Multi-stage sequential sampling models with finite or infinite time horizon and variable boundaries

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Abstract

The multi-stage decision model, aka multiattribute attention switching model, assumes a separate sampling process for each attribute and switching attention from one attribute to the next in a sequential fashion during one trial. Here the model is extended to finite and infinite time horizons and to non-constant boundaries. For a finite time horizon the model predicts a probability of not deciding within the available time. Two different families of non-constant boundaries are implemented, one with a nonlinear decrease, one with a constant boundary at the beginning and a linear decrease toward the deadline. Furthermore, it is shown how the stochastic process underlying each attribute of the multi-stage model (Wiener or Ornstein-Uhlenbeck process) can be discretized by a birth-death chain to implement all the relevant model features and how to provide speeded calculations. Several numerical examples are provided demonstrating the effect of the order of attribute processing (order schedule) and boundary properties. It is shown that, regardless of the time horizon or the non-constant boundaries, the order schedule is the determinant to predict a consistent choice probability/choice response time pattern including preference reversals and fast errors.

Keywords: Sequential sampling, multiattribute, attention switching time, time schedule, order schedule, infinite and finite time horizon, constant and non-constant boundaries, Ornstein-Uhlenbeck, Wiener, birth-death chain

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1. Introduction

Sequential sampling models of decision making have become the dominant approach to modeling decision processes in cognitive science. These models are designed to account for all three of the most basic dependent variables of cognitive psychology, which include choice, decision time, and confidence. Their application includes a variety of psychological tasks, from basic perceptual decision to complex preferential choice tasks. From early on, they were applied to identification and discrimination tasks (e.g. Ashby, 1983; Edwards, 1965; Heath, 1981; Laming, 1968; Link and Heath, 1975; Pike, 1973); memory retrieval (e.g. Ratcliff, 1978; Stone, 1960; Van Zandt et al., 2000) and classification (e.g., general recognition theory, (Ashby, 2000); exemplar-based random walk models of classification, (Nosofsky and Palmeri, 1997)) to account simultaneously for response times and accuracy data.

They have also been used for preferential decision tasks (e.g. decision field theory, Busemeyer and Townsend (1993); and multi-attribute decision field theory, Diederich (1997); Diederich and Busemeyer (1999)) and value based decision (Krajbich and Rangel, 2011) to account for choice response times and choice probabilities interpreted as preference strength; judgment and confidence ratings (Pleskac and Busemeyer, 2010); and to account for selling prices, certainty equivalents, and preference reversal phenomena (Busemeyer and Goldstein, 1992; Johnson and Busemeyer, 2005). More recently, they have been applied to combining perceptional decision making and preference (e.g. Diederich and Busemeyer, 2006; Diederich, 2008; Rorie et al., 2010; Gao et al., 2011). Furthermore, these models have been closely linked to measures from neuroscience such as multi-cell electrode recordings, EEG, and fMRI (e.g. Churchland et al., 2008; Ditterich, 2006; Gold and Shadlen, 2007; Ratcliff et al., 2007). Under fairly general conditions, these models also represent the optimal rule for making sequentially sampled decisions that balance decision accuracy with cost of sampling (e.g., Edwards, 1965; Rapoport and Burkheimer, 1971; Bogacz et al., 2006). In practical applications, sequential sampling models have been used to estimate parameters representing basic components of the decision process, such as discriminability, bias, and threshold criterion. Individual differences in these parameters are used to investigate how these parameters differ across age groups, psychopathology, and other populations (e.g. Thapar et al., 2003; White et al., 2010; Ratcliff et al., 2010).

The basic idea of all sequential sampling models is that, when a decision has to be made (a) noisy evidence for or against each choice option is sequentially sampled across time, (b) this evidence is accumulated across time, and (c) a final choice is made as soon as the evidence reaches a threshold, or a deadline has to be met. Choice probability is determined by the probability that evidence level crosses a threshold first for one option before another, and decision time is determined by the time required to reach a threshold. Confidence ratings following a choice can be determined from the strength of evidence that accumulates during a post-choice time interval. There are many specific versions of sequential models that differ according to precisely how evidence is accumulated, how the threshold criteria are set, and how confidence is derived. One class of sequential sampling models assumes that evidence for one option is at the same time evidence against the alternative option. Within this class, random walk models accumulate evidence in discrete time whereas diffusion models accumulate evidence in continuous time. The most commonly used version of the diffusion model is the Wiener diffusion model that linearly accumulates evidence without any decay (Ratcliff, 1978), but others include the Ornstein-Uhlenbeck model that linearly accumulates evidence with decay (Busemeyer and Townsend, 1993; Diederich, 1997), and the leaky competing accumulator (LCA) model (Usher and McClelland, 2001) that nonlinearly accumulates evidence with decay. Another class of sequential sampling models is widespread in psychology: accumulator and counter models. An accumulator/counter is established for each choice alternative separately, and evidence is accumulated in parallel. A decision is made as soon as one counter wins the race to reach one preset criterion. The accumulators/counters may or may not be independent. Poisson-counter models are prominent examples but random walk and diffusion models, one process for each alternative with a single criterion (absorbing boundary) for each process, can also be employed. Other accumulator models such as LATER (linear approach to threshold with ergodic rate) (Carpenter and Williams, 1995) and LBA (Linear Ballistic Accumulator) (Brown and Heathcote, 2005) assume a deterministic linear increase in evidence for one trial. Randomness in responses occurs by assuming a normal distribution across the linear accumulation and are not considered here further.

In the following we focus on random walk/diffusion models with one process and two decision criteria. For a review of both diffusion models and counter models see Ratcliff and Smith (2004).

Despite the great progress that has been made with the development and empirical testing of random walk/diffusion models, there remain some important limitations. One important limitation of many applications of random walk/diffusion models is that a single integrated source of evidence is assumed to be generating the evidence during the deliberation process leading to a decision. In particular, the integrated source may be based on multiple features or attributes, but all of these features or attributes are assumed to be combined and integrated into a single source of evidence, and this single source is used throughout the decision process until a final decision is reached. There are exceptions developed for very specific applications (e.g. Smith and Ratcliff, 2009; **?**) but by far, single source models predominate the field.

Another limitation is that most random walk/diffusion models cannot account for anticipatory and time-out responses. Trials with a shorter or longer than predefined response time threshold are typically eliminated from the data set.

Finally, most models assume constant decision criteria across the decision process. In some cases, however, it it possible that with elapsed time the boundaries are collapsing, which in neuroscience has been called "urgent signals" (e.g. Churchland et al., 2008; Ditterich, 2006) but see (Hawkins et al., 2015). We refer also to Zhang et al. (2014) for the inclusion of time-varying boundaries into a single-stage diffusion model.

In the following we will address these topics. To introduce notation, we begin by describing a stochastic process with its relation to psychological concepts. Second, the multi-stage decision model (aka multiattribute attention switching (MAAS) model) is introduced including time and order schedules, finite and infinite time horizons, and non-constant boundaries. Obviously, non-constant boundaries can also be applied to single-stage models. Third, to allow for efficient predictions we discretize the diffusion process (Wiener or Ornstein-Uhlenbeck) by a Markov chain model. Finally, we show the predictions of the model for various scenarios.

2. Sequential sampling approach

Sequential sampling models are stochastic processes, that is, a collection of random variables, representing the evolution of some system of random values over time. Two quantities are of foremost interest to psychologists: (1) the probability that the process eventually reaches one of the thresholds or boundaries for the first time (the criterion to initiate a response), i.e., *first passage or exit probability*; (2) the time it takes for the process to reach one of the boundaries for the first time, i.e., *first passage or exit time*. The former quantity is related to the observed relative frequencies, the latter usually to the observed mean choice response times or the observed choice response time distribution.

Let $X(t)$ denote the random variable representing the numerical value of the accumulated evidence at time t (for now we assume that we are in a continuous-time, continuous-state situation). For a binary choice between choice options A and B, the models assume that the decision process begins with an initial state of evidence $X(0)$. This initial state may either favor option A ($X(0) > 0$) or option B ($X(0) < 0$) or may be neutral with respect to A or B ($X(0) = 0$), or can be given as a probability distribution.

Upon presentation of the choice options, the decision maker sequentially samples information from the stimulus display over time, retrieves information from memory, or forms preferences, depending on the context. The small increments of evidence sampled at any moment in time are such that they either favor option A ($dX(t) > 0$) or option B ($dX(t) < 0$). The evidence is incremented according to a diffusion process:

$$dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dW(t).$$

Here, $\mu(x,t)$ is called the *effective drift rate* and describes the instantaneous rate of expected increment change at time t and state $x = X(t)$. The factor $\sigma(x,t)$ in front of the instantaneous increments $dW(t)$ of a standard Wiener process $W(t)$ is called the *diffusion coefficient*, and relates to the variance of the increments. This process continues until the magnitude of the cumulative evidence exceeds a threshold criterion. The process stops and option A is chosen as soon as the accumulated evidence reaches a criterion value for choosing A or it stops and chooses option B as soon as the accumulated evidence reaches a criterion value for choosing B. The probability p_A of choosing A over B is determined by the accumulation process reaching the criterion value or boundary for A before reaching the boundary for B, similarly for p_B.

The model is specified by making concrete assumptions on drift and diffusion rates, the criterion functions (discussed in the following) and the time within which a decision has to be made, i.e., decision interval $t \in [0, T_{end}]$, where $T_{end} \in (0, \infty]$ is a randomly or deterministically given final deadline.

For instance, a stochastic process with drift rate and diffusion coefficient

$$\mu(x,t) = \delta \qquad \sigma(x,t) = \sigma, \tag{1}$$

defines a time-homogeneous Wiener process with drift (setting $\delta = 0$ is the standard Wiener process). Intuitively, the drift rate δ reflects the tendency to approach one choice alternative over the other, and is related to the quality of evidence: The better the evidence discriminates between the choice options, the larger the value of δ, which determines the direction of the process. In our notation, a positive drift rate, $\delta > 0$, indicates that option A is chosen over option B more often, and a negative drift, $\delta < 0$, that B is chosen more often over A. The diffusion coefficient σ is considered in psychological applications a scaling factor and is often set to a constant, i.e. to 1.

Fixing the functional forms for effective drift rate and diffusion coefficient to

$$\mu(x,t) = \delta - \gamma x, \qquad \sigma(x,t) = \sigma, \tag{2}$$

defines a time-homogeneous Ornstein-Uhlenbeck process (OUP). Setting $\gamma > 0$ models evidence accumulation towards one of the choice options at a linearly decaying rate, that is, it induces a change of the effective drift rate $\mu(x,t) = \delta - \gamma x$ depending on the current state. The parameter is related to memory processes (e.g., forgetting, primacy and recency effects), leakage of information, similarities between choice alternatives, and conflict patterns (e.g. Busemeyer and Townsend, 1993; Diederich, 1997; Usher and McClelland, 2001). Setting $\gamma = 0$ reduces Eq. 2 to a Wiener process with drift (Eq. 1).

2.1. Stopping rules and criterion functions

A stopping rule constrains when and how the decision is made. Two stopping rules are mainly used in psychology. One is a *fixed stopping time* in which the time to make the decision is externally controlled fixed stopping time and the decision maker is forced to make a choice at (but not before) the deadline $t = T_{end}$, regardless of how much evidence has been accumulated towards either of the alternatives. If at $t = T_{end}$ the accumulated evidence $X(T_{end})$ is larger than 0, alternative A is chosen; if it is smaller than 0, B is chosen. No absorbing boundaries are assumed, and choice response times for both alternatives equal T_{end}, i.e., are deterministic rather than a random variable (Figure 1, A).[1] The other rule, which we focus on in this paper, is an *optional stopping time* in which the decision maker selects the time to make the decision. In this case, the response time is a random variable (Busemeyer and Diederich, 2002). The criterion functions $\theta_{A/B}(t)$ that define the stopping rule (also called deci-

[1] Note that accumulator models can be mimicked accordingly: A race between several alternatives (each trajectory presenting one alternative rather than one trial) occurs and the winner to determine the response may be the one with the highest value at time T_{end}. The Multialternative Decision Field Theory (Roe et al., 2001) and the LCA model (Usher and McClelland, 2001) assume exactly this.

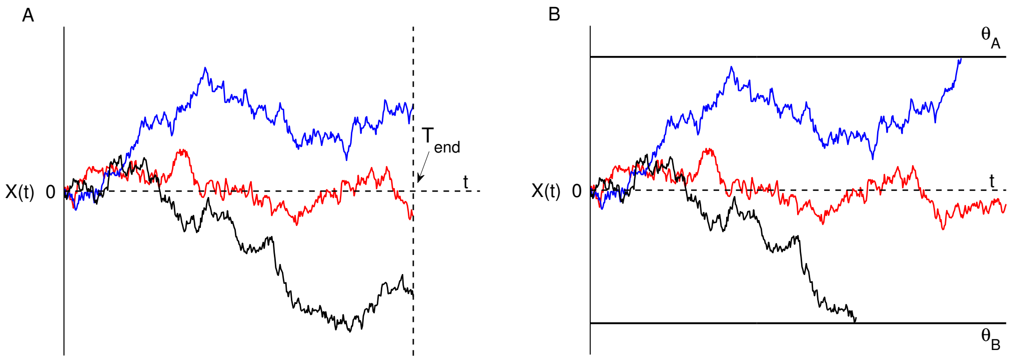

Figure 1: Stopping rules. (A) Fixed stopping time at T_{end}. (B) Optional stopping time at t.

sion boundaries) are typically assumed to be constant across the entire decision process ($\theta_{A/B}(t) = \theta_{A/B}$) (Figure 1, B).

The accumulation process continues until the magnitude of the cumulative evidence reaches a criterion bound. The process stops and an A response is initiated as soon as $X(t) \geq \theta_A$, or it stops and a B response is initiated as soon as $X(t) \leq \theta_B$. The decision criteria (absorbing boundaries in mathematical terms) reflect how much evidence is needed for the decision maker to come to a decision and are set by the decision maker prior to the decision task. They depend, among other things, on the time available for making a decision. Specifically, the criterion boundary is assumed to be an *increasing* function of the time limit. That is, with short time limits the boundaries are assumed to be narrow, and the time to reach it to initiate a response is short whereas with long or no time limits the boundaries are further apart and it takes longer to reach them to initiate a response. Assuming symmetric criteria, i.e., $\theta_A = -\theta_B$, around the starting point $X(0) = 0$, is equivalent to assuming no a priori bias.

