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A Priori L^2-Discretization Error Estimates for the State in Elliptic Optimization Problems with Pointwise Inequality State Constraints

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A PRIORI L^2-DISCRETIZATION ERROR ESTIMATES FOR THE STATE IN ELLIPTIC OPTIMIZATION PROBLEMS WITH POINTWISE INEQUALITY STATE CONSTRAINTS

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Abstract. In this paper, an elliptic optimization problem with pointwise inequality constraints on the state is considered. The main contribution of this paper are a priori L^2-error estimates for the discretization error in the optimal states. Due to the non separability of the space for the Lagrange multipliers for the inequality constraints, the problem is tackled by separation of the discretization error into two components. First, the state constraints are discretized. Second, with discretized inequality constraints, a duality argument for the error due to the discretization of the PDE is employed. For the second stage an a priori error estimate is derived with constants depending on the regularity of the dual problem. Finally, we discuss two cases in which these constants can be bounded in a favorable way; leading to higher order estimates than those induced by the known L^2-error in the control variable. More precisely, we consider a given fixed number of pointwise inequality constraints and a case of infinitely many but only weakly active constraints.

1. Introduction

In this paper, we are concerned with finite element discretization error estimates for convex elliptic optimal control problems with pointwise state constraints. Due to the presence of measures in the optimality system, pointwise state constraints pose a challenge in many questions of numerical analysis. Yet, even though the literature for state-constrained problems is less complete than for control constrained ones, some considerable progress has been made in the last years. In addition to the first results for state-constrained problems on plain convergence, cf. [5], as well as problems with only finitely many integral state constraints, cf. [4], we would like to mention, in particular, the results on error estimates for linear-quadratic control problems from [8] and [16], where the error estimates $\|\bar{q}-\bar{q}_h\|_{L^2(\Omega)} \le ch^{2-d/2-\varepsilon}$ for the optimal controls is shown in $d=2$ or $d=3$ space dimensions. In [8], the authors consider variational discretization of a purely state-constrained problem. Due to the lack of control constraints, this is equivalent to a control discretization with piecewise linear functions, which is considered in [16], with different techniques. In addition, [16] contains results for problems with additional control constraints and a piecewise constant control discretization. For a more complete overview on state-constrained optimal control problems and further references, we refer the reader to [11]. Let us also point out new results based on higher regularity of the Lagrange multipliers, which can be obtained under assumptions on the given data, in [7]. There, the authors obtain the rate $\|\bar{q}-\bar{q}_h\|_{L^2(\Omega)} \le ch|\log h|$ in two and three space dimensions. The same order of convergence is proven in the context of the

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so called variational discretization in [10, Corollary 3.3] if both the continuous and discrete control are uniformly bounded in $L^\infty(\Omega)$.

In different settings with only finitely many controls, leading to either completely finite-dimensional problems with finitely many state constraints, cf. [15], or so-called semi-infinite programming problems with infinitely many state constraints, cf. [13, 14], higher convergence rates can be established, if some structural assumptions are satisfied.

On the other hand, problems with control functions and finitely many constraints have also been of interest to the optimal control community. In [6], a nonconvex semilinear problem with finitely many pointwise inequality and equality constraints on the state, and pointwise box constraints on the control has been analyzed for piecewise constant control approximation. In [12], the authors prove optimal error estimates for a linear-quadratic problem with finitely many equality constraints on the state and piecewise linear control discretization.

A rate of convergence for the optimal controls allows to derive the same estimate for the error for the optimal states by means of a discretization error for the state equation as well as Lipschitz continuity of the solution operator of the PDE, cf. [8]. However, as numerical experiments in the same paper indicate, this convergence rate for the state is not necessarily optimal. In contrast to that, in the setting with finitely many equality state-constraints from [12], the authors could prove an optimal rate of convergence by means of applying the duality argument of the Aubin-Nitsche Trick to the whole optimality system. This is possible as the dual space for given finitely many constraints is separable, in contrast to the set $\mathcal{M}(\Omega)$ of regular Borel-measures appearing in the presence of pointwise constraints on all of $\overline{\Omega}$.

The present work is concerned with an extension of this technique, i.e., the duality argument for the optimality system, of [12] to the case of finitely many inequality constraints on the state. Moreover, we will directly consider the dependence of the appearing constants with respect to the number of inequality constraints. This will be done by providing an alternative proof to [12, Lemma 3] that provides the explicit dependence of the constant with respect to the number and distribution of the inequality constraints.

