Finite Volume Method on Moving Surfaces

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ABSTRACT. *In this paper an evolving surface finite volume method is introduced for the numerical resolution of a transport diffusion problem on a family of moving hypersurfaces. These surfaces are assumed to evolve according to a given motion field. The ingredients of the method are an approximation of the family of surfaces by a family of interpolating simplicial meshes, where grid vertices move on motion trajectories, a consistent finite volume discretization of the induced transport on the simplices, and a proper incorporation of a diffusive flux balance at simplicial faces. Existence, uniqueness and a priori estimates are proved for the discrete solution. Furthermore, a convergence result is formulated together with a sketch of the proof. Finally, first numerical results are discussed.*

KEYWORDS: *Finite volume method, evolving surfaces, transport diffusion equations*

1. Introduction

In many applications in materials science, biology and geometric modeling evolution problems do not reside on a flat Euclidean domain but on curved surfaces, which frequently evolve themselves over time. Furthermore, partial differential equations on the surface are often coupled with the evolution of the geometry itself. Examples are the spreading of thin liquid films or coatings on surfaces [ROY 02], transport and diffusion of a surfactant on interfaces in multiphase flow [JAM 04], surfactant driven thin film flow coupled on enclosed surface of lung alveoli coupled with the expansion or contraction of the alveoli [GRÜ 02], diffusion induced grain boundary motion [CAH 97], reaction diffusion equations for texture generation on surfaces [TUR 91]. Here, we suppose the evolution of the surface to be given *a priori* and study the finite volume discretization of diffusion on the resulting family of evolving surfaces as a model problem. This generalizes the already classical approach by Eymard, Gallouet and Herbin [EYM 00] on fixed Euclidean domains. We do not restrict to a surface propagation only in normal direction but allow for the practically relevant case of a tangential velocity component corresponding to the motion of material points in the surface. A finite volume approach is in particular beneficial in case of a later coupling with strong advection as in the case of surface tension induced Marangoni flow or density transport on interfaces. Finite volume methods on curved geometries have been discussed recently in [CAL 07] but to the best of our knowledge they have so far not been analysed on evolving surfaces. Our approach is closely related to the finite element approach by Dziuk and Elliott [DZI 07]. They consider a moving triangulation, where the nodes are propagating with the actual motion velocity, which effectively leads to space time finite element basis functions similar to the ELLAM approach [HER 90]. We consider as well a family of triangulated surfaces with nodes located on motion trajectories where the triangles are treated as finite volume cells. The resulting scheme immediately incorporates mass conservation. In this paper we prove discrete counterparts of continuous energy estimates related to those in the case of Euclidean domain in [EYM 00]. We also give a convergence result for the discrete solution and show some numerical results.

Mathematical model. We consider a family of compact, smooth, and oriented hypersurfaces $\Gamma(t) \subset \mathbb{R}^n$ ($n = 2, 3$) for $t \in [0, t_{max}]$ generated by a flux function $\Phi : \Gamma_0 \times [0, t_{max}] \to \mathbb{R}^n$ with $\Phi(\Gamma^0, t) = \Gamma(t)$ and $\Phi \in \mathbb{C}^1([0, t_{max}], \mathbb{C}^1(\Gamma_0)) \cap \mathbb{C}^0([0, t_{max}], \mathbb{C}^3(\Gamma_0))$. For simplicity we assume the reference surface Γ_0 to coincide with the initial surface $\Gamma(0)$. We denote by $v = \partial_t \Phi$ the velocity of material points and assume the decomposition $v = v_n \nu + v_{tgl}$ into a scalar normal velocity v_n in direction of the surface normal ν and a tangential velocity v_{tgl}. The evolution of a conservative material quantity u with $u(\cdot, t) : \Gamma(t) \to \mathbb{R}$, which is propagated with the surface and simultaneously undergoes a linear diffusion on the surface, is governed by the parabolic equation

$$\dot{u} + u \nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathcal{D}_0 \nabla_\Gamma u) = 0 \quad \text{on } \Gamma = \Gamma(t) \qquad [1]$$

