}\mu^\bullet|u_h^{n,m}(t)|_{H^1}^2, \qquad t \in I,$$
$$\varphi(0) = \|e_h^{n,m}(0)\|_{L^2}^2.$$

Then the following a-posteriori error-estimates hold true:

(23)
$$\|e_h^{n,m}(t)\|_{L^2}^2 \le \varphi(t), \quad \forall t \in I, \qquad \|e_h^{n,m}\|_{L^2(I,L^2)}^2 \le \int_0^T \varphi(s)\mathrm{d}s,$$

(24)
$$\|e_h^{n,m}\|_{L^2(I,H_D^1)}^2 \le \frac{1}{\xi_\bullet\mu_\bullet}\Big(\|u_h^{n,m}(0)-u_h(0)\|_{L^2}^2 + \eta^{-1}\|r_h^n\|_{L^2(I,V_h^*)}^2 + \delta^{-1}\Delta_m^{\mathrm{EIM}}\mu^\bullet|u_h^{n,m}|_{L^2(I,H^1)}^2 + \alpha\int_0^T\varphi(s)\mathrm{d}s + \beta\int_0^T\varphi(s)^{2/q}\mathrm{d}s\Big).$$

Let us point out that the EIM-error $\Delta_m^{\mathrm{EIM}}(u_h^{n,m})$ at $u_h^{n,m}$ cannot be computed without referring to the full number of degrees of freedom; however, computation of $\|\xi(u_h^{n,m}) - \xi_m^{\mathrm{EIM}}(u_h^{n,m})\|_{L^\infty}$ in the full degrees of freedom is still much cheaper than computation of the respective full stiffness-matrices associated with the nonlinear elliptic operator that would be required for the computation of r_h^n. In contrast, note that the H^1-semi-norm of $u_h^{n,m}$ required in Theorems 4.1 and 4.3 admits efficient online evaluation, because it is induced by a bilinear form whose matrix w.r.t. the basis of V_h^n can be precomputed and saved. Similarly, also the weight-matrices for the evaluation of the EIM-reduced residual can be precomputed and saved in the offline-phase.

To conclude this section, we shortly outline a possibility to relax Assumption 3.1 (2) in order to allow error estimation also for a discontinuous in time trajectory.

Remark 4.4. Let $u_h^{n,m}$ e.g. be given as

$$u_h^{n,m} := \mathbf{1}_{\{0\}} U_{h,0}^{n,m} + \sum_{\ell=1}^{N_t} \mathbf{1}_{(t_{\ell-1},t_\ell]} U_{h,\ell}^{n,m}, \qquad U_{h,\ell}^{n,m} \in V_h^n, \quad \ell = 0, ..., N_t,$$

for a partition $0 = t_0 < t_1 < ... < t_{N_t-1} < t_{N_t} = T$. Such $u_h^{n,m}$ might be obtained by applying the backward Euler method in its DG0-formulation to (Eq_h-RB_n-EIM_m). Since our error-estimates do not apply directly to $u_h^{n,m}$ due to discontinuity w.r.t. time, we replace $u_h^{n,m}$ by its piecewise linear and continuous w.r.t time interpolation $\hat{u}_h^{n,m}$ w.r.t. the same partition defined by $\hat{u}_h^{n,m}(t_\ell) := u_h^{n,m}(t_\ell) = U_{h,\ell}^{n,m}$ for $\ell = 0, ..., N_t$. Obviously, Theorems 4.1 and 4.3 apply to $\hat{u}_h^{n,m}$, and to obtain an estimate for the overall error we need to add the interpolation error $\hat{u}_h^{n,m} - u_h^{n,m}$. The latter can be computed explicitely:

$$\|u_h^{n,m} - \hat{u}_h^{n,m}\|^2_{L^\infty(I,L^2)} \le \max_{1\le \ell \le N_t} \|U_{h,\ell}^{n,m} - U_{h,\ell-1}^{n,m}\|^2_{L^2},$$

$$\|u_h^{n,m} - \hat{u}_h^{n,m}\|^2_{L^2(I,L^2)} \le \sum_{\ell=1}^{N_t} \frac{1}{3}(t_\ell - t_{\ell-1}) \|U_{h,\ell}^{n,m} - U_{h,\ell-1}^{n,m}\|^2_{L^2},$$

$$\|u_h^{n,m} - \hat{u}_h^{n,m}\|^2_{L^2(I,H^1)} \le \sum_{\ell=1}^{N_t} \frac{1}{3}(t_\ell - t_{\ell-1}) \|U_{h,\ell}^{n,m} - U_{h,\ell-1}^{n,m}\|^2_{H^1},$$