The extension of the results from [12] to inequality constraints with explicit tracking of the appearing constants, allows to have a different continuous problem associated to each mesh. In certain situations, this will allow us to obtain improved L^2-error estimates for the optimal state for a problem with pointwise state constraints in the whole domain.

To make the idea clear; the standard model for elliptic optimization problems with pointwise inequality constraints is given by finding $(\overline{q},\overline{u}) \in L^2(\Omega) \times H^1_0(\Omega)$ solving

$$(\mathbb{P}) \qquad \begin{cases} \min J(q,u) \\ \qquad \text{s.t.} \begin{cases} (\nabla u, \nabla \varphi) = (q,\varphi) & \forall\, \varphi \in H^1_0(\Omega), \\ u^-(x) \le u(x) \le u^+(x) & \text{a.e. in } \overline{\Omega}, \end{cases} \end{cases}$$

on a given domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$ with some given bounds u^-, u^+. The standard discretization of this problem using some finite element space $V_h \subset H^1_0(\Omega)$, here we consider the lowest order choice, is given by finding $(\overline{q}_h, \overline{u}_h) \in V_h \times V_h$

$$(\mathbb{P}_h) \qquad \begin{cases} \min J(q_h, u_h) \\ \quad \text{s.t.} \begin{cases} (\nabla u_h, \nabla \varphi_h) = (q_h, \varphi_h) \quad \forall\, \varphi_h \in V_h, \\ u^-(x_i) \le u_h(x_i) \le u^+(x_i) \quad i = 1, \dots, N, \end{cases} \end{cases}$$

where the x_i, $i = 1, \dots, N$, are the vertices of the triangulation. As outlined above, order optimal convergence rates can be derived for the control $\|\overline{q} - \overline{q}_h\|$. However, estimates for the error in L^2 of the state, i.e., $\|\overline{u} - \overline{u}_h\|$, obtained from this control estimate are suboptimal compared to the numerically observed convergence orders, as well as compared to the best-approximation to u in V_h. Using the properties of the discretization and a corresponding interpolation operator $\mathcal{I}_h$ it is clear, that the finitely many inequality constraints in $(\mathbb{P}_h)$ correspond to

$$\mathcal{I}_h u^-(x) \le u_h(x) \le \mathcal{I}_h u^+(x) \quad \text{in } \bar{\Omega}.$$

We now define an intermediate problem where only the constraints are discretized. It reads, find $(\overline{q}_N, \overline{u}_N) \in L^2(\Omega) \times H_0^1(\Omega)$

$$(\mathbb{P}^N) \qquad \begin{cases} \min J(q, u) \\ \quad \text{s.t.} \begin{cases} (\nabla u, \nabla \varphi) = (q, \varphi) \quad \forall\, \varphi \in H_0^1(\Omega), \\ u_i^- \le u(x_i) \le u_i^+ \quad i = 1, \dots, N, \end{cases} \end{cases}$$

where we do not necessarily require N to be the number of nodes, but a finite number that may depend on h. We will show, in Theorem 19, that an estimate of the form

$$\|\overline{u}_N - \overline{u}_h\| \le C_{h,N} h^{4-d} (|\ln(h)| + 1)^{7-2d}$$

holds with a constant $C_{h,N}$ depending on the regularity of the dual problem associated to the given choice of h, N, more precisely, the chosen set x_i of points for the inequality constraints. Finally, we will discuss cases in which this constant can be bounded independently of h, N and, moreover, an estimate for $\|\overline{u} - \overline{u}_N\|$, of at least the same order, is available.

The rest of the paper is structured as follows. In Section 2, we will introduce the precise setting of the continuous optimization problem under consideration. In Sections 3 and 4, we describe the two discretization steps and introduce the problems $(\mathbb{P}^N)$ and $(\mathbb{P}_h)$, respectively. Then, in Section 5, we collect several known results regarding the discretization error for the PDE. In Section 6, we show the above claimed error estimate between $\bar{u}_N$ and $\bar{u}_h$, the optimal states of $(\mathbb{P}^N)$ and $(\mathbb{P}_h)$. Here, we consider N pointwise state constraints, where N can either be fixed or mesh-dependent. We point out that in the latter case, we do not necessarily require the constraints to be prescribed in the grid points. The main result of this section is stated in Theorem 19. It is obtained using an appropriate dual problem, i.e., we apply the Aubin-Nitsche trick to the optimality system.