where $\dot{u} = \frac{d}{dt} u(t, x(t))$ is the (advective) material derivative of u, $\nabla_\Gamma \cdot v$ the surface divergence of the vector field v, $\nabla_\Gamma u$ the surface gradient of the scalar field u, and $\mathcal{D}_0$ a symmetric, coercive diffusion tensor on whole $\mathbb{R}^n$, whose restriction on the tangent plane is effectively incorporated in the model. For the ease of presentation we restrict ourselves here to the case of a closed surface without boundary.
Our results can easily be generalized to surfaces with boundary, on which we either impose Dirichlet or Neumann boundary conditions. For a discussion of existence, uniqueness and regularity of solutions we refer to [DZI 07] and the references therein.

2. Derivation of the finite volume scheme

Following Dziuk and Elliott [DZI 07] we consider a sequence of regular, surface triangulations $\{\Gamma_h^k\}_{k=0,\cdots,k_{max}}$ with Γ_h^k interpolating $\Gamma(t_k)$ with $t_k = k\tau$. Here, h indicates the maximal diameter of a triangle on the whole sequence of triangulations,

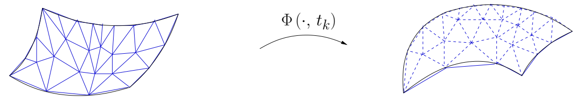

Figure 1. *Sketch of the reference triangulation Γ_h^0 interpolating Γ^0 and time 0 (left) and a triangulation Γ_h^k with the underlying continuous surface $\Gamma(t_k)$ (right).*

τ the time step size and k the index of a time step. All triangulations share the same grid topology, and given the set of vertices x_j^0 on the initial surface Γ_h^0 the vertices on Γ_h^k lie on motion trajectories. Thus, they are evaluated based on the flux function Φ, i.e. $x_j(t_k) = \Phi(x_j^0, t_k)$ (cf. Figure 1). In what follows a simplex of the triangulated surface Γ_h^k (a line segment for $n = 2$ or a triangle for $n = 3$) is denoted by S^k. As in the Euclidean case discussed in [EYM 00] we also assume that for all t_k for $k = 0, \cdots, k_{max}$ and all simplices $S^k \subset \Gamma_h^k$ there exists a point $X_S^k \in S^k$ and for each boundary simplex $\sigma^k \subset \partial S^k$ a point $X_\sigma^k \in \sigma^k$ such that $\overrightarrow{X_S^k X_\sigma^k}$ is perpendicular to σ^k with respect to the scalar product induced by the inverse of the diffusion tensor on the simplex S^k. Furthermore, we assume that these points can be chosen such that for two adjacent simplices S^k and L^k the corresponding points on the common edge $\sigma^k = S^k|L^k = \sigma^k$ coincide (cf. Figure 2). For a later comparison of discrete

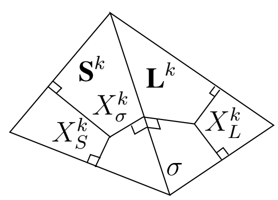

Figure 2. *A sketch of the local configuration of points X_S^k, X_L^k, and X_σ^k on two adjacent simplices S^k and L^k, which in general do not lie in the same plane.*

quantities on the triangulation Γ_h^k and continuous quantities on $\Gamma(t_k)$ we define a lifting operator from Γ_h^k onto $\Gamma(t_k)$ via the orthogonal projection $\mathcal{P}^k$ onto $\Gamma(t_k)$ in direction of the surface normal ν. For sufficiently small h this projection is uniquely defined and smooth. By $S^{l,k} := \mathcal{P}^k S^k$ we define the projection of a simplex $S^k \subset \Gamma_h^k$ and by $S^{l,k}(t) := \Phi(\Phi^{-1}(S^{l,k}, t_k), t)$ the propagation of $S^{l,k}$ during the $(k+1)$th time interval $[t_k, t_{k+1}]$. Furthermore, let us denote by m_S^k the $(n-1)$ dimensional measure of S^k. Then there exists a constant C depending on the smoothness of the flux function Φ such that $\left(1 - m_S^k \, (m_S^{k+1})^{-1}\right) \leq C\,\tau$ for all simplices S^k and all k.