The appearance of such jump-terms is what we may expect for an a-posteriori error for a discontinuous-in-time trajectory. Note that compared to classical a-posteriori error-estimates for discontinuous-in-time methods, see [**33**, **45**] for instance, we do not assume that $u_h^{n,m}$ is the solution to a discrete-in-time analogue to (Eq_h-RB_n).

5. Numerical Illustration for POD-MOR

In this final section of the paper we illustrate and compare the quality of our RB-EIM-a-posteriori error-estimates numerically for three prototypical test problems. Although the results of this paper apply to general RB-methods, our particular focus is on POD-MOR. Therefore, we restrict ourselves to reduced ansatz-spaces V_h^n spanned by a POD-basis of rank n in our numerical tests.

5.1. Test Problems and Technical Details. The two-dimensional domain $\Omega = [0,1]^2$ and the time interval $[0,T] = [0,1]$ are the same in all three test problems. We fix two discs $C_1 = B_{\frac{1}{5}}\left(\frac{1}{4}, \frac{1}{4}\right)$ and $C_2 = B_{\frac{1}{5}}\left(\frac{3}{4}, \frac{3}{4}\right)$, and the three boundary parts $\Gamma_1 = \{x \in \partial\Omega\colon x_2 = 1\}$, $\Gamma_2 = \{x \in \partial\Omega\colon x_1 = 0, x_2 < \frac{1}{2}\}$, $\Gamma_3 = \{x \in \partial\Omega\colon x_1 = 1, x_2 < \frac{1}{2}\}$. The nonlinearity is given by

$$\xi(u) = \frac{3}{4} + \frac{1}{2(1+e^{5u})}.$$

We introduce the three test problems P1-P3 by equipping the equation

$$\partial_t u - \nabla \cdot \xi(u) \nabla u = 10 \sin(2\pi t) \mathbf{1}_{C_1} - 10 \cos(2\pi t) \mathbf{1}_{C_2},$$

with the following boundary and initial conditions, respectively:

(P1) Pure homogeneous Dirichlet boundary conditions. and zero initial condition.

(P2) Pure homogeneous Neumann boundary conditions. and zero initial condition.

(P3) Mixed boundary conditions: homogeneous Dirichlet boundary condition $u = 0$ on $I \times \Gamma_1$, non-homogeneous Neumann conditions $\xi(u)\partial_n u = \sin(2\pi t)$ on $I \times \Gamma_2$, and $\xi(u)\partial_n u = -\cos(2\pi t)$ on $I \times \Gamma_3$, and natural boundary condition $\partial_n u = 0$ on the remaining part of the boundary. The initial condition is $[u(0)](x_1, x_2) := \frac{1}{10}(1 - x_1)$.

Space- and time-discretization. All computations are done utilizing `FEniCS` [**3**, **34**] and piecewise linear finite elements on a mesh generated by `mshr`, the mesh-generation tool of `FEniCS`, with $N_h = 5769$ degrees of freedom and maximum cell diameter $h_{\max} \approx 2.1 \cdot 10^{-2}$. The POD-basis is generated with snapshots coming from an (implicit) Crank-Nicolson solution of the equation with $N_t = 2500$ timesteps ("reference solution"). Hereby, the appearing nonlinear equations are solved by the built-in nonlinear solver of `FEniCS`. The same set of snapshots is also used to generate the EIM-approximation of the nonlinearity in a standard greedy procedure with L^∞-tolerance 10^{-6}. The POD-EIM-reduced equation is again solved utilizing the (implicit) Crank-Nicolson scheme with $N_t = 2500$ timesteps, whereby the nonlinear algebraic equations appearing in every timestep are solved by a standard Newton-method that is initialized with a semi-implicit Euler step as first guess ("reduced solution"). Approximate true $L^2(I, L^2)$-, $L^\infty(I, L^2)$, and $L^2(I, H^1)$-errors are computed with respect to a further numerical solution that is computed on the same finite element mesh, but with a four times higher number of timesteps than for the snapshot generation ("truth-solution"). Finally, to ensure comparability between the different test problems and norms, all errors and estimates are relative errors, i.e. the absolute error or error-estimate is divided by the corresponding norm of the truth-solution.