In Section 7, we discuss the implication of the main Theorem 19 in special situations where the constant $C_{h,N}$ can be bounded. These situations are, finitely many pointwise inequality constraints in a fixed given set of N points which may or may not coincide with the vertices of the mesh. Secondly, we consider the case where pointwise constraints are prescribed on $\overline{\Omega}$. In this case, we confine our discussion to a setting with only weakly active constraints.

In both cases, we eventually arrive at an error estimate of roughly order $\mathcal{O}(h^{4-d})$ for the states $\bar{u}$ and $\bar{u}_h$. To put these results in perspective, we point out that in space dimension $d = 2$, this is roughly order $\mathcal{O}(h^2)$, a result that was not available before. In space dimension $d = 3$, we obtain roughly the order $\mathcal{O}(h)$, which is stated in [10, Corollary 3.3] and [7], yet without the additional assumptions on uniform boundedness of the discrete solutions in the former and boundedness of the desired state, in the definition of J, in the latter case.

2. The Continuous Problem ($\mathbb{P}$) and its Properties

In this section, we will analyze the continuous state-constrained Problem ($\mathbb{P}$) with respect to existence and regularity of solutions, optimality conditions, and auxiliary properties that are needed in the subsequent analysis. In what follows, we will use standard notation for Lebesgue and Sobolev spaces. The norm and scalar product on L^2 will be given by $\|\cdot\|$ and $(\cdot,\cdot)$. Norms on the Sobolev spaces $W^{s,p}$ will be denoted by $\|\cdot\|_{s,p}$. Further, the duality pairing between the space of continuous functions and its dual $\mathcal{M}(\Omega)$, the space of all regular Borel measures, will be denoted by $\langle\cdot,\cdot\rangle$.

As a prototypical model, we consider the partial differential equation coupling the state $u \in H^1_0(\Omega) =: V$ and the control $q \in L^2(\Omega) =: Q$ to be Poisson's problem on a convex smooth or polygonal domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary Γ. As cost functional, we consider L^2-tracking with some given desired state $u^d \in L^2(\Omega)$ together with a Tikhonov regularization with parameter $\alpha > 0$ acting on q.

Remark 1. *To avoid technicalities, we consider H^2 regular problems to assert full approximation orders within the finite element approximation of the PDE. The required convexity of the domain in the case of a smooth boundary is only needed to assert that the standard finite element space is indeed a subspace of $H^1_0(\Omega)$; to avoid a technical, but possible, discussion of additional terms due to a non conforming discretization.*

Moreover, let two functions $u^{\pm} \in C(\bar{\Omega}) \cap H^2(\Omega)$ be given that satisfy $u^- < u^+$ as well as $u^-(x) < 0 < u^+(x)$ for all $x \in \partial\Omega$. We consider the following optimization problem

$$
\begin{aligned}
&\text{Minimize } J(q,u) = \frac{1}{2}\|u - u^d\|^2 + \frac{\alpha}{2}\|q\|^2 \\
&\text{s.t. } (\nabla u, \nabla\varphi) = (q, \varphi) \quad \forall \varphi \in V, \\
&\qquad u^-(x) \le u(x_i) \le u^+(x) \quad \forall\, x \in \bar{\Omega}.
\end{aligned} \tag{2.1}
$$

By unique solvability of the state equation, the control-to-state coupling defines a continuous linear map $S\colon Q \to V$ given by $S(q) = u(q)$. Moreover, by our assumptions on Ω the map S defines an isomorphism $Q \to V \cap H^2(\Omega)$. Note that $H^2 \hookrightarrow C(\overline{\Omega})$ holds due to well known Sobolev embeddings, cf. [1]. In the following, we will implicitly use embeddings from $V \cap H^2(\Omega)$ into $C(\overline{\Omega})$ as well as into $L^2(\Omega)$ without explicitly mentioning these.