Based on these notational preliminaries we can now derive a suitable finite volume discretization. Thus, let us integrate [1] on $\{(x,t)\,|\,t\in[t_k,\,t_{k+1}],\;\;x\in S^{l,\,k}(t)\}$. Using the Leibniz formula (cf. [DZI 07]), we obtain for the material derivative

$$\int_{t_k}^{t_{k+1}}\int_{S^{l,\,k}(t)} \dot u + u\nabla_\Gamma\cdot v = \int_{S^{l,\,k}(t_{k+1})} u - \int_{S^{l,\,k}(t_k)} u \approx m_S^{k+1}u(X_S^{k+1}) - m_S^k u(X_S^k)\,. \quad [2]$$

Next, on each simplex S^k we approximate the diffusion tensor $\mathcal{D}_0$ by $\mathcal{D}_S^k := \mathcal{D}_0(X_S^k)$. Integrating the elliptic term again over the temporal evolution of a lifted simplex patch and applying the Gauss theorem we derive the following approximation:

$$\begin{aligned}\int_{t_k}^{t_{k+1}}\int_{S^{l,\,k}(t)} \nabla_\Gamma\cdot(\mathcal{D}_0\nabla_\Gamma u) &= \int_{t_k}^{t_{k+1}}\int_{\partial S^{l,\,k}(t)} \mathcal{D}_0\nabla_\Gamma u\cdot\mu_{\partial S^{l,\,k}(t)} \\ &\approx \tau\sum_{\sigma^{k+1}\subset S^{k+1}} m_\sigma^{k+1}\lambda_{S|\sigma}^{k+1}\frac{u(X_\sigma^{k+1})-u(X_S^{k+1})}{d_{S|\sigma}^{k+1}} \quad [3]\end{aligned}$$

where $\mu_{\partial S^{l,\,k}(t)}$ is the co-normal on $\partial S^{l,\,k}(t)$ tangential to $\Gamma(t)$, σ^{k+1} a boundary simplex of S^{k+1}, m_σ^{k+1} the $(n-2)$-dimensional Hausdorff measure of the edge σ^{k+1}, $d_{S|\sigma}^{k+1} = \|X_S^{k+1}-X_\sigma^{k+1}\|$, and $\lambda_{S|\sigma}^{k+1} = \|\mathcal{D}_S^{k+1}\mu_S^{k+1}\|$. Now, we introduce discrete degrees of freedom u_S^k and u_σ^k for $u(X_S^k)$ and $u(X_\sigma^k)$, respectively. Then the discrete counterpart of the continuous flux balance

$$\int_{S^{l,k}(t)\cap L^{l,\,k}(t)} (\mathcal{D}_0\nabla_\Gamma u)|_{S^{l,k}(t)}\cdot\mu_{\partial S^{l,\,k}(t)} = -\int_{S^{l,k}(t)\cap L^{l,\,k}(t)} (\mathcal{D}_0\nabla_\Gamma u)|_{L^{l,k}(t)}\cdot\mu_{\partial L^{l,\,k}(t)}$$

on $S^{l,k}(t)\cap L^{l,\,k}(t)$ for two adjacent simplices S^k and L^k is given by

$$m_\sigma^{k+1}\frac{u_\sigma^{k+1}-u_S^{k+1}}{d_{S|\sigma}^{k+1}}\lambda_{S|\sigma}^{k+1} = -m_\sigma^{k+1}\frac{u_\sigma^{k+1}-u_L^{k+1}}{d_{L|\sigma}^{k+1}}\lambda_{L|\sigma}^{k+1}$$