Obviously both criteria to initiate a response are quite different. The latter operates on the evidence space, whereas the former is based on the time set. However, both criteria can also be combined. For instance, under short deadline conditions, the decision maker may employ internal fixed deadlines as well as the decision bounds to terminate the accumulation process (Diederich and Busemeyer, 2006; Diederich, 2008). This is

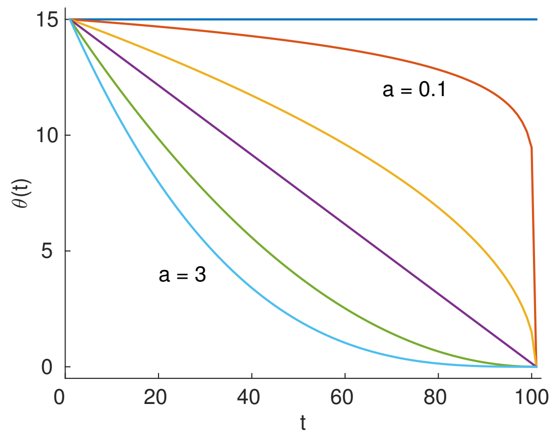

Figure 2: Non-constant boundaries. The shape is determined by Eq. 3 with $\theta(0) = 15$, $T_{end} = 100$, and $a = 0.1, 0.5, 1, 2, 3$ (from right to left). The special case, $a = 0$, results in a constant boundary, here the upper line at $\theta = 15$.

related to the deadline model in which the response time is determined either by the time needed to complete, e.g., a discrimination process, or by the arrival of a predetermined deadline, whichever comes first (e.g. Swensson, 1972; Yellot, 1971; Ruthruff, 1996, for a test of the model; see also Ratcliff and Rouder, 2000).

Another way to model the approaching deadline is to bring the decision horizons $\theta_{A/B}(t)$ closer to 0 as t approaches T_{end}. We present two such families of variable decision boundaries. The first family is given by

$$\theta_A(t) = \theta_A(0) \cdot (1 - t/T_{end})^{a_A}, \quad \theta_B(t) = \theta_B(0) \cdot (1 - t/T_{end})^{a_B}, \quad t \in [0, T_{end}], \quad (3)$$

where the constants $\theta_A(0) > 0 > \theta_B(0)$ stand for the initial decision horizons, and $a_{A/B} > 0$ characterize the shape of the decision horizons.

Figure 2 shows the resulting graphs of $\theta(t) = \theta_A(t)$ for several values $a = a_A$. A possible interpretation is that values $a > 1$ reflect the tendency of the decision maker to come to a decision sooner rather than later, possibly way before the actual deadline is approached, whereas values $a < 1$ indicate hesitation to make a decision too early. Finally, in case of $a = 1$, the decision horizon decreases linearly and steadily.

A second one-parameter family of decision horizon functions considered here is

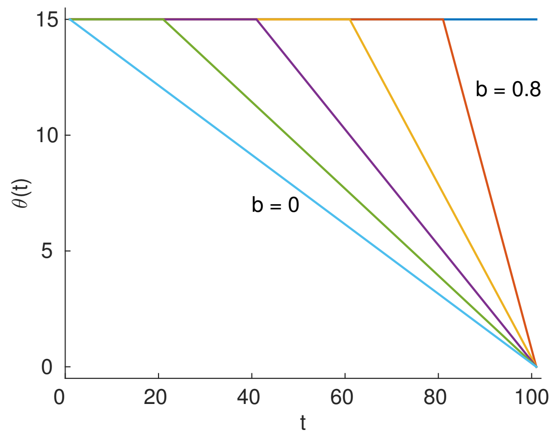

Figure 3: Non-constant boundaries. The shape is determined by Eq. 4 with $\theta(0) = 15$, $T_{end} = 100$, and $b = 0.8, 0.6, 0.4, 0.2, 0$ (from right to left). The special case, $b = 1$, results in a constant boundary, here the upper line at $\theta = 15$ and partly covered the the constant part of the remaining non-constant boundary examples.

given by

$$\theta_A(t) = \theta_A(0) \min(1, (1 - t/T_{end})/(1 - b)), \qquad t \in [0, T_{end}], \tag{4}$$

where $b \in [0, 1]$ is a parameter; similarly for $\theta_B(t)$. A possible interpretation is that only after a portion of time bT_{end}, the deadline T_{end} is announced, or the decision maker realizes only at this time that there is a deadline, and gradually lowers the decision horizon. Figure 3 shows the resulting graphs of $\theta(t) = \theta_A(t)$ for several values b.

3. Multi-stage decision model

Choice alternatives are often described by multiple features, dimensions, or attributes. For instance, visual objects may vary in color and size or in width and hight; crossmodal tasks involve different modalities, often with inter-stimulus asynchronies; consumer products are characterized by price and quality; in social priming experiments, race might serve as a bias in a perceptual discrimination task, and so on. Furthermore, experimental designs may involve several stages in which, for example, congruent or incongruent information is delivered sequentially. For those and similar situations a sequential sampling model that represents evidence for the different process stages might be more appropriate than combining and integrating all information into a single source of evidence that drives the diffusion process. Diederich (1995; 1997) and Diederich and Oswald (2014), developed a generalization of the single-stage sequential model, assuming that each attribute[2] of the stimulus arrangement is described by a separate sequential sampling process.

For each of the $k = 1, \ldots, K$ attributes we assume an Ornstein-Uhlenbeck process $X(t)$ defined by

$$dX(t) = (\delta_k - \gamma_k X(t))dt + \sigma_k dW(t). \tag{5}$$

The information sampling is attribute-by-attribute, i.e., the finitely many attributes are considered one-by-one for a certain period of time in some order and possibly with repetition. Each attribute appeals differently to the decision maker which is characterized by a set of attribute-dependent constants $\delta_k, \gamma_k, \sigma_k$ (in principle, these constants may also change with time, e.g., in a kind of learning process a later reconsideration of a certain attribute may have different appeal to the decision maker than it had at an earlier time).

The decision maker switches attention from one attribute to the next during the time course of one trial. For instance, in a crossmodal task (visual, auditory, tactile), Diederich (1995) assumed a serial process controlled by stimulus input at given stimulus onset asynchronies. That is, the order of attributes, here a light, followed by a tone, followed by a tactile vibration, as well as the point in time when a new attribute was added, here the tone presented at t_1 (t_1 ms after light onset) and the tactile vibration at t_2 (t_2 ms after light onset) was determined externally by the experimental setup. In the following we will call attention switches at predetermined, fixed times, and together with a predefined order of attributes, a *deterministic time and order schedule*. Often, however, neither the processing order of attributes nor the point in time when the de-

[2]For ease of communication we use "attribute" here in a very broad sense. It includes features or dimensions of the stimulus proper as well as information presented in different stages.

cision maker switches attention from one attribute to the next one are known or can be inferred from the experimental setup. For those cases, Diederich (1997) proposed a specific model in which attention switches from one attribute to the next with some probability. This model was further developed in Diederich and Oswald (2014) to include what we call a *random time and order schedule* by allowing also for randomly chosen attention switching times and attribute orders.

3.1. Time and order schedules

The specific order in which attributes are considered (*order schedule*) as well as at which times attention is switched from one attribute to another one (*time schedule*) is part of the model parameters, and may be given deterministically or randomly. Formally, we assume that attention switches from one attribute to the next in a sequence of *attention switching times*

$$T_0 = T_{start} = 0 < T_1 < T_2 < \ldots < T_L = T_{end}, \tag{6}$$

with T_{end} representing the maximum duration of the decision process. On a theoretical level, it is possible to assume $T_{end} = \infty$ (no finite deadline) and $L = \infty$ (infinite attention switching). We denote by $\Delta T_l = (T_{l-1}, T_l]$ the l-th attention time interval. A time and order schedule consists of a sequence $\{T_l\}_{l=1,\ldots,L}$ of attention switching times, and a sequence $\{k_l \in \{1,\ldots,K\}\}_{l=1,\ldots,L}$ of attribute indices which specifies that during the time interval ΔT_l the k_l-th attribute is considered. At attention switching time T_l, $l = 1,\ldots,L-1$, attention switches from attribute k_l to attribute k_{l+1}. How random time and order schedules can be generated has been discussed in Diederich and Oswald (2014).

Consequently, the process $X(t)$ determined by such a schedule is a *piecewise* OUP, with fixed parameters $\delta_{k_l}, \gamma_{k_l}, \sigma_{k_l}$ in each interval ΔT_l, satisfying the stochastic differential equation

$$dX(t) = (\delta_{k_l} - \gamma_{k_l} X(t))dt + \sigma_{k_l} dW(t), \qquad t \in \Delta T_l, \qquad l = 1,\ldots,L. \tag{7}$$

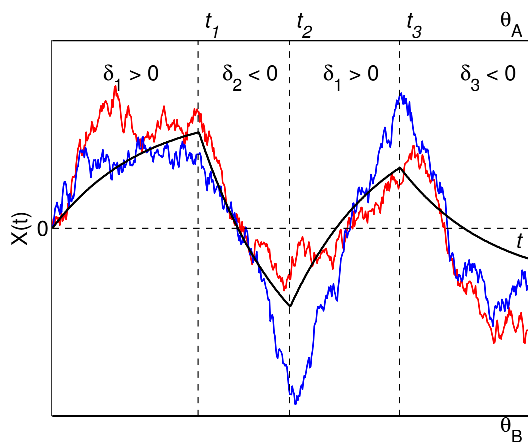

Figure 4: A piecewise OUP with three different attributes. The attribute order is $(1,2,1,3)$, attribute 1 is considered twice in the sequence of attribute consideration. Switching attention from one attribute to the next occurs at fixed times t_1, t_2, and t_3. The trajectories reflect the accumulation process for two different trials. The black solid lines indicate the deterministic trajectory of the process (set $\sigma_{k_l} = 0$ in Eq. 7).

Figure 4 shows an example with three different attributes ($K = 3$) and a deterministic time and order schedule of length $L = 4$ with switching times t_l independent of the trajectories, and attribute order $k_1 = 1$, $k_2 = 2$, $k_3 = 1$, $k_4 = 3$ (note that the first attribute is reconsidered once).

3.2. Finite and infinite time horizon

That the process is eventually absorbed at one of the decision boundaries implicitly assumes an infinite time horizon. In real life and, in particular, in experimental situation the time horizon is rather finite. Even if the experimental setup does not include explicitly time limits, a timeout is often installed. That is, a fixed deadline $T_{end} < \infty$ for the decision process is more realistic. When this deadline is not met, the trial is counted as a timeout trial. Rather than removing these timeout trials from the data set, the multi-stage decision model allows us to account for non-decision situations, i.e., to define a non-decision probability $p_N := 1 - p_A - p_B > 0$, by defining a stage accounting for the timeout trials.

3.3. *Decision times and choice probabilities*

The decision process ends with choosing alternative A if $X(t)$ hits the user specified decision horizons for the first time at $\theta_A(t)$. We assume that $\theta_A(t) \geq 0$ for $t \in (0, T_{end}]$ is a non-increasing function of time. Similarly, the process stops with choosing alternative B if $X(t)$ equals for the first time a given non-decreasing function $\theta_B(t) \leq 0$, $t \in (0, T_{end}]$. This leads us to the definition of the following quantities of interest which are often accessible in experiments: The decision times for A and B are random variables formally defined by

$$t_A := \begin{cases} t^*, & \text{if } \exists\, t^* \in [0, T_{end}] :\; X(t^*) \geq \theta_A(t^*),\; \theta_B(t) < X(t) < \theta_A(t),\; t \in [0, t^*), \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$t_B := \begin{cases} t^*, & \text{if } \exists\, t^* \in [0, T_{end}] :\; X(t^*) \leq \theta_B(t^*),\; \theta_B(t) < X(t) < \theta_A(t),\; t \in [0, t^*), \\ +\infty, & \text{otherwise,} \end{cases}$$

respectively. Their realizations can be observed from single trials, while their conditional cumulative distribution functions

$$F_{t_A}(t) = \mathbf{P}(t_A \leq t | t_A < \infty), \qquad F_{t_B}(t) = \mathbf{P}(t_B \leq t | t_B < \infty), \qquad t \in [0, T_{end}],$$

and probability density functions can be approximately reconstructed from repeated trials. The same is true for their moments, in particular, for the choice probabilities

$$p_A = \mathbf{P}(t_A < +\infty), \qquad p_B = \mathbf{P}(t_B < +\infty),$$

and the average times for deciding on A or B,

$$\mathbf{E}(t_A) = \int_0^{T_{end}} t \; dF_{t_A}(t), \qquad \mathbf{E}(t_B) = \int_0^{T_{end}} t \; dF_{t_B}(t),$$

respectively.

3.4. Implementation problems

Unfortunately, closed-form expressions for the above quantities are available only in special cases, e.g., when only one attribute (meaning a single process) is considered, and the decision boundaries are constant (Borodin and Salminen, 2002). The general situation of many attention intervals $L > 1$, or even general time-dependent drift and dispersion coefficients, and non-constant decision boundaries requires the numerical solution of certain partial differential and integral equations. An excellent primer on how to determine the first passage time and first passage probabilities for non-constant decision boundaries is provided in Smith (2000); see also Buonocore et al. (1987, 1990); Sacerdote et al. (2014). We are not aware of any general purpose implementation of this approach[3].

Instead of going this way, we use a consistent approximation of the above continuous model by a discrete-time, discrete-state random walk model (Diederich, 1997; Diederich and Busemeyer, 2003; Diederich and Oswald, 2014), which is flexible enough to account for nonstationary and nonlinear properties but can also be adapted to the situation described above of non-constant time decision horizons $\theta_{A/B}(t)$. Another reason for this choice is the simplicity of implementation and versatility of finite-state discrete-time Markov chain models.