Remark 2. *We point out that by continuity of the states $u(q)$ and the conditions on $u^{\pm}$, the constraints can only be active in an interior subdomain Ω_0 of Ω and the distance of active points to $\partial\Omega$ can be estimated by a constant that depends only on the given data.*

We define a reduced formulation of the optimal control problem in the usual way, i.e., we consider

$$(\mathbb{P})\qquad \begin{aligned} &\text{Minimize } j(q) := J(q, S(q)) \\ &\qquad \text{s.t. } u^-(x) \leq S(q)(x) \leq u^+ \quad \forall\, x \in \bar{\Omega}. \end{aligned}$$

The existence of an optimal solution to Problem ($\mathbb{P}$) is obtained in a standard way since one shows by straight forward calculations that the set of feasible controls,

$$Q_{\text{feas}} := \{q \in Q\colon\ u^-(x) \leq S(q)(x) \leq u^+(x), \quad \forall x \in \bar{\Omega}\},$$

is not empty. The - by strict convexity - unique optimal control will be denoted by $\overline{q}$, with associated optimal state $\overline{u}$. Moreover, there exists a Slater point q_τ and a real number τ such that $u^- + \tau \leq S(q_\tau) = u_\tau \leq u^+ - \tau$. It is then clear that also $\|u^+ - u^-\|_\infty \geq 2\tau$.

Based on the Slater condition, we can now formulate the first order necessary optimality conditions for Problem ($\mathbb{P}$). Note that by convexity, these are also sufficient.

Theorem 3. *A control $\overline{q} \in Q_{feas}$ with associated state $\overline{u} = S(\overline{q})$ is an optimal solution of Problem ($\mathbb{P}$) if and only if there exist nonnegative Lagrange multipliers $\overline{\mu}^+, \overline{\mu}^- \in \mathcal{M}(\Omega)$ as well as an adjoint state $\overline{z} \in W_0^{1,s}(\Omega)$, $s < \frac{d}{d-1}$, such that the adjoint equation*

$$(\nabla\varphi, \nabla\overline{z}) = (\varphi, \overline{u} - u^d) + \langle \overline{\mu}^+ - \overline{\mu}^-, \varphi\rangle \quad \forall\, \varphi \in W_0^{1,s'}(\Omega) \tag{2.2}$$

with $\frac{1}{s} + \frac{1}{s'} = 1$, the gradient equation

$$\alpha\overline{q} + \overline{z} = 0, \tag{2.3}$$

and the complementary slackness conditions

$$\langle \overline{\mu}^+, \overline{u} - u^+\rangle = \langle \overline{\mu}^-, u^- - \overline{u}\rangle = 0 \tag{2.4}$$

are fulfilled.

Proof. The proof follows along the lines of [3]. □

For further reference, we split the adjoint state $\bar{z} = \bar{z}_0 + \bar{z}_\mu$ into a regular part $z_0 \in H_0^1(\Omega)$ solving

$$(\nabla\varphi, \nabla\bar{z}_0) = (\overline{u} - u^d, \varphi) \quad \forall\ \varphi \in H_0^1(\Omega),$$

and a less regular part $\bar{z}_\mu \in W_0^{1,s}(\Omega)$, $s < \frac{d}{d-1}$, given by

$$(\nabla\varphi, \nabla\bar{z}_\mu) = \langle \bar{\mu}^+ - \bar{\mu}^-, \varphi\rangle \quad \forall\ \varphi \in W_0^{1,s'}(\Omega). \tag{2.5}$$

We note, that for some cases it is suitable to consider simply a feasible point. We will denote such a point by q_0 which is compatible with the definition of the Slater point q_τ, although q_0 is not a Slater point itself.

Lemma 4. *There exists a constant $C > 0$ such that the following bounds on the variables given by Theorem 3 hold:*

$$\|\overline{q}\| + \|\overline{u}\| + \|\overline{z}_0\| \leq C(\|u^d\| + \|q_0\|) =: \overline{C}_0,$$
$$\begin{aligned} \|\overline{\mu}^+\|_{\mathcal{M}(\Omega)} + \|\overline{\mu}^-\|_{\mathcal{M}(\Omega)} &\leq \frac{C}{\tau}(\|u^d\| + \|q_0\| + \|q_\tau\|)(\|u^d\| + \|q_0\|) \\ &=: \overline{C}_\tau. \end{aligned}$$

Without loss of generality, we assume $1 \leq \overline{C}_0 \leq \overline{C}_\tau$.