for the edge $\sigma = S^k\cap L^k$. Hence, we can cancel out the degrees of freedom u_σ^{k+1} on edges and based on the approximations for the parabolic term in [2] and the elliptic term in [3] we obtain the finite volume scheme

$$m_S^{k+1}u_S^{k+1} - m_S^k u_S^k - \tau\sum_{\sigma^{k+1}\subset S^{k+1}}\frac{m_\sigma^{k+1}\lambda_{S|\sigma}^{k+1}\lambda_{L|\sigma}^{k+1}}{d_{L|\sigma}^{k+1}\lambda_{S|\sigma}^{k+1}+d_{S|\sigma}^{k+1}\lambda_{L|\sigma}^{k+1}}\,(u_L^{k+1}-u_S^{k+1}) = 0\,, \quad [4]$$

which requires the solution of a linear system of equations for the cell-wise solution values u_S^{k+1} for $k = 0,\cdots,K-1$ and for given initial data u_S^0 at time $t_0 = 0$.
Remark: Different from the finite volume method on Euclidean domains in [EYM 00], all coefficients depend on the geometric evolution, and thus in particular change over time. A comparison of the discrete and continuous solution requires a mapping from the sequence of triangulations $\{\Gamma_h^k\}$ onto the continuous family of surfaces $\{\Gamma(t)\}_{t\in[0,t_{max}]}$

Let us associate with the components u_S^k on the simplices S^k of the triangulation Γ_h^k a piecewise constant function u^k with $u^k|_{S^k} = u_S^k$ and let $\mathcal{V}_h^k$ be the space of these functions on Γ_h^k. On this function space, we can define a discrete energy semi-norm:

Definition 2.1 (Discrete energy semi-norm) *For $u^k \in \mathcal{V}_h^k$ we define*

$$\|u^k\|^2_{1,\,\Gamma_h^k} = \sum_{\substack{\sigma=S^k\cap L^k\\ S^k,\,L^k\in\Gamma_h^k}} m_\sigma (d^k_{S|L})^{-1}\mathcal{M}^k_\sigma \,(u^k_L \;-\; u^k_S)^2 \qquad [5]$$

where $\mathcal{M}^k_\sigma = \dfrac{\lambda^k_{S|\sigma}\lambda^k_{L|\sigma}\, d^k_{S|L}}{d^k_{L|\sigma}\lambda^k_{S|\sigma} + d^k_{S|\sigma}\lambda^k_{L|\sigma}},\;\; d^k_{S|L} \;=\; d^k_{S|\sigma} \,+\, d^k_{L|\sigma}.$

Before we prove suitable *a priori* estimates, let us verify existence and uniqueness of the discrete solution.

Theorem 2.2 *The discrete problem* [4] *has a unique solution.*

Proof System [4] has a unique solution u^{k+1}, if the kernel of the corresponding linear operator is trivial. To see this, we assume $u^k = 0$ in [4] and multiply the equation by u^k_S for each simplex $S^k \subset \Gamma^k_h$. Summing up over all simplices and taking into account the symmetry of the second term in [4] with respect to the two simplices S^k, L^k intersecting at the edge $\sigma = S^k \cap L^k$ we obtain

$$\|u^{k+1}\|^2_{\mathbb{L}^2\left(\Gamma_h^{k+1}\right)} + \tau\|u^{k+1}\|^2_{1,\,\Gamma_h^{k+1}} = 0$$

from which $u^{k+1} = 0$ immediately follows. □

3. *A priori* estimates and convergence

In what follows we will prove an energy estimate and a $\mathbb{H}^1(\mathbb{L}^2)$ and $\mathbb{L}^\infty(\mathbb{H}^1)$ *a priori* estimate.