4. Model discretization

In the following we present a discrete-time, finite-state space Markov chain (MC) model that approximates the described continuous, piecewise OUP model. Both time and state space are now discrete. The discretization is facilitated by two parameters:

[3] After submission we became aware of recent work on software: Drugowitsch (2014), and Verdonck et al. (2015), whose implementations follow the approach from Smith (2000), and Srivastavaa et al. (2015), who use piecewise constant approximations to diffusion-type processes by Wiener processes with drift, and piecewise constant approximations to time-varying decision boundaries to deal with problems as discussed in this paper.

$\Delta > 0$ is the constant step-size for the spatial resolution of the range of evidence values, and $\xi \geq 1$ is an auxiliary parameter specifying the underlying random walk model. Because the size of the resulting finite state space has a major impact on the computational complexity, we will choose Δ as large as possible. It turns out that already moderate values for Δ and state space sizes, as used in the numerical tests below, lead to results of sufficient accuracy. Evidence accumulation now happens only at fixed time stamps, belonging to a grid that is uniform[4] during each attention time interval ΔT_l. The resulting probability transition matrices are chosen such that at each discrete time stamp the actual evidence value is increased or decreased by Δ, or stays the same (i.e., we use a trinomial tree random walk model by setting $\xi > 1$, the binomial case $\xi = 1$ is included as a partial case). The corresponding transition probabilities are chosen such that convergence to the continuous model is guaranteed as $\Delta \to 0$. The transition matrices are thus tri-diagonal, and all quantities of interest (exit probabilities, exit time distributions and their expectations) can be computed cheaply, with an overall complexity that is roughly of order $\Delta^{-3}T_{end}$ (further savings for the case of constant decision criterion values $\theta_{A/B}$ are possible, see Diederich (1997); Diederich and Busemeyer (2003); Diederich and Oswald (2014) and Section 5.1 below).

It is worth noting that considering such a discrete OUP model may well be warranted without any reference to a continuous-time, continuous-state limit in mind. For instance, attribute-related information may be available only at certain moments in time (this is typical for certain laboratory experiments but also in some economics and finance scenarios). Also, evidence may be accumulated in discrete numerical units, and not on a continuous scale. We will, however, not dwell on this issue further.

With $\Delta > 0$, $\xi \geq 1$, and a fixed time and order schedule given, the piecewise OUP

[4]This is because the parameter σ, which may change between attention time intervals, enters the relation between Δ and the time step-size τ, necessary to achieve convergence of the discrete MC model to the continuous one as $\Delta \to 0$

$X(t)$ defined by (7) is approximated by a discrete time, finite state space Markov chain

$$X_n \approx X(t_n), \qquad n = 1,2,\ldots,N, \qquad X_0 = X(0) = 0,$$

taking values in a finite (but time-dependent) state space

$$S_n = \{x_i := i\Delta :\ i \in \mathscr{I}_n\}, \qquad \mathscr{I}_n = \{-m_{B,n},\ldots,m_{A,n}\} \subset \mathbb{Z}, \tag{8}$$

where the limits of the current index set $\mathscr{I}_n$ are defined from the decision horizon values at t_n as follows:

$$-m_{B,n}\Delta \leq \theta_B(t_n) < (-m_{B,n}+1)\Delta \leq (m_{A,n}-1)\Delta < \theta_A(t_n) \leq m_{A,n}\Delta.$$

Thus, the largest and smallest x_i in S_n are considered absorbing states at t_n in our MC model, reaching (or exceeding) them means decision for alternative A and B, respectively. The set of non-absorbing states at t_n is denoted by S_n^*, the corresponding index set is $\mathscr{I}_n^* = \{-m_{B,n}+1,\ldots,m_{A,n}-1\}$.

The discrete time stamps t_n are defined according to

$$t_n = T_{l-1} + (n - n_{l-1})\tau_l, \qquad n = n_{l-1},\ldots,n_l, \quad l = 1,\ldots,L,$$

where the constant time-step τ_l characteristic for each attention time interval ΔT_l is chosen as the value closest to

$$\tau_l \approx \Delta^2/(\xi\sigma_{k_l})^2$$

for which $n_l := n_{l-1} + (T_l - T_{l-1})/\tau_l$ is an integer ($n_0 = 0$, $N = n_L$). This choice of step-size τ_l is standard for matching the trinomial tree model to the piecewise OUP under consideration.

For $n = n_{l-1}+1,\ldots,n_l$, corresponding to the l-th attention interval, i.e., when $t_n \in \Delta T_l$, the transition probabilities $p_{n,j,i} := \mathbf{P}(X_n = x_i | X_{n-1} = x_j)$ describing the transition from X_{n-1} to X_n are defined as

$$p_{n,j,i} = \begin{cases} \xi^{-2}(1-(\delta_{k_l}-\gamma_{k_l}x_j)\Delta/\sigma_l^2)/2, & j = i+1, \\ \xi^{-2}(1+(\delta_{k_l}-\gamma_{k_l}x_j)\Delta/\sigma_l^2)/2, & j = i-1, \\ 1 - p_{n,i,i+1} - p_{n,i,i-1}, & j = i, \\ 0, & |j-i| > 1. \end{cases} \tag{9}$$

This corresponds to a random walk where in each small time interval $(t_{n-1}, t_n]$ evidence towards alternative A is increased by Δ, decreased by Δ, or left unchanged with certain (non-negative) probabilities. The formulas ensure convergence of the discrete process X_i to the continuous process $X(t)$ if $\Delta \to 0$ (for fixed $\xi \geq 1$).[5]

The resulting transition matrix is denoted by $\mathbf{P}_n$. Note that, for $t_n \in \Delta T_l$ the entries of $\mathbf{P}_n$ depend only on k_l, the index of the attribute associated with the l-th attention interval, however, the size of $\mathbf{P}_n$ may change if the decision horizons $\theta_{A/B}(t_n)$ are non-constant, forcing the state spaces to shrink. In other words, for $t_n \in \Delta T_l$, the transition matrices are submatrices (depending on $S_n \subset S_{n-1}$) of a matrix $\mathbf{P}^{(k_l)}$ solely depending on the k_l-th attribute associated with the l-th attention time interval, with entries given by (9).

Knowing the transition probability matrices $\mathbf{P}_n$ allows us to compute the probability vectors Z_n with entries

$$Z_{n,i} := \mathbf{P}(X_n = x_i), \qquad i \in \mathscr{I}_n^*, \tag{10}$$

corresponding to the non-absorbing states at time t_n from the previous Z_{n-1} by matrix-vector multiplication

$$\tilde{Z}_n' = Z_{n-1}'\tilde{\mathbf{P}}_n, \qquad n = 1, \ldots, N, \qquad Z_n = \tilde{Z}_n'|_{\mathscr{I}_n'}, \tag{11}$$

where at start Z_0 is a unit column vector with index set $\mathscr{I}_0^*$ and $Z_{0,0} = 1$ corresponding to our assumption $X(0) = 0$. The remaining notation is as follows: $\tilde{\mathbf{P}}_n$ stands for the submatrix of $\mathbf{P}_n$ as defined by Eq. 9 corresponding to the index set $\mathscr{I}_{n-1}^* \times \mathscr{I}_{n-1}$, and $\tilde{Z}_n$ is an auxiliary column vector with index set $\mathscr{I}_{n-1}$. This costs $\mathrm{O}(|S_{n-1}|)$ elementary operations per multiply, and overall leads to a computational effort of $\mathrm{O}(\Delta^{-3}T_{end})$ flops (see the definition of the state spaces S_n and of the step-sizes τ_l).

[5] In order to satisfy the natural requirement that the $p_{n,j,i}$ always belong to $[0,1]$ and sum up to 1 for fixed j, the discretization parameter Δ cannot be taken arbitrarily large. The concrete limitations depend on the process parameters (and decision thresholds), and can be computed from Eq. 9. To ensure robustness, in the implementation $p_{n,j,i}$ values violating these constraints are appropriately modified. In the simulations reported below, the value of Δ was always small enough, and Eq. 9 was used as is.

Moreover, carrying out the multiplication in Eq. 11 recursively delivers approximations to all quantities of interest such as choice probabilities, expected choice response times, and exit time distributions. Indeed, define

$$p_{A,n} = \sum_{i \in \mathscr{I}_{n-1} : x_i \geq \theta_A(t_n)} \tilde{Z}_{n,i}, \qquad p_{B,n} = \sum_{i \in \mathscr{I}_{n-1} : x_i \leq \theta_B(t_n)} \tilde{Z}_{n,i}. \tag{12}$$

Note that the values $\tilde{Z}_{n,i}$ entering the formulas for $p_{A/B,n}$ correspond to states x_i that are outside the non-absorbing part S_n^* of S_n. Also, the probability

$$\mathbf{P}(X_n \in S_n^*) = \mathbf{P}(\theta_B(t_n) < X_n < \theta_A(t_n)) = \sum_{i \in \mathscr{I}_n^*} Z_{n,i}$$

that the random walk does not hit or exceed the decision boundaries during the time interval generally decreases if n increases (this also explains why we avoid the term "probability distribution vector" for Z_n). With $p_{A/B,n}$ defined, we find that

$$p_A \approx \hat{p}_A := \sum_{n=1}^{N} p_{A,n}, \qquad p_B \approx \hat{p}_B := \sum_{n=1}^{N} p_{B,n}, \tag{13}$$

are approximations of the choice probabilities, and

$$\mathbf{E}(t_A) \approx \hat{t}_A := \hat{p}_A^{-1} \sum_{n=1}^{N} p_{A,n} t_n, \qquad \mathbf{E}(t_B) \approx \hat{t}_B := \hat{p}_B^{-1} \sum_{n=1}^{N} p_{B,n} t_n, \tag{14}$$

approximations of the expected choice response times (assuming non-zero values for $\hat{p}_{A/B}$). Moreover, approximations to the cumulative distribution function $F_{t_A}(t)$ of the choice response time t_A for alternative A can be computed by

$$F_{t_A}(t_n) \approx \hat{p}_A^{-1} \sum_{m=1}^{n} p_{A,m}, \qquad \text{if } \hat{p}_A > 0, \tag{15}$$

similarly approximations for $F_{t_B}(t_n)$ are available[6].

[6]In the case of non-constant decision boundaries, some post-processing and smoothing is necessary to produce faithful approximations to the probability density functions since the rough discretization of $\theta_{A/B}(t)$ determining the state spaces S_n in Eq. 8 leads to visible oscillations in the time series $\{p_{A/B,n}\}_{n=1,\ldots,N}$.

Unless $S_N^* = \emptyset$, we end up with a positive value for the probability

$$p_N := 1 - p_A - p_B = \sum_{i \in \mathscr{I}_N^*} Z_{N,i}$$

of not coming to a decision by T_{end}, since by definition of S_n we have $0 \in S_N^*$ and generally $z_{N,0} > 0$. If $\theta_{A/B}(T_{end}) = 0$ (a case that by default should enforce a decision), we have $S_N^* = \emptyset$ and consequently $p_N = 0$.

5. Choice probabilities and decision times

In the following we present in more detail how to determine the choice probabilities, the response time distributions, and the mean response times for choosing alternatives A and B and provide numerical examples (predictions). Due to limited space we focus on a deterministic time and order schedule. For random schedules with constant boundaries we refer to Diederich and Oswald (2014).

5.1. Constant boundaries

In the case of constant decision boundaries $\theta_{A/B}(t) = \theta_{A/B}$, some simplifications are possible. In order not to overload the exposition, we only give a brief introduction to the matrix notation as used in Diederich (1997); Diederich and Busemeyer (2003); Diederich and Oswald (2014), and refer to these papers for further details. Since the state space does not depend on t_n, the transition probability matrices $\mathbf{P}_n$ needed for the recursion in Eq. 11 will have fixed size and only depend on the current attribute. Therefore, we will drop the subscript n, introduce the fixed index sets

$$\mathscr{I}^* := \{i = -m_B + 1, \ldots, m_A - 1\}, \qquad \mathscr{I} := \{i = -m_B, \ldots, m_A\}$$

related to the sets of non-absorbing states and to the state space S, respectively (compare Eq. 8). The integers $m_{A/B}$ are given by the condition

$$-m_B\Delta \leq \theta_B < (-m_B + 1)\Delta \leq (m_A - 1)\Delta < \theta_A \leq m_A\Delta.$$

If the k-th attribute is considered during the time interval $(t_{n-1}, t_n]$ then the part of the transition probability matrix needed in Eq. 11 depends only on the parameters of this attribute. It will be denoted by $\tilde{\mathbf{P}}^{(k)}$, and is given by

$$\tilde{\mathbf{P}}^{(k)} = \left[\; R_{B,k} \; \middle| \; Q_k \; \middle| \; R_{A,k} \; \right], \tag{16}$$

where the square submatrix

$$Q_k = \begin{pmatrix} p^{(k)}_{-m_B+1,-m_B+1} & p^{(k)}_{-m_B+1,-m_B+2} & 0 & \cdots & 0 & 0 \\ p^{(k)}_{-m_B+2,-m_B+1} & p^{(k)}_{-m_B+2,-m_B+2} & p^{(k)}_{-m_B+2,-m_B+3} & \cdots & 0 & 0 \\ 0 & p^{(k)}_{-m_B+3,-m_B+2} & p^{(k)}_{-m_B+3,-m_B+3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p^{(k)}_{m_A-2,m_A-2} & p^{(k)}_{m_A-2,m_A-1} \\ 0 & 0 & 0 & \cdots & p^{(k)}_{m_A-1,m_A-2} & p^{(k)}_{m_A-1,m_A-1} \end{pmatrix}, \tag{17}$$

with entries $p^{(k)}_{i,j} = p_{n,i,j}$ from Eq. 9 for $k = k_l$, corresponds to the non-absorbing states $\theta_B < x_i < \theta_A$ with index set $\mathscr{I}^*$. Its dimension is

$$N_\theta := m_A + m_B - 1 \approx (\theta_A - \theta_B)/\Delta.$$

The vectors $R_{A,k}$ and $R_{B,k}$ contain the transition probabilities to the absorbing states $x_{-m_B} \leq \theta_B$ and $x_{m_A} \geq \theta_A$, respectively. They are used for computing the exit probabilities, whereas multiplying the probability vector at t_{n-1} by Q_k yields the vector Z_n containing the probabilities for being in one of the non-absorbing states at time t_n.