Proof. Utilizing $J(\overline{q},\overline{u}) \le J(q_0,u_0)$, and the given form of the cost functional, the assertion is shown for $\overline{q}$ and $\overline{u}$. Elliptic regularity proves the result for $\overline{z}_0$.

To see the bound on $\overline{\mu}^{\pm}$, we proceed by meanwhile standard calculations with the optimality system and test the adjoint equation (2.2) with $\overline{u}-u_\tau \in V\cap H^2(\Omega)$. We obtain, utilizing (2.4) and the definition of the Slater point,

$$\begin{aligned}(\nabla(\overline{u}-u_\tau),\nabla\overline{z}) - (\overline{u}-u_\tau,\overline{u}-u^d) &= \langle\bar{\mu}^+,\overline{u}-u_\tau\rangle + \langle\bar{\mu}^-,u_\tau-\overline{u}\rangle\\ &= \langle\bar{\mu}^+,u^+-u_\tau\rangle + \langle\bar{\mu}^-,u_\tau-u^-\rangle\\ &\ge \tau(\|\bar{\mu}^+\|_{\mathcal{M}(\Omega)} + \|\bar{\mu}^-\|_{\mathcal{M}(\Omega)}),\end{aligned}$$

noting that the supports of $\bar{\mu}^+$ and $\bar{\mu}^-$ are disjoint. Using the state equation, we conclude

$$\begin{aligned}\tau(\|\bar{\mu}^+\|_{\mathcal{M}(\Omega)} + \|\bar{\mu}^-\|_{\mathcal{M}(\Omega)}) &\le (\overline{q}-q_\tau,\overline{z}) - (\overline{u}-u_\tau,\overline{u}-u^d)\\ &\le (\overline{q},\overline{z}) - (\overline{u},\overline{u}-u^d) - (q_\tau,\overline{z}) + (u_\tau,\overline{u}-u^d)\\ &\le C(\|u^d\|+\|q_0\|)^2 + C\|q_\tau\|(\|u^d\|+\|q_0\|).\end{aligned}$$

This shows the assertion. □

Note that the optimality system from Theorem 3 implies additional $W^{1,s}$-regularity of the optimal control $\overline{q}$, obtained by means of the gradient equation (2.3).

Corollary 5. *With the notation of Lemma 4, for any $s < \frac{d}{d-1}$, we have $\overline{q} \in W^{1,s}(\Omega)$, $\overline{u},\overline{z}_0 \in V\cap H^2(\Omega)$ and there exists a constant $C>0$ such that*

$$\begin{aligned}\|\overline{u}\|_{2,2} + \|\overline{z}_0\|_{2,2} &\le C(\|u^d\|+\|q_0\|),\\ \|\overline{q}\|_{1,s} &\le C\overline{C}_\tau\end{aligned}$$

holds.

Proof. The assertion follows from the previous Lemma 4 together with elliptic regularity. □

3. The Problem ($\mathbb{P}^N$) with Finitely Many State Constraints and its Properties

In this section, we will introduce a problem with finitely many pointwise state constraints, relying on the same assumptions and using the same notations as in the previous section. Recall that the states $u = S(q)$ are continuous with homogeneous Dirichlet boundary conditions and thus the constraints can only be active in an interior subdomain of Ω denoted by Ω_0. We will therefore consider some given, fixed points $x_i \subset \Omega_0$, $i \in \mathcal{B}_N = \{1,\ldots,N\}$ in which we impose pointwise inequality constraints, and the two vectors $u^- < u^+ \in \mathbb{R}^N$ are obtained from evaluating $u^{\pm}$ at $x_i \in \Omega_0$, $i = 1,\ldots,N$ with $\operatorname{dist}(\Omega_0,\Gamma) > 0$. We point out that we do not require the x_i to be grid points, yet to have pairwise positive distance.