Theorem 3.1 (discrete $\mathbb{L}^\infty(\mathbb{L}^2)$, $\mathbb{L}^2(\mathbb{H}^1)$ energy estimate) *Let $\{u^k\}_{k=1,\cdots,k_{max}}$ be the discrete solution fulfilling* [4] *with given initial data* u^0*, then there exists a constant* C*, such that*

$$\sup_k \|u^{k+1}\|^2_{\mathbb{L}^2\left(\Gamma_h^{k+1}\right)} + \sum_k \tau\|u^{k+1}\|^2_{1,\,\Gamma_h^{k+1}} \le C\|u^0\|^2_{\mathbb{L}^2\left(\Gamma_h^0\right)}. \qquad [6]$$

Proof For every cell $S^k \in \Gamma_h^{k+1}$, multiply the corresponding equation [4] by u_S^{k+1} and thereafter, sum the new equations over all $S^k \in \Gamma_h^k$ to obtain

$$\sum_{S^k\in\Gamma_h^k}\left(m_S^{k+1}\,(u_S^{k+1})^2 - m_S^k u_S^k u_S^{k+1}\right) + \tau \sum_{\substack{\sigma^k=S^k\cap L^k\\ S^k,\,L^k\in\Gamma_h^k}} m_\sigma^{k+1} d^{k+1}_{S|L}\mathcal{M}^{k+1}_\sigma\left(\frac{u_L^{k+1}-u_S^{k+1}}{d^{k+1}_{S|L}}\right)^2 = 0$$

which leads to

$$\frac{1}{2}\|u^{k+1}\|^2_{\mathbb{L}^2(\Gamma_h^{k+1})} + \tau\|u^{k+1}\|^2_{1,\,\Gamma_h^{k+1}} \;\le\; \frac{1}{2}\,\|u^k\|^2_{\mathbb{L}^2(\Gamma_h^k)} \;+\; \frac{1}{2}\delta(h,\tau)\tau\|u^{k+1}\|^2_{\mathbb{L}^2(\Gamma_h^{k+1})}$$

where $\delta(h,\tau) = \sup_k \sup_S \left| m^k_S \left(m^{k+1}_S\right)^{-1} - 1 \right| \tau^{-1}$. Due to the smoothness of the flux function, we observe that $\delta(h,\tau)$ is uniformly bounded by some δ for sufficiently small h and τ. Finally, we apply a Gronwall type argument and achieve

$$\sup_k \|u^k\|^2_{\mathbb{L}^2(\Gamma^k_h)} + \sum_k \tau \|u^{k+1}\|^2_{1,\,\Gamma^{k+1}_h} \leq \frac{3}{2} e^{2\delta t_{max}} \|u^0\|^2_{\mathbb{L}^2(\Gamma^0_h)}$$

which proves the claim. □

Theorem 3.2 (discrete $\mathbb{H}^1(\mathbb{L}^2)$, $\mathbb{L}^\infty(\mathbb{H}^1)$ energy estimate) *For the discrete solution $\{u^k\}_{k=1,\cdots,k_{max}}$ of* [4] *with given initial data u^0, then there exist a constant C, such that*

$$\sum_k \tau \|\partial^\tau_t u^{k+1}\|^2_{\mathbb{L}^2\left(\Gamma^{k+1}_h\right)} + \sup_k \|u^{k+1}\|^2_{1,\,\Gamma^{k+1}_h} \leq C\left(\|u^0\|^2_{\mathbb{L}^2\left(\Gamma^0_h\right)} + \|u^0\|^2_{1,\,\Gamma^0_h} \right), \quad [7]$$

where $\partial^\tau_t u^{k+1} := \frac{u^{k+1}-u^k}{\tau}$ is defined as a difference quotient in time.

Proof For every simplex $S^k \in \Gamma^{k+1}_h$, multiply [4] by $\partial^\tau_t u^{k+1}$ and thereafter, sum over all simplices to obtain