A particular path $\{X_n\}_{n\geq 0}$ with $X_0 \in S$ is absorbed at $m_A\Delta$ (decision for A) if there is an integer $n_A > 0$ (the decision time index) such that $X_{n_A} = m_A\Delta$ but $X_n \in S^*$ for all $n < n_A$. If it is never absorbed at $m_A\Delta$, we set $n_A = \infty$. The decision time index n_B is similarly defined. Then, by definition,

$$\hat{p}_A := \mathbf{P}(n_A < \infty) = \sum_{l=1}^{L} p_{A,l}, \qquad p_{A,l} := \mathbf{P}(n_{l-1} < n_A \leq n_l), \quad l = 1, \ldots, L,$$

similarly for $\hat{p}_B$. Note that we do not exclude the case of an infinite time horizon $T_{end} = T_L = \infty$, in which case we silently assume $n < n_L = \infty$ in the definition of $p_{A,L}$.

Let the $N_\theta \times 1$ column vectors Z_n, $n \geq 0$, contain the probabilities $Z_{n,i} = \mathbf{P}(X_n = x_i)$, $i \in \mathscr{I}^*$ (the initial distribution Z_0 must be provided). With the n_l and k_l given by the time and order schedule, we have

$$Z_n' = \begin{cases} Z_0' Q_{k_1}^n, & 0 = n_0 < n \leq n_1, \\ Z_0' Q_{k_1}^{n_1} Q_{k_2}^{n-n_1}, & n_1 < n \leq n_2, \\ \quad \dots & \\ Z_0' Q_{k_1}^{n_1} \dots Q_{k_2}^{n_{L-1}-n_{L-2}} Q_{k_{L-1}}^{n-n_{L-1}}, & n_{L-1} < n < \infty. \end{cases}$$

Moreover, the probability of choosing alternative A at time $t_n \in \Delta T_l$, $l = 1, \dots, L$, is

$$\begin{aligned} \mathbf{P}(n_A = n) &= \mathbf{P}(X_{n-1} = x_{m_A-1}) P(X_n = x_{m_A} | X_{n-1} = x_{m_A-1}) \\ &= Z_{n-1,m_A-1} p_{m_A-1,m_A}^{(k_l)} = Z_{n-1}' R_{A,k_l}, \end{aligned}$$

where $R_{A,k}$ is the $N_\theta \times 1$ column vector with the last entry $r_{m_A-1} = p_{m_A-1,m_A}^{(k)}$, and $r_i = 0$ for the remaining $i = -m_B + 1, \dots, m_A - 2$.

With these formulas at hand, we get expressions for $p_{A,l}$ and $\hat{p}_A$ in a more compact form, as shown in (Diederich, 1997). Denote $\Delta n_l = n_l - n_{l-1}$, $l = 1, \dots, L$. Then for these l

$$p_{A,l} = Z_{n_{l-1}}' \left(\sum_{r=0}^{\Delta n_l - 1} Q_{k_l}^r \right) R_{A,k_l}, \qquad Z_{n_l}' = Z_{n_{l-1}}' Q_{k_l}^{\Delta n_l}, \tag{18}$$

whereas in the case $n_L = \infty$ (infinite time horizon), the formulas for $l = L$ are replaced by

$$p_{A,L} = Z_{n_{L-1}}' \left(\sum_{r=0}^{\infty} Q_{k_L}^r \right) R_{A,k_L} = Z_{n_{L-1}}' (I - Q_{k_L})^{-1} R_{A,k_L}, \qquad Z_{n_L}' = 0. \tag{19}$$

Because

$$Z_{n_{l-1}}' \left(\sum_{r=0}^{\Delta n_l - 1} Q_{k_l}^r \right) = Z_{n_{l-1}}' (I - Q_{k_l}^{\Delta n_l})(I - Q_{k_l})^{-1} = (Z_{n_{l-1}}' - Z_{n_l}')(I - Q_{k_l})^{-1},$$

a recursive evaluation of the Z_{n_l}, $p_{A.l}$, and eventually of $\hat{p}_A$ can be orchestrated by a linear algebra operations involving a few tridiagonal matrices. We note that a direct evaluation of $p_{A,l}$ using Eq. 18 might be even faster (or at least feasible) for reasonable N_θ and Δn_l.

Similar formulas also hold for p_B (just replace $R_{A,k}$ by the corresponding $R_{B,k}$), and for the conditional expected decision times

$$\begin{aligned}\hat{t}_A &= \mathbf{E}(t_{n_A}|n_A<\infty) = \sum_{l=1}^{L}\sum_{n=n_{l-1}+1}^{n_l} t_n\mathbf{P}(n_A=n)/\hat{p}_A,\\ \hat{t}_B &= \mathbf{E}(t_{n_B}|n_B<\infty) = \sum_{l=1}^{L}\sum_{n=n_{l-1}+1}^{n_l} t_n\mathbf{P}(n_B=n)/\hat{p}_B,\end{aligned}$$

where $t_n = T_{l-1}+(n-n_{l-1})\tau_{k_l}$ for $n=n_{l-1}+1,\dots,n_l$, $l=1,\dots,L$. Substituting this together with the formulas for $\mathbf{P}(n_A=n)$ and the recursion for Z_n, we obtain

$$\begin{aligned}&\sum_{n=n_{l-1}+1}^{n_l} t_n Z'_{n_{l-1}} Q_{k_l}^{n-1-n_{l-1}} R_{A,k_l}\\ &\quad = T_{l-1}p_{A,l} + \tau_{k_l} Z'_{n_{l-1}}\left(\sum_{r=1}^{\Delta n_l} r Q_{k_l}^{r-1}\right) R_{A,k_l}\\ &\quad = T_{l-1}p_{A,l} + \tau_{k_l} Z'_{n_{l-1}}[(I-Q_{k_l}^{\Delta n_l})(I-Q_{k_l})^{-2} - \Delta n_l Q_{k_l}^{\Delta n_l}(I-Q_{k_l})^{-1}]R_{A,k_l}\\ &\quad = T_{l-1}p_{A,l} + \tau_{k_l}[(Z'_{n_{l-1}} - Z'_{n_l})(I-Q_{k_l})^{-1} - \Delta n_l Z'_{n_l}](I-Q_{k_l})^{-1})R_{A,k_l}.\end{aligned}$$

If $n_L=\infty$ and $l=L$, the above formula has to be replaced by

$$\begin{aligned}\sum_{n=n_{L-1}+1}^{\infty} t_n Z'_{n_{L-1}} Q_{k_L}^{n-1-n_{L-1}} R_{A,k_L} &= T_{L-1}p_{A,L} + \tau_{k_L} Z'_{n_{L-1}}\left(\sum_{r=1}^{\infty} r Q_{k_L}^{r-1}\right) R_{A,k_L}\\ &= T_{L-1}p_{A,L} + \tau_{k_L} Z'_{n_{L-1}}(I-Q_{k_L})^{-2} R_{A,k_L}.\end{aligned}$$

Therefore, for $T_{end}<\infty$, we arrive at

$$\hat{t}_A = \hat{p}_A^{-1}\left(\sum_{l=1}^{L} T_{l-1}p_{A,l} + \sum_{l=1}^{L}\tau_{k_l}[(Z'_{n_{l-1}} - Z^T_{n_l})(I-Q_{k_l})^{-1} - \Delta n_l Z'_{n_l}](I-Q_{k_l})^{-1}R_{A,k_l}\right).$$

The term with $l=L$ in the last sum has to be replaced by $\tau_{k_L} Z^T_{n_{L-1}}(I-Q_{k_L})^{-2}R_{A,k_L}$ if $T_{end}=\infty$. A similar formula holds for $\hat{t}_B$, by replacing R_{A,k_l} with R_{B,k_l}, and $\hat{p}_A$ by $\hat{p}_B$. Compared to the evaluation of choice probabilities, the computation of $\hat{t}_{A/B}$ only requires the solution of one additional tridiagonal linear per attribute switch system corresponding to a matrix vector multiplication by $(I-Q_{k_l})^{-1}$.

5.2. Numerical examples: Constant boundaries

In the following we present predictions of the model for choice alternatives with $K = 2$ attributes and with finite and infinite time horizon T_{end}.

Throughout this section we fix the following parameters: $\sigma = 1$, $\xi = 1$, $\Delta = \frac{1}{4}$, $\theta_A = -\theta_B = 15$; and the process always starts at the neutral position $X(0) = 0$ between choice alternatives A and B. For simplicity, we set $\gamma_1 = \gamma_2 = 0$, i.e. we assume a Wiener process instead of an OUP. Furthermore, the two attributes are considered only once (i.e., $L = 2$).

Figure 5 shows the choice probabilities and mean choice response times as a function of the attention switching time T_1 for the attribute considered first with finite and infinite time horizon for the second attribute for two order schedules. In case of an infinite time horizon (lines and dashed lines), the first attribute is considered until time $t = T_1$ time units, and the second attribute $T_2 = T_{end} = \infty$. In case of a finite time horizon (dotted lines), the time is set to $T_2 = T_{end} = 500$. That is, attribute k_1 is considered first for T_1 time units, after which attribute k_2 is considered during the remaining $T_2 - T_1 = 500 - T_1$ time units. In this case there is a positive probability that none of the alternatives have been chosen in the given time frame. The drift parameters for attributes 1 and 2 are $\delta_1 = 0.1$ and $\delta_2 = 0.01$, respectively. The left panels show the predictions of the order schedule $k_1 = 1$, $k_2 = 2$; the right panels the predictions of the order schedule $k_1 = 2$, $k_2 = 1$.

Consider the order schedule $k_1 = 1, k_2 = 2$ with $\delta_1 = 0.1$ and $\delta_2 = 0.01$ for attribute 1 and 2, respectively, first (left panels). Regardless of the time horizon, the model predicts for both scenarios faster response times for the more frequently chosen alternative, here A. Compared to an infinite time horizon the probabilities for choosing A and B in a finite time horizon are reduced (almost by a constant amount) and a no-decision (time-out) is predicted in about 10 percent of the cases (probability for choosing none of the alternatives) for small T_1. The mean choice response times for A (the more frequently chosen alternative) are slightly longer for the infinite time horizon than for the finite time horizon but similar in shape as a function of T_1. The mean

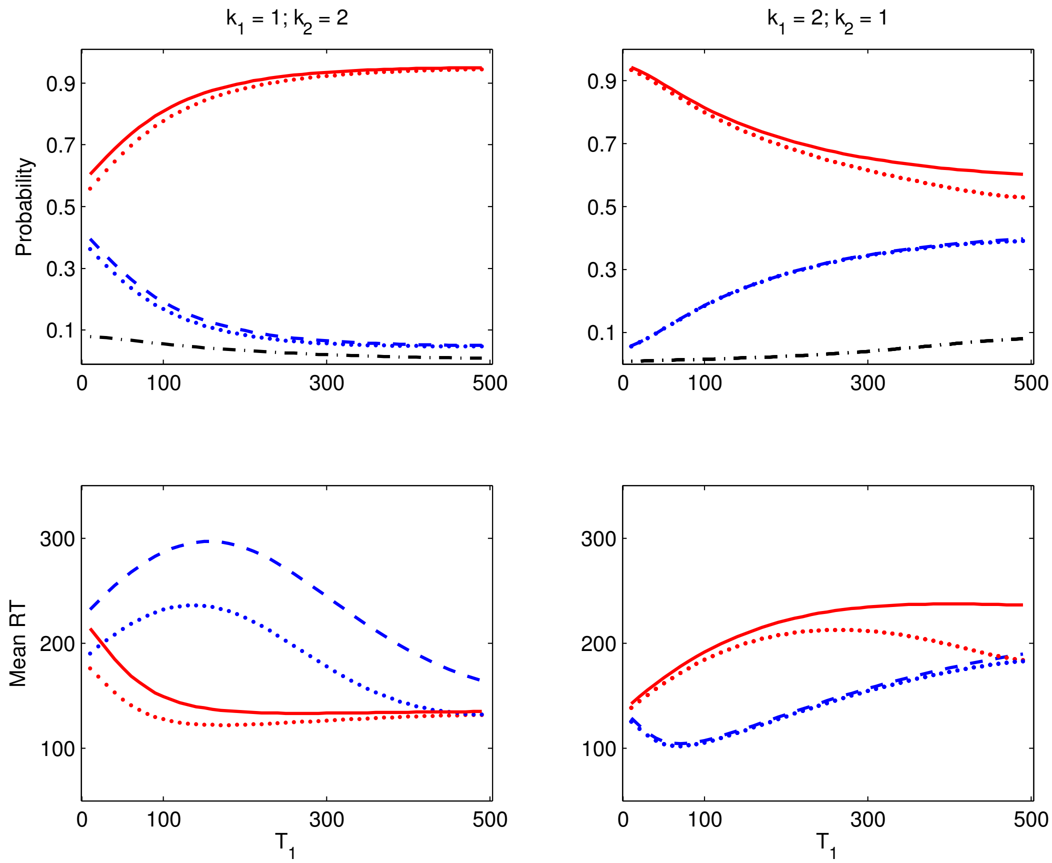

Figure 5: Choice probabilities and mean choice response times as a function of attention switching time T_1 for two order schedules (k_1, k_2) with infinite ($T_2 = T_{end} = \infty$) and finite ($T_2 = T_{end} = 500$) time horizon. Solid (red) and dashed (blue) lines show the predictions with infinite time horizon for choosing option A and B, respectively. Dotted lines (red and blue) show the predictions with finite time horizon for A and B, respectively. The (black) dot-dashed lines in the upper panels indicate the probability for choosing none of the options. The drift parameters for attributes 1 and 2 are $\delta_1 = 0.1$ and $\delta_2 = 0.01$, respectively.

choice response times for B (the less frequently chosen alternative) differ substantially more for both time horizons. The overall shapes are, however, similar.