Estimation of the required parameters. Parameters like $\xi_\bullet, \mu_\bullet, |\xi'|_\infty$ etc. are known from the problem data. The solution-dependent parameters are found as follows: The norms of u_h are computed exactly based on the truth-solution in order to give the possibility to determine whether our estimates are sharp or not under the exact data. However, in real applications we would have to estimates those norms appropriately. The quality of the error-estimates –as absolute values– will heavily deteriorate in case of "safe" (i.e. large) estimates for the parameters. The same might happen in case of just inconvenient problem data due to the exponential terms in the estimates. However, we would like to point out that one might still hope in such a case that the relative behavior of the estimates, i.e. whether they decrease/increase by some factor, provides some information on the quality of the reduced model. Although we compute the EIM-error Δ_m^{EIM} as defined in Subsection 4.1 by accessing the full number of degrees of freedom, we did not observe significant time consumption for this.

Choice of the exponent p. In order to obtain expressive results we had to use relatively large values for p, e.g. $p = 16$. Therefore, choosing p according to the requirements of [**7**], i.e. only slightly larger than d in general, seems to be difficult.

Estimates for Approach I (Theorem 4.1). For Approach I we determine the parameters $\varepsilon, \eta, \delta$ in such a way that the simpler estimates for the $L^2(I, H^1)$-error in Corollary 4.2 become optimal, and plug in the same parameters into the estimates from Theorem 4.1. Integrals with respect to time (residuals or weighted residuals

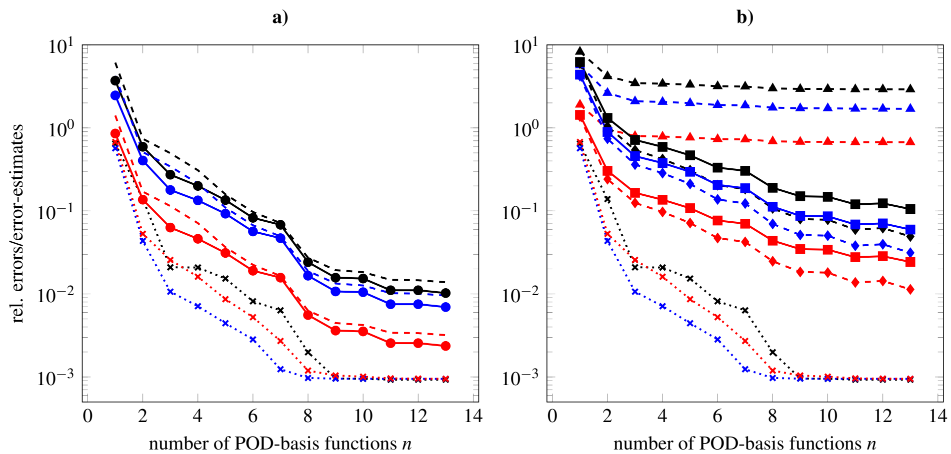

FIGURE 1. Test problem P1 (homogeneous Dirichlet boundary conditions): **a)** Estimates from Approach I (●: Theorem 4.1 with optimized parameters, dashed lines: Corollary 4.2). **b)** Estimates from Approach II (▲: $p = 6$, ■: $p = 16$, ◆, $p = 32$). $L^\infty(I, L^2)$-, $L^2(I, L^2)$-, and $L^2(I, H^1)$-errors are displayed in black, blue, and red, respectively. Approximate true errors w.r.t. the truth-solution are included in dotted lines.

in the formulas of Theorem 4.1) are evaluated using Gauss-quadrature of order 2 on every subinterval given by the timesteps.