$$\begin{aligned}
0 \quad = \quad & \tau \sum_S m^{k+1}_S \left(\frac{u^{k+1}_S - u^k_S}{\tau} \right)^2 + \sum_S \left(m^{k+1}_S - m^k_S\right) u^k_S \left(\frac{u^{k+1}_S - u^k_S}{\tau} \right) \\
& + \sum_{\substack{\sigma^k = S^k \cap L^k \\ S^k,\, L^k \in \Gamma^k_h}} \Bigg(m^k_\sigma \, d^{k+1}_{S|L} \, \mathcal{M}^{k+1}_\sigma \left(\frac{u^{k+1}_L - u^{k+1}_S}{d^{k+1}_{S|L}} \right)^2 \\
& \qquad - m^k_\sigma \, \mathcal{M}^{k+1}_\sigma \left(\frac{u^{k+1}_S - u^{k+1}_L}{d^{k+1}_{S|L}} \right) \left(u^k_S - u^k_L\right) \Bigg).
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
& \tau \sum_S m^{k+1}_S \left(\frac{u^{k+1}_S - u^k_S}{\tau} \right)^2 + \left(\|u^{k+1}\|_{1,\,\Gamma^{k+1}_h} \right)^2 \\
& = \sum_{\substack{\sigma^k = S^k \cap L^k \\ S^k,\, L^k \in \Gamma^k_h}} \sqrt{d^k_{S|L} m^k_\sigma \mathcal{M}^k_\sigma} \left(\frac{u^k_S - u^k_L}{d^k_{S|L}} \right) \cdot \\
& \qquad \sqrt{\frac{d^k_{S|L} m^{k+1}_\sigma \mathcal{M}^{k+1}_\sigma}{d^{k+1}_{S|L} m^k_\sigma \mathcal{M}^k_\sigma}} \sqrt{d^{k+1}_{S|L} m^{k+1}_\sigma \mathcal{M}^{k+1}_\sigma} \left(\frac{u^{k+1}_S - u^{k+1}_L}{d^{k+1}_{S|L}} \right) \\
& \qquad - \sum_{S^k} \left(1 - \frac{m^k_S}{m^{k+1}_S} \right) \tau^{-1} \frac{\sqrt{m^{k+1}_S}}{\sqrt{m^k_S}} \tau \sqrt{m^k_S} u^k_S \left(\sqrt{m^{k+1}_S} \frac{u^{k+1}_S - u^k_S}{\tau} \right)
\end{aligned}$$

Applying the Cauchy and the Young inequality, we finally obtain

$$\begin{aligned}
& \tau \, \|\partial^\tau_t \, u^{k+1}\|^2_{\mathbb{L}^2\left(\Gamma^{k+1}_h\right)} + \frac{1}{2} \|u^{k+1}\|^2_{1,\,\Gamma^{k+1}_h} - \frac{1}{2} \|u^k\|^2_{1,\,\Gamma^k_h} \\
& \leq \frac{1}{2} \, \gamma \, \tau \|u^{k+1}\|^2_{1,\,\Gamma^{k+1}_h} + \alpha \tau \|u^k\|_{\mathbb{L}^2\left(\Gamma^k_h\right)} \|\partial^\tau_t \, u^{k+1}\|_{\mathbb{L}^2\left(\Gamma^{k+1}_h\right)} \qquad [8]
\end{aligned}$$

with $\gamma = \sup_k \sup_{\substack{\sigma^k = S^k \cap L^k \\ S^k,\, L^k \in \Gamma_h^k}} \left| \frac{m_\sigma^{k+1}}{m_\sigma^k} \frac{\mathcal{M}_\sigma^{k+1}}{\mathcal{M}_\sigma^k} \frac{d_{S|L}^k}{d_{S|L}^{k+1}} - 1 \right| \tau^{-1}$ and $\alpha = \delta(h,\tau) \sup_k \sup_{S^k} \sqrt{\frac{m_S^{k+1}}{m_S^k}}$.
Next, we apply again Young inequality on equation [8] and sum over all time steps. Finally, an application of Theorem 3.1 lead us to the desired estimate. □

Now, we consider error estimates for the finite volume solution $u^k \in \mathcal{V}_h^k$. Already in the derivation of the scheme we made use of a lifting operation from the discrete surfaces Γ_h^k onto the continuous surfaces $\Gamma(t^k)$. Here, we use the pull back of the continuous solution u at time t_k under this lift $u^{-l}(X_S^k, t_k) := u(\mathcal{P}^k(X_S^k), t_k)$. The consistency of the scheme depends on the proper choice of the nodes X_S^k and in particular their relation in time. Let us assume that $\|\Phi(\Phi^{-1}(\mathcal{P}^k(X_S^k), t_k), t_{k+1}) - \mathcal{P}^{k+1}(X_S^{k+1})\| \leq Ch\tau$. This condition can easily be verified for constant diffusivity and acute meshes with X_S^k being the orthocenter of S^k. Then the following convergence theorem holds.