Reversing the order schedule ($k_1 = 2, k_2 = 1$) (right panels) the model predicts faster response times for the less frequently chosen alternative. This is a consistent pattern for particular drift rate constellations and represents a very important characteristic of the multi-stage model. If both drift rates point into the same directions and the drift rate of the attribute considered first is larger (in absolute value, i.e. more evidence) than the drift rate of the attribute considered second, then the multi-stage model *always* predicts faster mean response time to the more frequently chosen alternative. If, however, the drift rate of the attribute considers first is smaller (in absolute value, i.e., less evidence) than the drift rate of the attribute considered second, then the multi-stage model *always* predicts faster mean response times to the less frequently chosen alternative, B. The psychological interpretation of the pattern is that if alternative B (often the incorrect one) is chosen, the answer tends to be fast before later on, new information gives even more evidence in favor of alternative A. These patterns hold, regardless of the underlying distribution of T (Diederich and Oswald, 2014) and, as shown here, regardless of the time horizon. The finite time horizon has only a small effect on the choices for A for larger T_1.

The choice probability/choice response patterns for an alternative with conflicting attributes, i.e. one is in favor of alternative A and the other in favor of choosing alternative B, is a bit more complex but also shows a consistent pattern. For demonstration, consider Figure 6 with $\delta_1 = -0.1$ and $\delta_2 = 0.03$. The left panels correspond to the predictions for order schedule $k_1 = 1, k_2 = 2$, the right ones of a finite time horizon for order schedule $k_1 = 2, k_2 = 1$. Regardless of the time horizon and order schedule, a preference reversal as a function of the attention switching time T_1 is predicted. That is, the probabilities for choosing one alternative over the other change from below (above) 0.5 to above (below) 0.5 as attention time for the attribute considered first increases. The larger $|\delta|$ is of the attribute considered first, the sooner the reversal occurs as a function of the attention time (Diederich, 2015). The model predicts slow response times for the more frequently chosen alternative *before* the preference reversal and faster responses for the more frequently chosen alternative *after* the preference reversal (e.g. in Figure 6, left panel, alternative B).

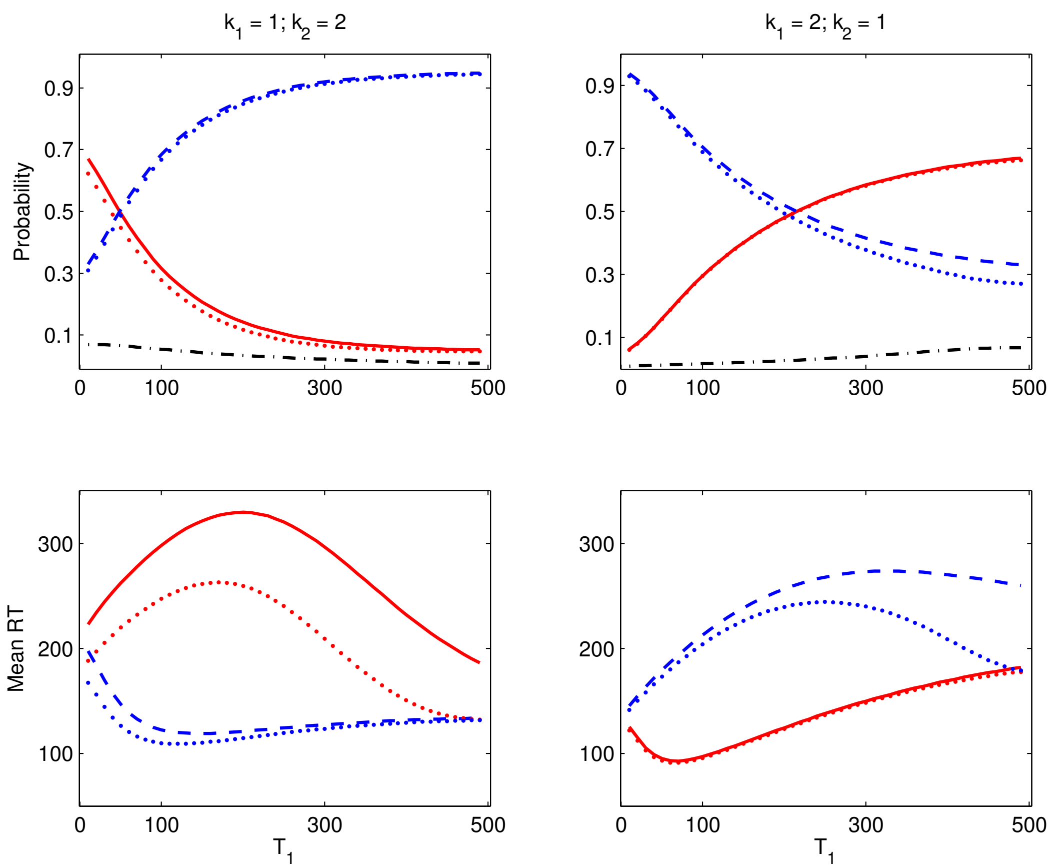

Figure 6: Choice probabilities and mean choice response times as a function of attention switching time T_1 for two order schedules (k_1, k_2) with infinite ($T_2 = T_{end} = \infty$) and finite ($T_2 = T_{end} = 500$) time horizon. Solid (red) and dashed (blue) lines show the predictions with infinite time horizon for choosing option A and B, respectively. Dotted lines (red and blue) show the predictions with finite time horizon for A and B, respectively. The (black) dot-dashed lines in the upper panels indicate the probability for choosing none of the options. The drift parameters for attributes 1 and 2 are $\delta_1 = -0.1$ and $\delta_2 = 0.03$, respectively.

Finally we present the probability density functions (pdf) and cumulative density functions (cdf) for $\delta_1 = 0.1$ and $\delta_2 = 0.01$ with three different switching times $T_1 =$

30, 50, and 100 (Figure 5.2 from top to bottom) for order schedule ($k_1 = 1, k_2 = 2$) (left panels) and order schedule ($k_1 = 2, k_2 = 1$) (right panels) and infinite time horizon (compare to Figure 5). The distributions are skewed as found in many experimental response time data; the switching times from the first attribute to the second attribute, however, are clearly reflected in the distributions.

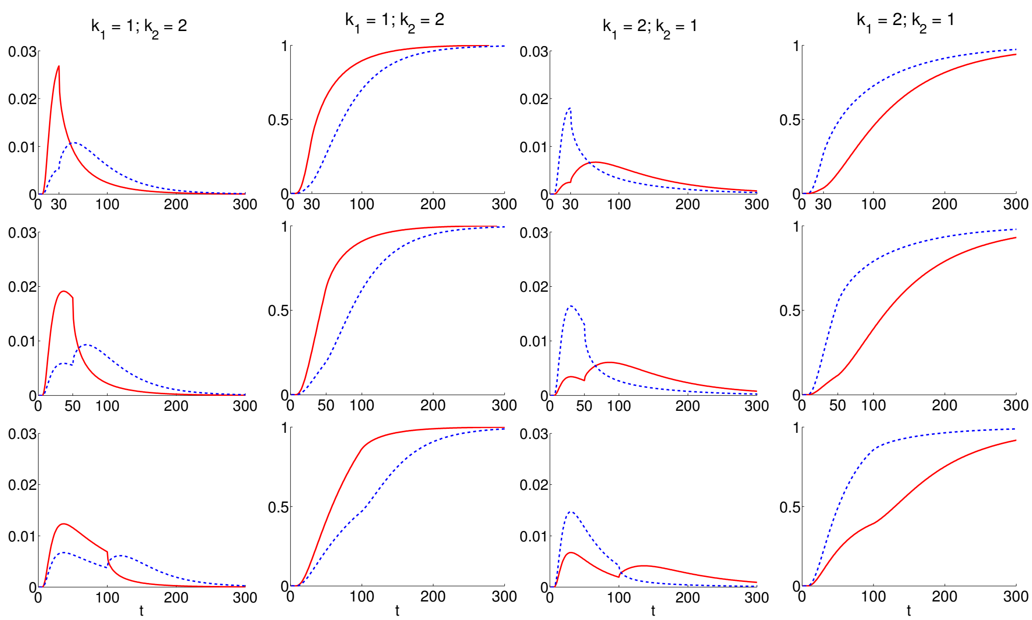

Figure 7: Probability density functions and cumulative density functions for choosing options A ((red) lines) and B ((blue) dashed line) with $\delta_1 = 0.1$ and $\delta_2 = 0.01$ with three different switching times $T_1 = 30, 50$, and 100 (from top to bottom) for order schedule ($k_1 = 1, k_2 = 2$) (left panels) and order schedule ($k_1 = 2, k_2 = 1$) (right panels) and infinite time horizon.

5.3. Non-constant boundaries

Similar to the case of constant decision boundaries situation we can determine choice probabilities and choice response times for infinite and finite time horizons, the latter with allowing for a no-decision option with probability $p_N = 1 - p_A - p_B$. In addition, we can also implement another decision rule: When at a given deadline the accumulated evidence is larger than a criterion value $\theta_A(T_{end})$ decide for A, when the accumulated evidence is smaller than a criterion value $\theta_B(T_{end})$ decide for B. If evidence at that time is between these criterion values choose A or B with probability 0.5. That is,

$$
\begin{aligned}
p^{+}_{A,N} &:= \sum_{i:\,\theta_A(T_{end})\leq x_i\in S_N} Z_{N,i} + \frac{1}{2}\sum_{i:\,x_i\in S_N,\,\theta_B(T_{end})<x_i<\theta_A(T_{end})} Z_{N,i},\\
p^{+}_{B,N} &:= \sum_{i:\,\theta_B(T_{end})\geq x_i\in S_N} Z_{N,i} + \frac{1}{2}\sum_{i:\,x_i\in S_N\,\theta_B(T_{end})<x_i<\theta_A(T_{end})} Z_{N,i},
\end{aligned}
$$

added to the choice probabilities p_A and p_B computed by the Eqs. 13 and 14. The constants $\theta_B(T_{end}) \leq 0 \leq \theta_A(T_{end})$ may reflect last-minute decision making (if such is observed in measured data) or may be relevant for modeling decision processes with externally controlled stopping procedures.

In contrast to the case of constant boundaries $\theta_{A/B}$, where we relied on the shortcuts presented in Section 5.1, the implementation of the model for non-constant decision boundaries is directly based on the recursion in Eq. 11 for the probability vectors Z_n defined in Eq. 10. Choice probabilities, mean choice response times, and conditional cumulative distribution functions (cdfs) of exit times are determined from Eqs. 12 to 15. With non-constant decision boundaries, the state space shrinks according to the specific boundary assumed for the process which is reflected in the size of state probability vector.

There is one drawback of our discretization scheme if it comes to the approximation of conditional probability density functions (pdfs) for exit times using the formulas

$$
f_{t_A}(t_n) \approx p_{A,n}/\hat{p}_A \;\; (\text{if } \; \hat{p}_A > 0), \qquad f_{t_B}(t_n) \approx p_{B,n}/\hat{p}_B \;\; (\text{if } \; \hat{p}_B > 0), \tag{20}
$$

in the case of non-constant decision boundaries: Each time one of the threshold functions $\theta_{(A/B)}(t)$ crosses a spatial grid value x_i during a time interval $(t_{n-1}, t_n]$, the state space shrinks at t_n, creating at least one additional absorbing state at t_n, and an additional entry of the probability vector $\tilde{Z}_n$ enters the summation for determining $p_{A/B,n}$ in Eq. 12. This leads to relatively large, visible spikes and oscillations in the graphical display of approximate pdfs.

To reduce the observed oscillations in the time series of exit probabilities $p_{A,n}$ and $p_{B,n}$, we have implemented an ad hoc modification of the exit boundary rule. The new formula for $p_{A,n}$ we use is

$$p_{A,n} = \sum_{i:x_i>\theta_A(t_n)} z_{n,i} + \frac{x_{i^*+1} - \theta_A(t_n)}{\Delta} z_{n,i^*}, \tag{21}$$

where i^* is the largest integer i such that $x_i \leq \theta_A(t_n)$. The rationale of this modification is to already assign a significant part of the probability of the state closest to the exit boundary to the current exit probability. To keep the probability balance, afterwards the value of z_{n,i^*} is reduced to

$$z_{n,i^*} := \frac{\theta_A(t_n) - x_{i^*}}{\Delta} z_{n,i^*}.$$

The decision boundary for B is treated similarly. This should reduce the spikes and oscillations in the time series of exit probabilities. We refer to the Section 6 for numerical evidence, and further postprocessing steps.