Estimates for Approach II (Theorem 4.3). Based on several tries we choose the following parameters:

$$\eta = \frac{1}{10}(1 - \Delta_m^{\mathrm{EIM}})\xi_\bullet\mu_\bullet, \qquad \varepsilon = \frac{9}{10}\frac{1 - \Delta_m^{\mathrm{EIM}}}{\mu^\bullet(2\xi^\bullet)^{1-\frac{2}{q}}|\xi'|_\infty^{\frac{2}{q}}c_p}, \qquad \delta = \frac{\xi_\bullet\mu_\bullet}{\mu^\bullet}.$$

Note that optimization of the parameters as in Approach I is not possible because we do not have an explicit formula at hand. The ODE for the evaluation of φ is solved utilizing the backward difference formulae solver (BDF) within the `solve_ivp`-routine from `scipy.integrate`, with relative tolerance `rtol`$=10^{-6}$, and absolute tolerance `atol`$=10^{-3} \cdot \|u_h(0) - u_h^n(0)\|_{L^2}^2$. The maximal allowed step size is the same as the size of timesteps in the reduced model. We found that among other methods (Runge-Kutte with 2/3 and 4/5 stages, Radau) this choice delivered the best results. However, it is clear that the numerical approximation of φ is challenging (in particular for small p or small initial values), which might influence the reliability of the results.

5.2. Discussion of the results. Figures 1-3 show the results of our experiments. It can be seen that Approach I yields better results the smoother the truth-solution is: Test problems P1 and P2 (homogeneous boundary conditions) perform better than the problem with mixed boundary conditions (Test problem P3). Moreover, we observe that the a-posteriori error-estimates of both approaches start stagnating at about the same point at which also the true errors stagnate due to time-discretization. This fact will be advantageous in practice: Having a

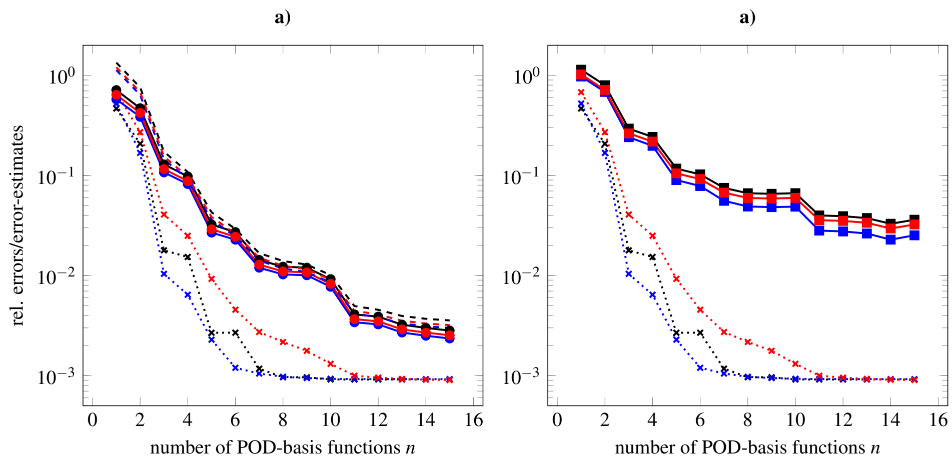

FIGURE 2. Example P2 (homogeneous Neumann boundary conditions): **a)** Estimates from Approach I (•: Theorem 4.1 with optimized parameters, dashed lines: Corollary 4.2). **b)** Estimates from Approach II for $p = 16$. $L^\infty(I, L^2)$-, $L^2(I, L^2)$-, and $L^2(I, H^1)$-errors are displayed in black, blue, and red, respectively. Approximate true errors w.r.t. the truth-solution are included in dotted lines.