Theorem 3.3 (error estimate) *Under the assumptions listed above, the error $E^k := \sum_S \left(u^{-l}\left(X_S^k, t_k\right) - u_S^k\right) \chi_S$ between the above pull back of the continuous solution $u(\cdot, t_k)$ of* [1] *at time t_k and the finite volume solution u^k of* [4] *is estimated by*

$$\|E^k\|^2_{\mathbb{L}^2(\Gamma_h(t_k))} \leq C(h+\tau)^2 \qquad [9]$$

for a constant C and all $k = 0, \cdots, k_{max}$.

Sketch of Proof Following the finite element error analysis in [DZI 07] we consider the pull back of a simplex wise flux formulation of [1] (cf. the middle term in [2] and [3]) onto the discrete surface Γ_h^k and subtract from this the discrete problem [4]. Then we multiply this by the error $u^{-l}\left(X_S^k, t_k\right) - u_S^k$ and sum over all simplices $S^k \subset \Gamma_h^k$. Now, we combine techniques from [EYM 00] and estimates based on our geometric assumptions to establish an estimate of the error term $\|E^{k+1}\|^2_{\mathbb{L}^2(\Gamma_h^{k+1})}$ by $\|E^k\|^2_{\mathbb{L}^2(\Gamma_h^k)}$ plus consistency terms involving τ and h. Using a similar Gronwall type argument as in the proof of Theorem 3.1 we establish the stated estimate. □

4. Numerical results

In Figure 3 we depict some numerical results obtained by our finite volume algorithm., which demonstrate the interplay between diffusion and advection due to geometric deformation.

5. References

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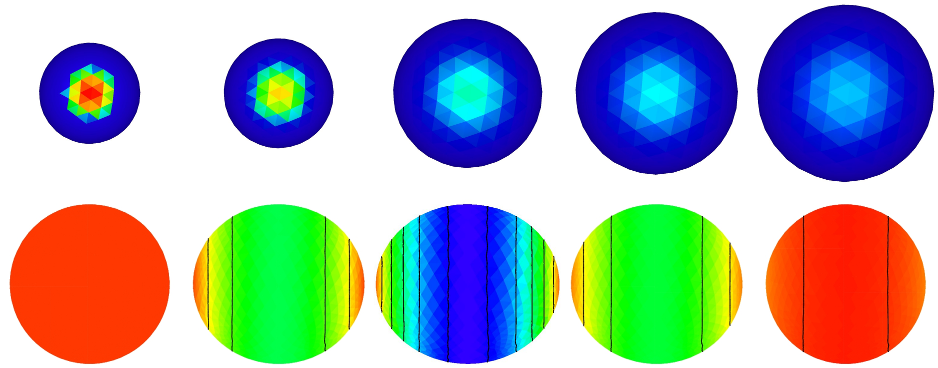

Figure 3. *The heat equation ($\mathcal{D}_0 = Id$) is solved on an expanding sphere for initial data with local support on a coarse grid (top row). The density is rendered on different time steps of the evolution based on color coding ranging from blue to red. In the bottom row, an anisotropic expansion and later reverse contraction of a sphere with constant initial data computed on a finer grid is depicted. Again we plot different time steps, which show that an inhomogeneous density is observed with maxima on the less stretched poles during expansion followed by an advective concentration of density close to the symmetry plane during the contraction phase. We have drawn isolines at constant intervals for a better visualisation of this effect.*

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