5.4. Numerical examples: Non-constant boundaries

In the following we consider only situations in which the decision maker makes the decision within the given time frame $T_{end} = 500$ by assuming $\theta_A(T_{end}) = \theta_B(T_{end}) = 0$. Consequently, $p_N = 0$. As for the examples with constant boundary we fix the following parameters: $\xi = 1$, $\Delta = \frac{1}{4}$, $\sigma = 1$, $\theta_A = -\theta_B = 15$. The process always starts at the neutral position $X(0) = 0$ between choice alternatives A and B. We show the predictions of the model with non-constant boundaries according to Eq. 3 with parameter values $a = 0, 0.1, 0.5, 1, 2, 3$ (cf. Figure 2) and predictions according to Eq. 4 with parameter values $b = 1, 0.8, 0.6, 0.4, 0.2, 0$ (cf. Figure 3). For comparison we use two δ parameter value sets from the previous examples. Figures 8 and 9 show choice probabilities and mean choice response times with $\delta_1 = 0.1, \delta_2 = 0.01$ and order schedule $k_1 = 2, k_2 = 1$ as a function of the switching time T_1 for the above a and b parameter values, respectively. The color code is the same as for Figures 2 and 3. For both types of non-constant decision boundaries the choice probabilities for A decrease as a increases (b decreases); likewise, the predicted mean response times decrease as a increases (b decreases). The cases $a = 0$ in Figure 8 and $b = 1$ in Figure 9 should be

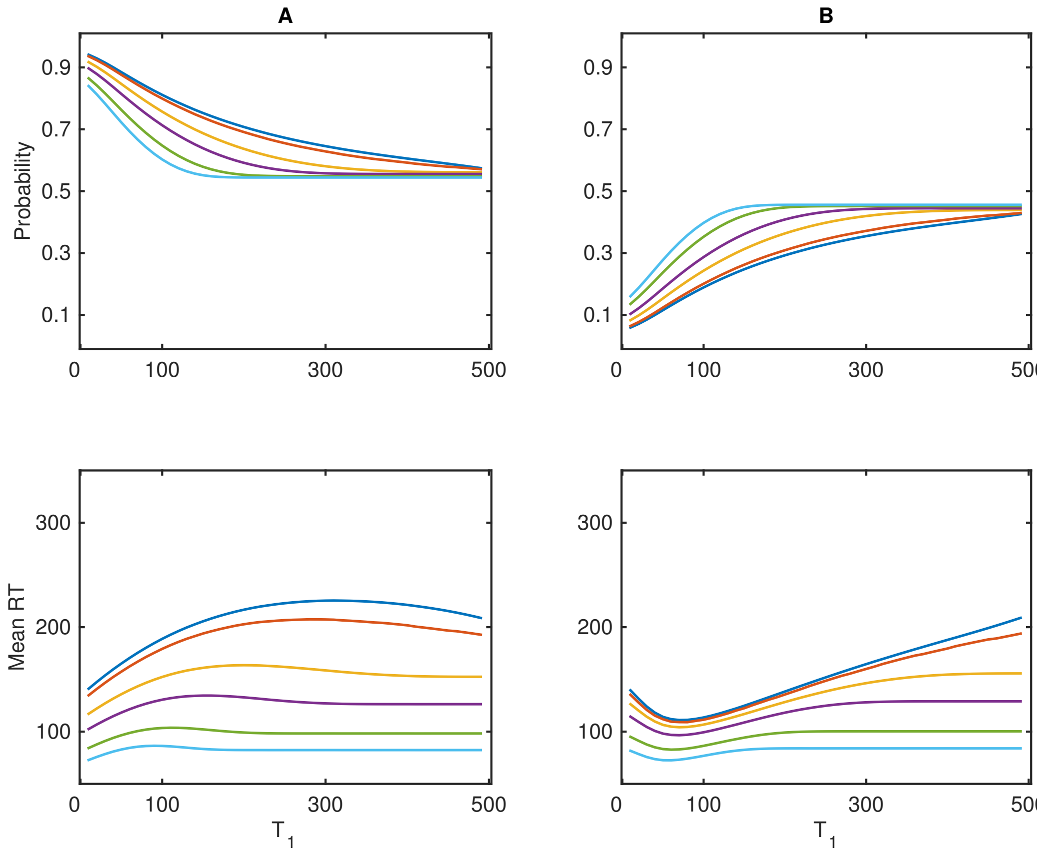

Figure 8: Choice probabilities and mean choice response times for options A (left) and B (right) as a function of the switching time T_1 with $\delta_1 = 0.1, \delta_2 = 0.01$ and order schedule $k_1 = 2, k_2 = 1$ for non-constant boundaries with $a = 0, 0.1, 0.5, 1, 2, 3$. The probability for choosing A decreases as a increases; the mean choice response time for A and B decreases as a increases.

compared also with Figures 5, right panels, for constant boundaries with infinite and finite time horizon.

Figures 10 and 11 show the predicted choice probabilities and mean choice response times with $\delta_1 = -0.1, \delta_2 = 0.03$ and order schedule $k_1 = 1, k_2 = 2$ as a function of the switching time T_1 for the above a and b parameter values, respectively. The color code is the same as for Figures 2 and 3. For both non-constant boundary models the choice probabilities change little as a function of a respectively b. The predicted mean response times, however, decrease as a increases (b decreases). Compare the

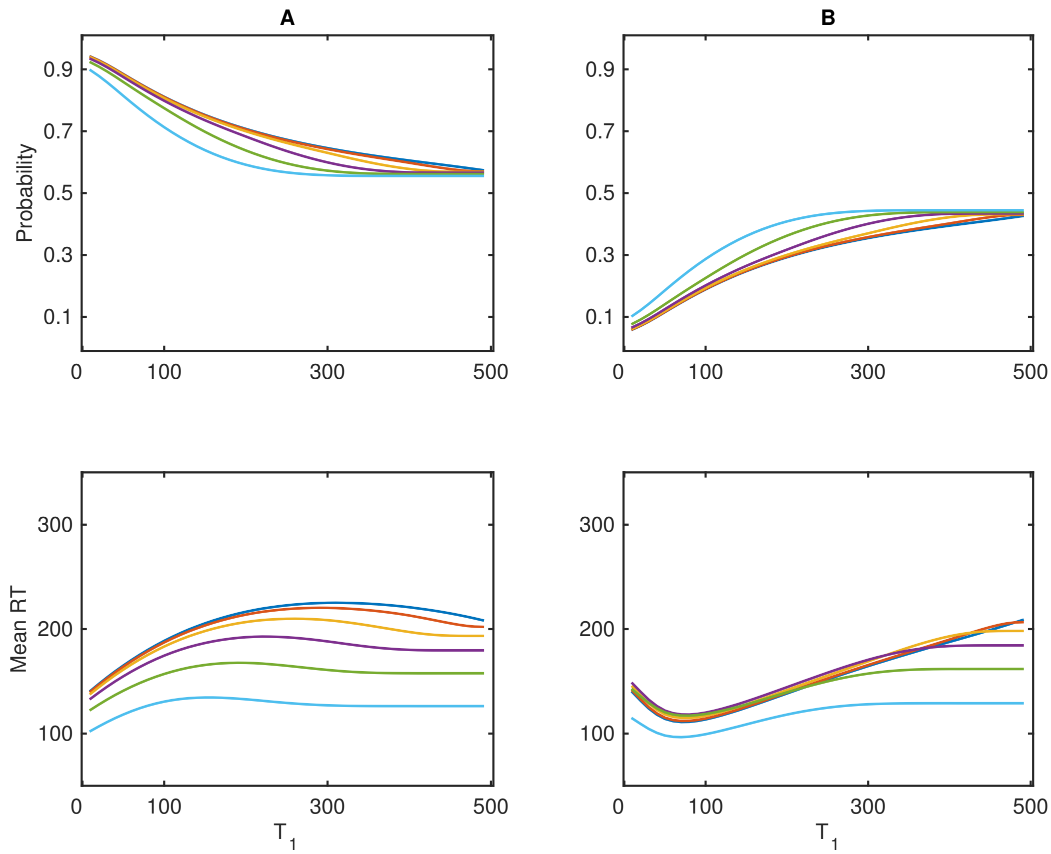

Figure 9: Choice probabilities and mean choice response times for options A (left) and B (right) as a function of the switching time T_1 with $\delta_1 = 0.1, \delta_2 = 0.01$ and order schedule $k_1 = 2, k_2 = 1$ for non-constant boundaries with $b = 1, 0.8, 0.6, 0.4, 0.2, 0$. The probability for choosing A decreases as a decreases; the mean choice response time for A and B decreases as a decreases.

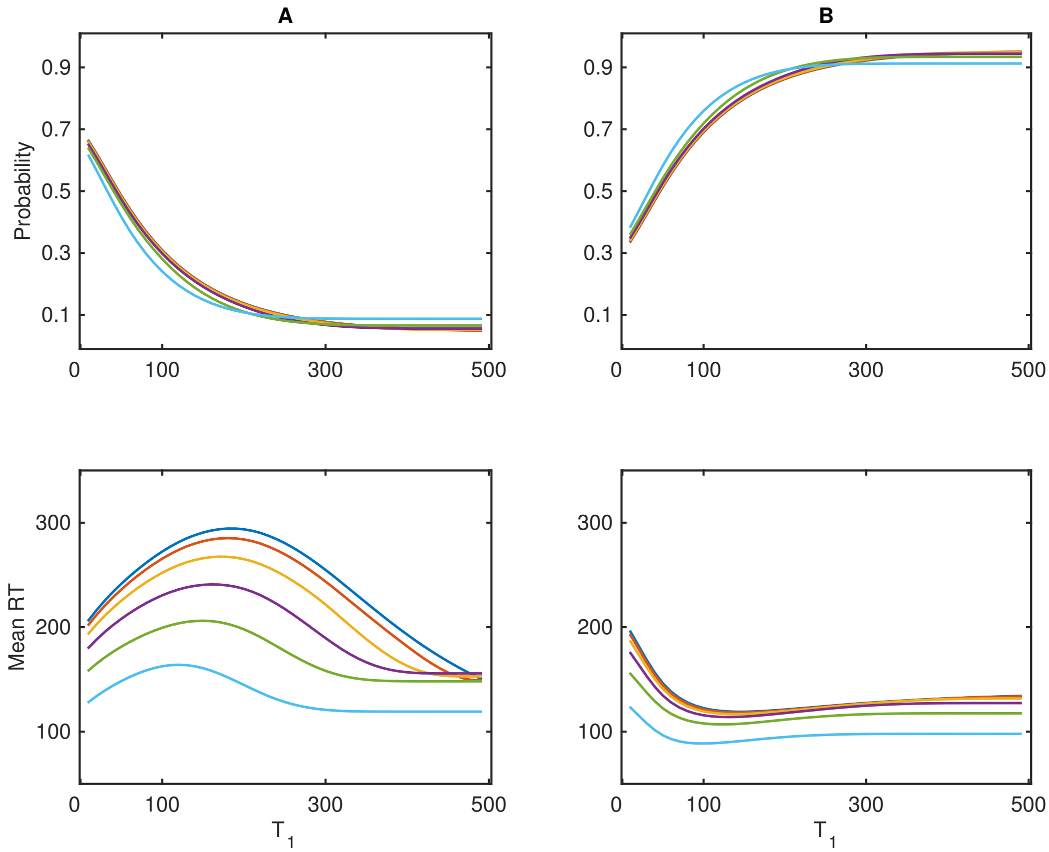

Figure 10: Choice probabilities and mean choice response times for options A (left) and B (right) as a function of the switching time T_1 with $\delta_1 = -0.1, \delta_2 = 0.03$ and order schedule $k_1 = 1, k_2 = 2$ for non-constant boundaries with $a = 0, 0.1, 0.5, 1, 2, 3$. The probabilities are only slightly affected by a; the mean choice response time for A and B decreases as a increases.

cases $a = 0$ in Figure 10 and $b = 1$ in Figure 11 also with Figures 6, right panels, for constant boundaries with infinite and finite time horizon.

Figure 12 presents the probability density functions and cumulative density functions for $\delta_1 = 0.1$ and $\delta_2 = 0.01$ with three different switching times $T_1 = 30, 50$, and 100 (from top to bottom) for order schedule $(k_1 = 1, k_2 = 2)$ (left panels) and order schedule $(k_1 = 2, k_2 = 1)$ (right panels) with finite time horizon $T_{end} = 300$ and non-constant boundaries $\theta_A(t) = -\theta_B(t) = 15(300 - t)$.

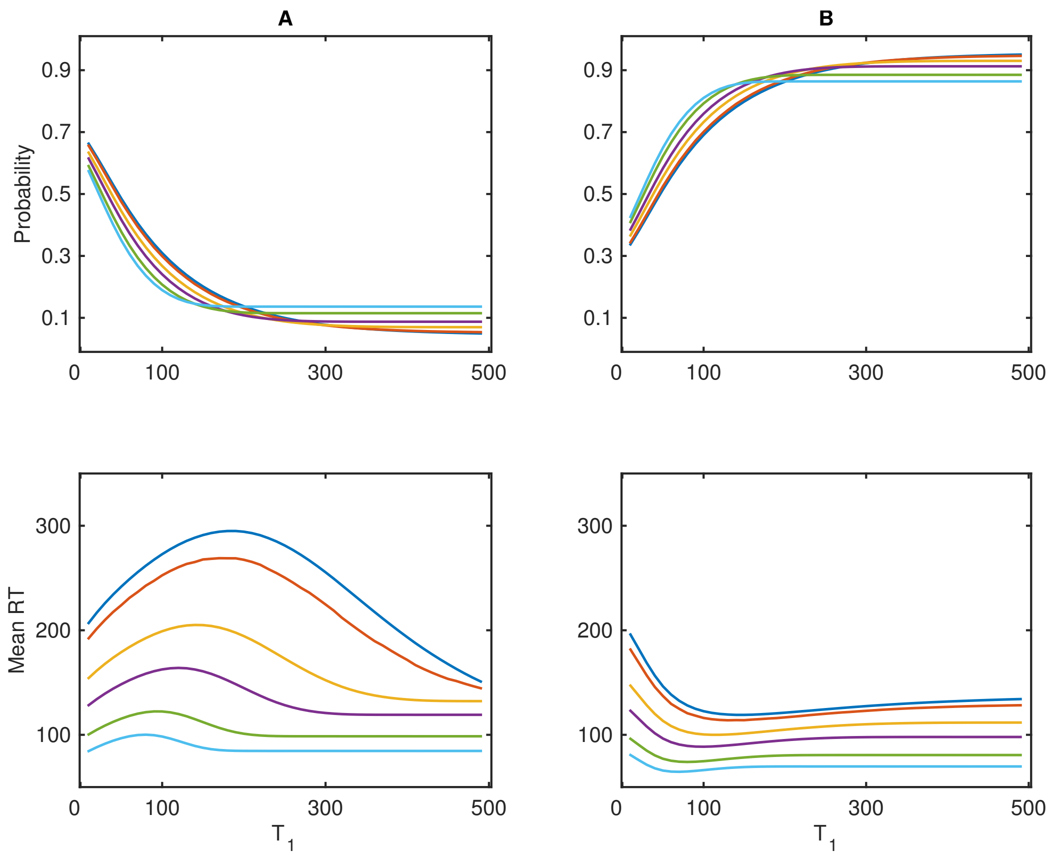

Figure 11: Choice probabilities and mean choice response times for options A (left) and B (right) as a function of the switching time T_1 with $\delta_1 = -0.1, \delta_2 = 0.03$ and order schedule $k_1 = 1, k_2 = 2$ for non-constant boundaries with $b = 1, 0.8, 0.6, 0.4, 0.2, 0$. The probability for choosing A decreases as a decreases and then increases for larger T_1; the mean choice response time for A and B decreases as a decreases.