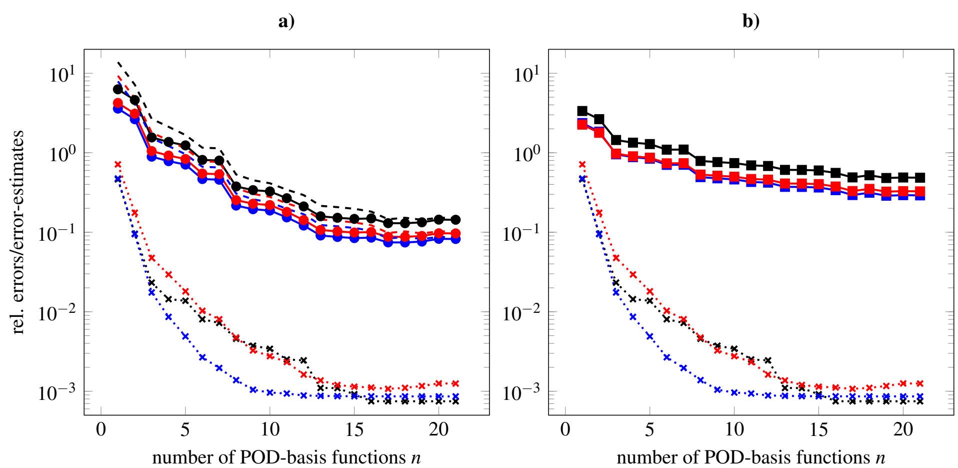

FIGURE 3. Example P3 (mixed boundary conditions): **a)** Estimates from Approach I (•: Theorem 4.1 with optimized parameters, dashed lines: Corollary 4.2). **b)** Estimates from Approach II for $p = 16$. $L^\infty(I, L^2)$-, $L^2(I, L^2)$-, and $L^2(I, H^1)$-errors are displayed in black, blue, and red, respectively. Approximate true errors w.r.t. the truth-solution are included in dotted lines.

Computing times for	Example P1	Example P2	Example P3
number of EIM-basis functions	28	36	49
Setup EIM-reduced model	50-57%	70%	99-157%
POD-EIM-reduced model	1% (0.9%)	1% (1.1%)	1-4% (1.6%)
Approach I	2-3% (3%)	2-4% (4%)	3-6% (5%)
Approach I optimized	3-6% (6%)	3-9% (8%)	5-12% (10%)
Approach II	3-15% (9-15%)	4-22% (18%)	6-15% (11%)

TABLE 1. Computing times for the setup of the EIM-reduction of the nonlinearity, the evaluation of the POD-EIM-reduced model, and the error-estimates, respectively. 100% correspond to the time that is required to compute the snapshots ("reference solution"). We show the range of times observed in the experiments from Figures 1-3, and in brackets we give the time observed for $n = 13$ POD-basis functions.

combined POD- and time-discretization-error prevents us from choosing an unnecessarily large POD-basis whose accuracy is below the error level inferred from time-discretization anyway.

How much Approach II depends on the choice of the exponent p can be seen in Figure 2b). The estimates stagnate very early for small p, i.e. Approach II unfortunately does not yield reasonable results in that case. For large p the estimates seem to get closer to the values of Approach I. In this sense one might interpret Approach II as a modification of Approach I that trades strength of the required assumption (bigger p means stronger assumption) against quality of results (smaller p means less meaningful results and numerical instability).

For the computing times observed in our numerical experiments we refer to Table 1: The evaluation of the POD-EIM-reduced model is about 25- to 100-times faster than the evaluation of the full model. We believe that even higher speedups might be possible in case of finer finite element discretization. Compared to the computing time for the full model, evaluation of the a-posteriori error-estimates from Approach I is quite cheap: Evaluation of the POD-EIM-reduced model together with computation of an error-estimate still yields a speedup of factor at least 10. As expected, evaluation of the estimates from Approach II needs slightly more time.

References

[1] Alessandro Alla, Michael Hinze, Philip Kolvenbach, Oliver Lass, and Stefan Ulbrich, *A certified model reduction approach for robust parameter optimization with PDE constraints*, Adv. Comput. Math. **45** (2019), no. 3, 1221–1250. MR3955717

[2] Alessandro Alla and J. Nathan Kutz, *Nonlinear model order reduction via dynamic mode decomposition*, SIAM J. Sci. Comput. **39** (2017), no. 5, B778–B796. MR3696057

[3] Martin S. Alnæs, Jan Blechta, Johan Hake, August Johansson, Benjamin Kehlet, Anders Logg, Chris Richardson, Johannes Ring, Marie E. Rognes, and Garth N. Wells, *The fenics project version 1.5*, Archive of Numerical Software **3** (2015), no. 100.

[4] H. Amann, *Maximal regularity for nonautonomous evolution equations*, Adv. Nonlinear Stud. **4** (2004), no. 4, 417–430. MR2100906

[5] E. Aria, M. Fahl, and E.W. Sachs, *Trust-region proper orthogonal decomposition for flow control* (2000).

[6] Maxime Barrault, Yvon Maday, Ngoc Cuong Nguyen, and Anthony T. Patera, *An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 9, 667–672. MR2103208

[7] Lucas Bonifacius and Ira Neitzel, *Second order optimality conditions for optimal control of quasilinear parabolic equations*, Math. Control Relat. Fields **8** (2018), no. 1, 1–34. MR3810865

[8] Eduardo Casas and Konstantinos Chrysafinos, *Analysis and optimal control of some quasilinear parabolic equations*, Math. Control Relat. Fields **8** (2018), no. 3-4, 607–623. MR3917455

[9] ———, *Numerical analysis of quasilinear parabolic equations under low regularity assumptions*, Numer. Math. **143** (2019), no. 4, 749–780. MR4026371

[10] Dominique Chapelle, Asven Gariah, and Jacques Sainte-Marie, *Galerkin approximation with proper orthogonal decomposition: new error estimates and illustrative examples*, ESAIM Math. Model. Numer. Anal. **46** (2012), no. 4, 731–757. MR2891468

[11] Saifon Chaturantabut and Danny C. Sorensen, *Nonlinear model reduction via discrete empirical interpolation*, SIAM J. Sci. Comput. **32** (2010), no. 5, 2737–2764. MR2684735

[12] Sever Silvestru Dragomir, *Some Gronwall type inequalities and applications*, Nova Science Publishers, Inc., Hauppauge, NY, 2003. MR2016992

[13] Martin Drohmann, Bernard Haasdonk, and Mario Ohlberger, *Reduced basis approximation for nonlinear parametrized evolution equations based on empirical operator interpolation*, SIAM J. Sci. Comput. **34** (2012), no. 2, A937–A969. MR2914310

[14] Jan Francu, *Monotone operators. A survey directed to applications to differential equations*, Apl. Mat. **35** (1990), no. 4, 257–301. MR1065003

[15] Carmen Grässle and Michael Hinze, *POD reduced-order modeling for evolution equations utilizing arbitrary finite element discretizations*, Adv. Comput. Math. **44** (2018), no. 6, 1941–1978. MR3880338

[16] Carmen Grässle, Michael Hinze, Jens Lang, and Sebastian Ullmann, *POD model order reduction with space-adapted snapshots for incompressible flows*, Adv. Comput. Math. **45** (2019), no. 5-6, 2401–2428. MR4047008

[17] Carmen Grässle, Michael Hinze, and Nicolas Scharmacher, *POD for optimal control of the Cahn-Hilliard system using spatially adapted snapshots*, Numerical mathematics and advanced applications—ENUMATH2017, 2019, pp. 703–711. MR3977026

[18] Carmen Gräßle, Martin Gubisch, Simone Metzdorf, Sabrina Rogg, and Stefan Volkwein, *Pod basis updates for nonlinear pde control*, at - Automatisierungstechnik **65** (29 May. 2017), no. 5, 298 –307.

[19] Martin Gubisch, Ira Neitzel, and Stefan Volkwein, *A-posteriori error estimation of discrete POD models for PDE-constrained optimal control*, Model reduction of parametrized systems, 2017, pp. 213–234. MR3702350

[20] Martin Gubisch and Stefan Volkwein, *POD a-posteriori error analysis for optimal control problems with mixed control-state constraints*, Comput. Optim. Appl. **58** (2014), no. 3, 619–644. MR3217251

[21] ———, *Proper orthogonal decomposition for linear-quadratic optimal control*, Model reduction and approximation, 2017, pp. 3–63. MR3672145

[22] Robert Haller-Dintelmann and Joachim Rehberg, *Maximal parabolic regularity for divergence operators including mixed boundary conditions*, J. Differential Equations **247** (2009), no. 5, 1354–1396. MR2541414

[23] Jan S. Hesthaven, Gianluigi Rozza, and Benjamin Stamm, *Certified reduced basis methods for parametrized partial differential equations*, SpringerBriefs in Mathematics, Springer, Cham; BCAM Basque Center for Applied Mathematics, Bilbao, 2016. BCAM SpringerBriefs. MR3408061

[24] Michael Hinze and Denis Korolev, *A space-time certified reduced basis method for quasilinear parabolic partial differential equations*, Preprint. arXiv:2004.00548 (2020).

[25] ———, *Reduced basis methods for quasilinear elliptic PDEs with applications to permanent magnet synchronous motors*, Preprint. arXiv:2002.04288v1 (2020).

[26] Fabian Hoppe and Ira Neitzel, *Convergence of the SQP method for quasilinear parabolic optimal control problems*, Optim. Eng. (2020).

[27] ———, *Optimal control of quasilinear parabolic PDEs with state constraints*, 2020. Available as INS Preprint No. 2004.