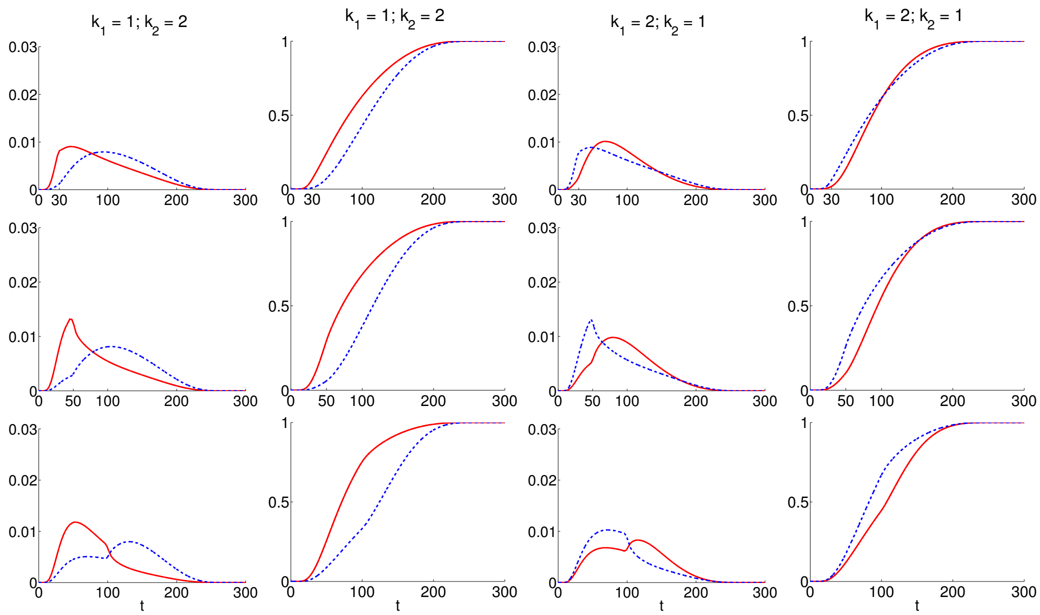

Figure 12: Probability density functions and cumulative density functions for choossing options A ((red) lines) and B ((blue) dashed line) with $\delta_1 = 0.1$ and $\delta_2 = 0.01$ with three different switching times $T_1 = 30, 50$, and 100 (from top to bottom) for order schedule ($k_1 = 1, k_2 = 2$) (left panels) and order schedule ($k_1 = 2, k_2 = 1$) (right panels) with finite time horizon and non-constant boundary.

6. Approximation quality

To demonstrate the convergence of our discrete approach to the continuous model we consider two numerical examples. We furthermore discuss the influence of the parameter ξ.

The first example includes a single standard Wiener process ($\delta = \gamma = 0$, $\sigma = 1$), finite time horizon $T_{end} = 4$, and constant decision boundaries: $\theta_A = 1.2$, $\theta_B = 0.8$. For this model, all quantities of interest can be expressed analytically (Borodin and Salminen, 2002), even though their evaluation still involves numerical effort (we have used the series representations for exit time pdfs from Sacerdote et al. (2014, Equation (6)) but conditioned on $t_A \leq T_{end}$ resp. $t_B \leq T_{end}$ to compute values for choice probabilities $p_{A/B}$ and mean choice response times $\mathbf{E}(t_{A/B})$ within double precision). The example shows that the Markov chain approximation delivers highly accurate approx-

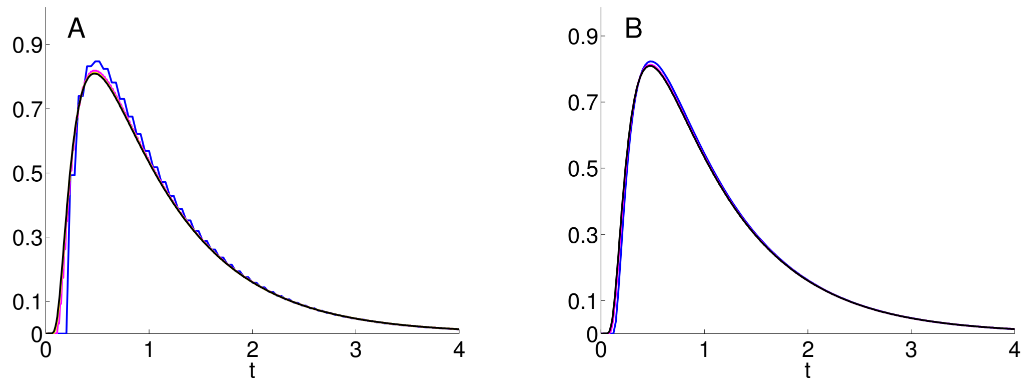

Figure 13: Conditional pdfs for mean choice response times $\hat{t}_A$ for a Wiener process with drift and constant decision horizons for different spatial resolutions (A: binomial model $\xi = 1$, B: trinomial model $\xi = 1.3$).

imations to the conditional pdfs for relatively small state space sizes N_θ (naturally, Δ is chosen such that the values $\theta_{A/B}$ are among the grid points, i.e., the grid matches the decision thresholds exactly). The approximations, shown for $N_\theta = 10, 20, 40, 80, 160$, are slightly better when using the trinomial model ($\xi = 1.3$, Figure 13, B) than the binomial model ($\xi = 1$, Figure 13, A) (see also Table 1). To avoid oscillations inherent to the binomial model, the time series $p_{A/B,n}$, $n = 1, \ldots, N$, have been smoothed by simply averaging neighboring values (this explains the visible stair-casing effect for small N_θ which disappears in the trinomial case). Table 1 lists the deviation of the computed approximations $\hat{p}_{A/B}$ and $\hat{t}_{A/B}$ from the "true" choice probabilities $p_{A/B}$ and mean choice response times $\mathbf{E}(t_{A/B})$ as a function of N_θ. The observed convergence is of order 2, i.e., doubling the value of N_θ results in an error reduction by roughly a factor 4. The errors in the trinomial case are smaller than in the binomial case (by a factor of about 3), at the cost of slightly increasing the number of discrete time steps.

We conclude that already very rough discretizations with less than 50 discretization points deliver good fits to the continuous model. Further increasing N_θ is probably warranted only in special applications. Note, however, that very complicated models and rapidly changing decision horizons may necessitate larger N_θ, not so much for probabilities and expected choice response times but for probability density function.

ξ	N_θ	$\hat{p}_A - p_A$	$\hat{p}_B - p_B$	$\hat{t}_A - \mathbf{E}(t_A)$	$\hat{t}_B - \mathbf{E}(t_B)$
	10	0.000171111	0.000171111	0.001542179	0.001097739
	20	0.000043095	0.000043096	0.000388195	0.000276308
1.3	40	0.000010794	0.000010794	0.000097214	0.000069194
	80	0.000002700	0.000002700	0.000024314	0.000017306
	160	0.000000675	0.000000675	0.000006079	0.000004327
	10	0.000481003	0.000481004	0.004264672	0.003039619
	20	0.000123197	0.000123197	0.001089836	0.000776705
1.0	40	0.000030984	0.000030984	0.000273938	0.000195225
	80	0.000007757	0.000007757	0.000068577	0.000048872
	160	0.000001940	0.000001940	0.000017150	0.000012222

Table 1: Error decay for choice probabilities and mean choice response times with respect to doubling state space size N_θ.

The second example includes three stages with a fixed order schedule and time schedule

$$0 < T_1 = 40 < T_2 = 70 < T_3 = T_{end} = 100 \qquad (L = 3),$$

and parameters

$$\delta_1 = \delta_3 = 0.2,\ \delta_2 = -0.4,\ \sigma_l = 1,\ \gamma_l = 0,\ l = 1,2,3.$$

That is, the continuous model consists of three Wiener processes with drift rates 0.2 (favoring A) for the first 40 and last 30 time units, and drift rate -0.4 (more strongly favoring B) for the second 30 time units. The decision horizon is given by

$$\theta_A(t) = -\theta_B(t) = 15(100 - t),$$

i.e., the boundaries decay linearly towards T_{end} leading to $p_N = 0$. Table 2 shows computed approximate values for choice probabilities $\hat{p}_A$ and expected choice response times $\hat{t}_A$ and $\hat{t}_B$ as a function of the initial state size $N_\theta = 50 \cdot 2^{-m}$, $m = 0, \ldots, 6$. Note that this is equivalent to decreasing $\Delta = 0.6 \cdot 2^{-m}$. We utilized the trinomial model

($\xi = 1.3$); the numerical results for the binomial model ($\xi = 1$, not shown here) are qualitatively the same.

The values in the left half of Table 2 (computed without boundary modification) show that expected response times are overestimated, and converge monotonically with order one (i.e., more slowly than in the case of constant decision boundaries). The values shown in the right half of Table 2 are computed with the ad hoc boundary modification described earlier (Eq. 21) and the expected response times are underestimated, and converge in an increasing fashion (the empirical order of convergence is also one, but the errors are much smaller in absolute value). This show that already for small state space sizes very good approximations can be computed if one uses the proposed boundary modification in Eq. 21, and that further improvements can be expected from incorporating extrapolation ideas.

	Boundary modification					
	without			with		
N_θ	$\hat{p}_A$	$\hat{t}_A$	$\hat{t}_B$	$\hat{p}_A$	$\hat{t}_A$	$\hat{t}_B$
50	0.602763	34.3505	58.7283	0.616360	33.1365	57.8138
100	0.607392	33.9082	58.3315	0.614780	33.2832	57.8468
200	0.609986	33.6782	58.1362	0.613827	33.3598	57.8862
400	0.611367	33.5628	58.0423	0.613332	33.4010	57.9145
800	0.612052	33.5058	57.9960	0.613049	33.4239	57.9312
1600	0.612403	33.4768	57.9728	0.612907	33.4355	57.9401
3200	0.612582	33.4621	57.9611	0.612836	33.4413	57.9447

Table 2: Convergence of choice probabilities and expected choice response times with respect to increasing state space size N_θ.

Figure 14 shows plots of the obtained pdfs (first three rows) and pdfs (last row) for $N_\theta = 50, 200, 800$ (from left to right). The first row shows the pdfs obtained without boundary modification, the second row the ones obtained with boundary modification according to Eq. 21. As can be seen, the application of the modified boundary rule greatly reduced oscillations but did not remove them completely. However, a simple post-processing can be further applied to remedy these discretization artifacts, as shown in the third row. In order to produce these pdfs, we approximated the time series of the exit probabilities $p_{A/B,n}$ by a constant value in each time interval during which the size of the current state space S_n remained constant. These constant values were chosen such that the cumulative exit probability was preserved in each such interval. In a second step, to obtain more pleasant displays, this piecewise constant function was in turn approximated by a non-negative piecewise linear function. The cdfs, shown in the fourth row of Figure 14 do not vary too much with or without boundary modification or post-processing, and cannot be distinguished for $m \geq 3$. This is important for applications because parameters are often estimated from cdfs (or quantiles or percentiles) rather than from pdfs.

7. Conclusion

The multi-stage decision aka *multiattribute attention switching* (MAAS) model assumes that attributes are processed sequentially and each attribute process is characterized by a separate stochastic sequential sampling process. It extends single-stage models which assume that all information is collapsed prior to the accumulation process regardless of whether it stems from a single source or from different sources. By defining attributes in this context very broadly, ranging from dimensions of the stimuli to experimental designs such as information given piecewise, time-out period, or cues, the model presented here provides a framework for many applications. Moreover, the model is extended here to incorporate various time horizon and boundary conditions. Absorption in the end at one decision boundary and initiating a response for A or B implicitly assumes an infinite time horizon. In real life and, in particular, in experimental situation a finite time horizon for the decision process is more realistic. Time limits are often installed either explicitly, for instance, by invoking time constraints as experimental conditions and trials not meeting the deadline are not counted, or implicitly, for instance, by removing trials with longer response times from the data set afterwards.

The multi-stage decision model allows us to account for non-decision situations by defining a non-decision probability. Furthermore, decision boundaries are almost always assumed to be constant throughout the decision process. However, research in neuroscience suggests that with elapsed time the boundaries might be collapsing (e.g. Churchland et al., 2008; Ditterich, 2006). Note that Hawkins et al. (2015) concluded that it is not necessary to assume collapsing boundaries in perceptual decision making because the diffusion model with constant boundaries performs as well as several alternatives with non-constant boundaries. This might be true for choice probabilities. However, it seems to depend very much on the specific parameters. In the multi-stage model behavior depends, again, on the absolute drift rate value for the attribute considered first and second. If more evidence is provided later in the process (the attribute considered second) the non-constant boundaries have a profound effect on the choice probabilities as well. Furthermore, constant boundaries might exist for relatively fast responses, for instance, in perception. But the situation will be different for preferential choice situations in which the decision maker contemplates about the alternatives longer and eventually wants to come to an end. Or a deadline is announced during the deliberation process of which the decision maker was not aware of. Here we showed that non-constant boundaries, also related to experimental designs, can be invoked in the model framework.

One reviewer was concerned with model identifiability. Apparently most choice response time models do not use time varying decision bounds because the distributions can be perfectly mimicked with a fixed decision bound and other varying parameters. This seems in line with Hawkins et al. (2015). The primary pursue of the present paper is not on the model's parameter identification but rather to provide a comprehensive mathematical model (framework) that can be reduced to adapt to particular modelling tasks if a practical situation provides information on what aspects to concentrate on. Such reduced models would then be used for identification tasks. Furthermore, pursuing sensitivity analysis can enable us to select those parameters which have measurable impact on observable quantities. But this will be future work. Furthermore, varying parameters to mimic non-constant boundaries leads to the same problem. Those parameters might not even be related to experimental condition such as time constraints or psychological concepts such as satisficing but merely improve the quantitative fit.