[28] L. Iapichino, S. Ulbrich, and S. Volkwein, *Multiobjective PDE-constrained optimization using the reduced-basis method*, Adv. Comput. Math. **43** (2017), no. 5, 945–972. MR3720386

[29] Eileen Kammann, Fredi Tröltzsch, and Stefan Volkwein, *A posteriori error estimation for semilinear parabolic optimal control problems with application to model reduction by POD*, ESAIM Math. Model. Numer. Anal. **47** (2013), no. 2, 555–581. MR3021698

[30] K. Kunisch and S. Volkwein, *Galerkin proper orthogonal decomposition methods for parabolic problems*, Numer. Math. **90** (2001), no. 1, 117–148. MR1868765

[31] ______, *Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics*, SIAM J. Numer. Anal. **40** (2002), no. 2, 492–515. MR1921667

[32] Karl Kunisch and Stefan Volkwein, *Proper orthogonal decomposition for optimality systems*, M2AN Math. Model. Numer. Anal. **42** (2008), no. 1, 1–23. MR2387420

[33] Wenbin Liu, Heping Ma, Tao Tang, and Ningning Yan, *A posteriori error estimates for discontinuous Galerkin time-stepping method for optimal control problems governed by parabolic equations*, SIAM J. Numer. Anal. **42** (2004), no. 3, 1032–1061. MR2113674

[34] Anders Logg, Kent-Andre Mardal, Garth N. Wells, et al., *Automated solution of differential equations by the finite element method*, Springer, 2012.

[35] H. Meinlschmidt, C. Meyer, and J. Rehberg, *Optimal control of the thermistor problem in three spatial dimensions, Part 1: Existence of optimal solutions*, SIAM J. Control Optim. **55** (2017), no. 5, 2876–2904. MR3702856

[36] ______, *Optimal control of the thermistor problem in three spatial dimensions, Part 2: Optimality conditions*, SIAM J. Control Optim. **55** (2017), no. 4, 2368–2392. MR3682174

[37] Hannes Meinlschmidt and Joachim Rehberg, *Hölder-estimates for non-autonomous parabolic problems with rough data*, Evol. Equ. Control Theory **5** (2016), no. 1, 147–184. MR3485929

[38] Elizabeth Qian, Martin Grepl, Karen Veroy, and Karen Willcox, *A certified trust region reduced basis approach to PDE-constrained optimization*, SIAM J. Sci. Comput. **39** (2017), no. 5, S434–S460. MR3716566

[39] Sabrina Rogg, Stefan Trenz, and Stefan Volkwein, *Trust-region pod using a-posteriori error estimation for semilinear parabolic optimal control problems*, Technical Report 359, Konstanzer Schriften in Mathematik, 2017.

[40] Tomas Roubicek, *Nonlinear partial differential equations with applications*, International Series of Numerical Mathematics, vol. 153, Birkhäuser Verlag, Basel, 2005. MR2176645

[41] M. Schuh, *Adaptive Trust-Region POD Methods and Their Applications in Finance*, Ph.D. Thesis, 2012.

[42] S. Selberherr, *Analysis and simulation of semiconductor devices*, Springer-Verlag, 1984.

[43] John R. Singler, *New POD error expressions, error bounds, and asymptotic results for reduced order models of parabolic PDEs*, SIAM J. Numer. Anal. **52** (2014), no. 2, 852–876. MR3190754

[44] Kathrin Smetana and Mario Ohlberger, *Hierarchical model reduction of nonlinear partial differential equations based on the adaptive empirical projection method and reduced basis techniques*, ESAIM Math. Model. Numer. Anal. **51** (2017), no. 2, 641–677. MR3626414

[45] Vidar Thomée, *Galerkin finite element methods for parabolic problems*, Second, Springer Series in Computational Mathematics, vol. 25, Springer-Verlag, Berlin, 2006. MR2249024

[46] F. Tröltzsch and S. Volkwein, *POD a-posteriori error estimates for linear-quadratic optimal control problems*, Comput. Optim. Appl. **44** (2009), no. 1, 83–115. MR2556846

[47] Stefan Volkwein, *Proper orthogonal decomposition: Theory and reduced-order modelling*, Lecture Notes, University of Konstanz (201201).

[48] Eberhard Zeidler, *Nonlinear functional analysis and its applications. II/B*, Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron. MR1033498