Regardless of what the conditions are, the multi-stage model predicts a very complex but consistent pattern of choice probability and mean choice response times. A large range of different parameter values showed the following patterns: If two attributes both favor alternative A, and the first attribute that is considered provides more evidence for choosing A than the second ($\delta_1 > \delta_2$), then the model predicts *always* shorter response times for the more frequently chosen alternative A, regardless of the assumed underlying attention time distribution, the time horizon (infinite or finite), or boundary conditions (constant or non-constant). If the order of processing these attributes is reversed, i.e., the attribute that favors alternative A less is considered first ($\delta_2 > \delta_1$), then the model *always* predicts faster responses for the less frequently chosen alternative B, again regardless of the assumed underlying attention time distribution, time horizon, or boundary condition.[7] As Jones and Dzhafarov (2014) pointed out, the predictions of various sequential sampling models rest upon the specific assumptions made about the assumed probability distributions. Notably, the model presented here is falsifiable without assuming specific distributions. Rather than relying on statistical assumptions to ensure an observed response pattern we rely on assumptions about cognitive processes such as attention switching and salience.

[7] A formal proof is not provided but we are convinced that the statement holds for all parameters values.

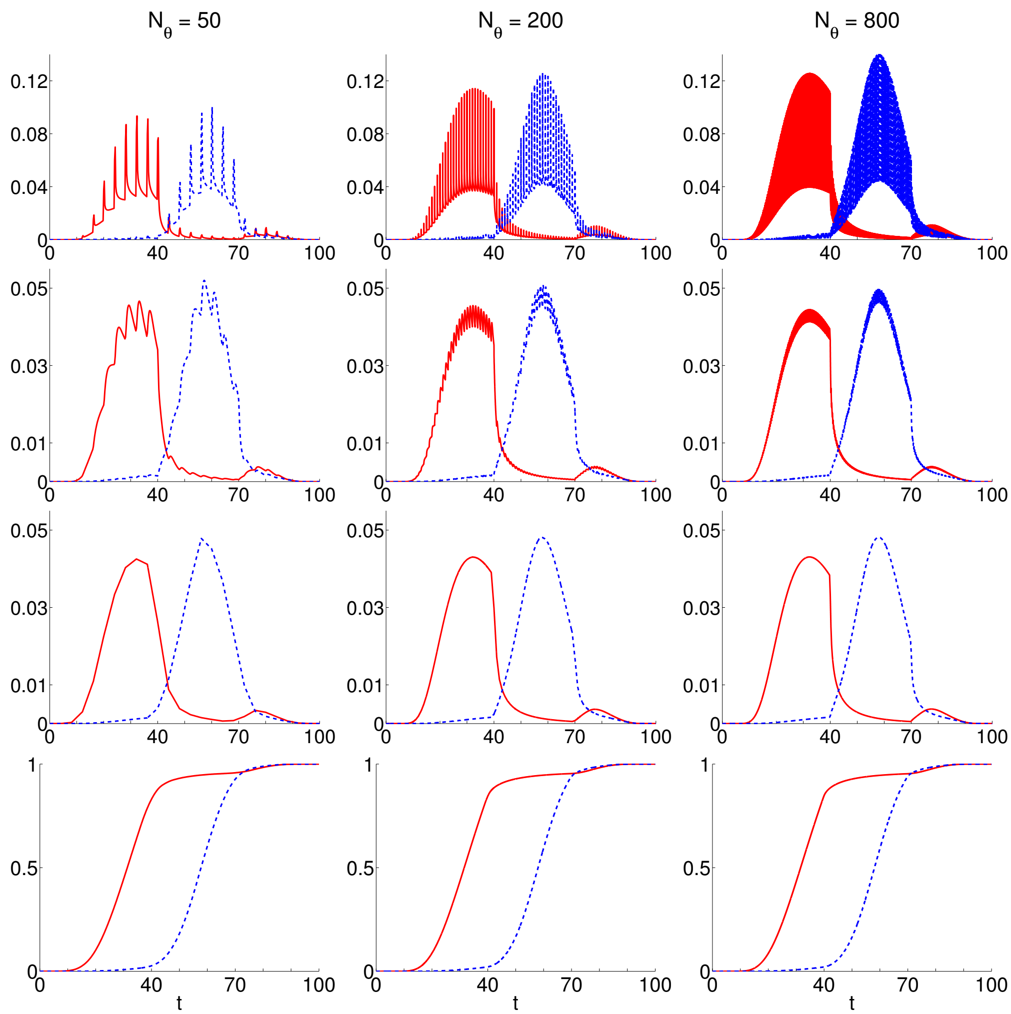

Figure 14: Approximations to pdfs and cdfs of choice response times. The columns show results for different state space magnitudes N_θ. The first three rows show pdfs obtained without and with boundary modification and after post-processing, respectively. The last row shows the cdfs which are not significantly affected by neither the state space magnitude nor by the boundary treatment or post-processing.

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Ashby, F., 1983. A biased random walk model for two choice reaction times. Journal of Mathematical Psychology 27, 277–297.

Ashby, F., 2000. A stochastic version of general recognition theory. Journal of Mathematical Psychology 44, 310–329.

Bogacz, R., Brown, E., Moehlis, J., Holmes, P., Cohen, J., 2006. The physics of optimal decision making: A formal analysis of models of performance in two-alternative forced-choice tasks. Psychological Review 113, 700–765.

Borodin, A., Salminen, P., 2002. Handbook of Brownian motion. Facts and Formulae, 2nd Edition. Birkhäuser, Basel.

Brown, S., Heathcote, A., 2005. A ballistic model of choice response time. Psychological Review 112, 117–128, doi:10.1037/0033-295X.112 .1.117.

Buonocore, A., Giorno, V., Nobile, A., Ricciardi, L., 1990. On the two-boundary first-crossing-time problem for diffusion processes. Journal of Applied Probability 27, 102–114.

Buonocore, A. V., Nobile, A., Ricciardi, L., 1987. A new integral equation for the evaluation of first-passage-time probability densities. Advances in Applied Probability 19, 1784–1800.

Busemeyer, J., Diederich, A., 2002. Survey of decision field theory. Mathematical Social Sciences 43, 345–370.

Busemeyer, J., Goldstein, W., 1992. Linking together different measures of preference: A dynamic model of matching derived from decision field theory. Organizational Behavior and Human Decision Processes 52, 370–396.

Busemeyer, J., Townsend, J., 1993. Decision field theory: A dynamic-cognitive approach to decision-making in an uncertain environment. Psychological Review 100 (3), 432–459.

Carpenter, R. H. S., Williams, M. L. L., 1995. Neural computation of log likelihood in the control of saccadic eye movements. Nature 377, 59–62.

Churchland, A., Kiani, R., Shadlen, M., 2008. Decision-making with multiple alternatives. Nature Neuroscience 11 (6), 693–702.

Diederich, A., 1995. Intersensory facilitation of reaction time: Evaluation of counter and diffusion coactivation models. Journal of Mathematical Psychology 39 (2), 197–215.

Diederich, A., 1997. Dynamic stochastic models for decision making with time constraints. Journal of Mathematical Psychology 41 (3), 260–274.

Diederich, A., 2008. A further test on sequential sampling models accounting for payoff effects on response bias in perceptual decision tasks. Perception and Psychophysics 70 (2), 229–256.

Diederich, A., 2015. A multistage attention-switching model account for payoff effects on perceptual decision tasks with manipulated processing order. Decision-Doi.org/10.1037/dec0000041.

Diederich, A., Busemeyer, J., 1999. Conflict and the stochastic dominance principle of decision making. Psychological Science 10 (4), 353–359.

Diederich, A., Busemeyer, J., 2003. Simple matrix methods for analyzing diffusion models of choice probability, choice response time and simple response time. Journal of Mathematical Psychology 47 (3), 304–322.

Diederich, A., Busemeyer, J., 2006. Modeling the effects of payoff on response bias in a perceptual discrimination task: Threshold-bound, drift rate-change, or two-stage-processing hypothesis. Perception and Psychophysics 68 (2), 194–207.

Diederich, A., Oswald, P., 2014. Sequential sampling model for multiattribute choice alternatives with random attention time and processing order. Frontiers in Human Neuroscience 8 (697).

Ditterich, J., 2006. Evidence for time-variant decision making. European Journal of Neuroscience 24, 3628–3641.

Drugowitsch, J., 2014. C++ diffusion model toolset with python and matlab interfaces. GitHub repository:404 https://github.com/jdrugo/dm, commit: 5729cd891b6ab37981ffacc02d04016870f0a998.

Edwards, W., 1965. Optimal strategies for seeking information: Models for statistics, choice reaction times, and human information processing. Journal of Mathematical Psychology 2, 312–329.

Gao, J., Tortell, R., J.L., M., 2011. Dynamic integration of reward and stimulus information in perceptual decision-making. PLoS ONE 6 (3), doi:10.1371.

Gold, J., Shadlen, M., 2007. The neural basis of decision making. Annual Review of Neuroscience 30, 535–574.

Hawkins, G., Forstmann, B., Wagenmakers, E., Ratcliff, R., Brown, S., 2015. Revisiting the evidence for collapsing boundaries and urgency signals in perceptual decision-making. The Journal of of Neuroscience 35 (6), 2476–2484.

Heath, R., 1981. A tandem random walk model for psychological discrimination. British Journal of Mathematical and Statistical Psychology 34, 76–92.

Johnson, J., Busemeyer, J., 2005. A dynamic, stochastic, computational model of preference reversal phenomena. Psychological Review 112 (4), 841–861.

Jones, M., Dzhafarov, E. N., 2014. Unfalsifiability and mutual translatability of major modeling schemes for choice reaction time. Psychological Review 121, 1–32.

Krajbich, I., Rangel, A., 2011. Multialternative drift-diffusion model predicts the relationship between visual fixations and choice in value-based choice. Proceedings of the National Academy of Sciences 108 (33), 13852–13857.

Laming, D., 1968. Information theory of choice reaction times. Academic Press, Oxford, England.

Link, S., Heath, R., 1975. A sequential theory of psychological discrimination. Psychometrika 40, 77–105.

Nosofsky, R., Palmeri, T., 1997. An exemplar based random walk model of speeded classification. Psychological Review 104, 266–300.

Pike, A., 1973. Response latency models for signal detection. Psychological Review 80, 53–68.

Pleskac, T., Busemeyer, J., 2010. Two-stage dynamic signal detection: A theory of choice, decision time, and confidence. Acta Neurobiologiae Experimentalis 117 (3), 864–901.

Rapoport, A., Burkheimer, G. J., 1971. Models for deferred decision making. Journal of Mathematical Psychology 8, 508–538.

Ratcliff, R., 1978. A theory of memory retrieval. Psychological Review 85, 59–108.

Ratcliff, R., Hasegawa, Y., Hasegawa, Y., Smith, P., Segraves, M., 2007. Dual diffusion model for single-cell recording data from the superior colliculus in a brightness-discrimination task. Journal of Neurophysiology 97, 1756–1774.

Ratcliff, R., Rouder, J., 2000. A diffusion model account of masking in two-choice letter identification. Journal of Experimental Psychology: Human Perception and Performance 26 (1), 127–140.

Ratcliff, R., Smith, P., 2004. A comparison of sequential sampling models for two-choice reaction time. Psychological Review 111 (2), 333–367.

Ratcliff, R., Thapar, A., McKoon, G., 2010. Individual differences, aging, and iq in two-choice tasks. Cognitive Psychology 60, 127–157.

Roe, R., Busemeyer, J., Townsend, J., 2001. Multialternative decision field theory: A dynamic connectionist model of decision-making. Psychological Review 108, 370–392.

Rorie, A., Gao, J., McClelland, J., Newsome, W., 2010. Integration of sensory and reward information during perceptual decision-making in lateral intraparietal cortex (lip) of the macaque monkey. PLoS ONE 5 (2), doi:10.1371.

Ruthruff, E., 1996. A test of the deadline model for speed-accuracy tradeoffs. Perception & Psychophysics 58 (1), 56–64.

Sacerdote, L., Telve, O., Zucca, C., 2014. Joint densities of the first hitting times of a diffusion process through two time dependent boundaries. arXiv:1403.1756v1 [math.PR].

Smith, P., 2000. Stochastic dynamic models of response time and accuracy: A foundational primer. Journal of Mathematical Psychology 44 (3), 408–463.

Smith, P. L., Ratcliff, R., 2009. An integrated theory of attention and decision making in visual signal detection. Psychological Review 116, 283–317.

Srivastavaa, V., Feng, S., Cohen, J., Ehrich Leonarda, N., Shenhavd, A., 2015. First passage time properties for time-varying diffusion models: A martingale approach.

Stone, M., 1960. Models for choice-reaction time. Psychometrika 25, 251–260.

Swensson, R., 1972. The elusive tradeoff: Speed vs accuracy in visual discrimination tasks. Perception & Psychophysics 12, 16–32.

Thapar, A., Ratcliff, R., McKoon, G., 2003. A diffusion model analysis of the effects of aging on letter discrimination. Psychology and Aging 18, 415–429.

Usher, M., McClelland, J., 2001. The time course of perceptual choice: The leaky, competing accumulator model. Psychological Review 108, 550–592.

Van Zandt, T., Colonius, H., Proctor, R., 2000. A comparison of two reaction time models applied to perceptual matching. Psychonomic Bulletin and Review 7 (2), 208–256.

Verdonck, S., Meers, K., Tuerlinckx, F., 2015. Efficient simulation of diffusion-based choice rt models on cpu and gpu. Behavior Research Methods.

White, C. N., Ratcliff, R., Vasey, M. W., McKoon, G., 2010. Dusing diffusion models to understand clinical disorders. Journal of Mathematical Psychology 54, 39–52.

Yellot, J., 1971. Correction for fast guessing and the speed-accuracy tradeoff in choice reaction time. Journal of Mathematical Psychology 8, 159–199.

Zhang, S., Lee, M., Vandekerckhove, J., Gunter, M., Wagenmakers, E.-J., 2014. Time-varying boundaries for diffusion models of decision making and response time. Frontiers in Quantitative Psychology and Measurement 5:1364.