We consider the following optimization problem

$$\begin{aligned}&\text{Minimize } J(q,u)\\ &\quad\text{s.t. } (\nabla u,\nabla\varphi) = (q,\varphi) \quad \forall\varphi\in V,\\ &\qquad u_i^- \le u(x_i) \le u_i^+ \quad i\in\mathcal{B}_N.\end{aligned} \tag{3.1}$$

Since the problem is governed by the same partial differential equation as Problem (2.1), we use the same control-to-state operator and obtain the reduced problem formulation

$$(\mathbb{P}^N)\qquad \begin{aligned}&\text{Minimize } j(q)\\ &\quad\text{s.t. } u_i^- \le S(q)(x_i) \le u_i^+ \quad i \in \mathcal{B}_N.\end{aligned}$$

Obviously, the set of feasible controls,

$$Q^N_{\text{feas}} := \{q \in Q\colon\ u_i^- \le S(q)(x_i) \le u_i^+, \quad i \in \mathcal{B}_N\},$$

is not empty since it contains the set Q_{feas}. The - by strict convexity - unique optimal control will be denoted by $\overline{q}_N$, with associated optimal state $\overline{u}_N$. Also, the Slater point q_τ for Problem (2.1) is a Slater point for Problem $(\mathbb{P}^N)$ independent of N as well. It is clear that also $|u_i^+ - u_i^-| \ge 2\tau$.

Optimality conditions for Problem $(\mathbb{P}^N)$ can now be obtained analogously to Problem $(\mathbb{P})$. However, since there are only finitely many constraints, the Lagrange multipliers have a specific structure, compare [3] for problems with only finitely many active points. We denote the associated space of Lagrange multipliers by

$$\mathcal{M} = \mathcal{M}_N = \{\mu \in \mathcal{M}(\Omega) \,|\, \mu = \sum_{i\in\mathcal{B}_N} \mu_i \delta_{x_i}\},$$

where δ_{x_i} denotes the Dirac measure associated to $x_i \in \Omega_0$. In particular, the norm on $\mathcal{M}$ is given by

$$|\mu|_{\mathcal{M}} = \|\mu\|_{C^*} = \sup_{v\in C(\overline{\Omega_0})} \frac{\langle v,\mu\rangle}{\|v\|} = \sum_{i\in\mathcal{B}_N} |\mu_i|.$$

In contrast to $(\mathbb{P})$ the space $\mathcal{M} \subset \mathcal{M}(\Omega)$ is separable. We obtain the following analogue to Theorem 3.

Theorem 6. *A control $\overline{q}_N \in Q_{feas}$ with associated state $\overline{u}_N = S(\overline{q}_N)$ is an optimal solution of Problem $(\mathbb{P}^N)$ if and only if there exist nonnegative Lagrange multipliers $\overline{\mu}_N^+, \overline{\mu}_N^- \in \mathcal{M} = \mathcal{M}_N$ as well as an adjoint state $\overline{z}_N \in W_0^{1,s}(\Omega)$, $s < \frac{d}{d-1}$, such that the adjoint equation*

$$(\nabla\varphi, \nabla\overline{z}_N) = (\varphi, \overline{u}_N - u^d) + \langle \overline{\mu}_N^+ - \overline{\mu}_N^-, \varphi\rangle \quad \forall\, \varphi \in W_0^{1,s'}(\Omega) \tag{3.2}$$

with $\frac{1}{s} + \frac{1}{s'} = 1$, the gradient equation

$$\alpha\overline{q}_N + \overline{z}_N = 0, \tag{3.3}$$

and the complementary slackness conditions

$$\langle \overline{\mu}_N^+, \overline{u}_N - u^+\rangle = \langle \overline{\mu}_N^-, u^- - \overline{u}_N\rangle = 0 \tag{3.4}$$

are fulfilled.

Proof. As for Theorem 3, the proof follows along the lines of [3]. □

For any $i \in \mathcal{B}_N$, we have the associated Green's function $z_i \in W_0^{1,s}(\Omega)$, $s < \frac{d}{d-1}$ given by

$$(\nabla\varphi, \nabla z_i) = \varphi(x_i) \quad \forall\, \varphi \in W_0^{1,s'}(\Omega). \tag{3.5}$$

With this we can split the adjoint state as $\overline{z}_N = \overline{z}_{0,N} + \sum_{i\in\mathcal{B}_N} (\overline{\mu}_{i,N}^+ - \overline{\mu}_{i,N}^-) z_i$ with $\mu_{i,N}^\pm \in \mathbb{R}_+$ where $\mu_N^\pm = \sum_{i\in\mathcal{B}_N} \mu_{i,N}^\pm \delta_{x_i}$ and $\overline{z}_{0,N} \in H_0^1(\Omega)$ solves

$$(\nabla\varphi, \nabla\overline{z}_{0,N}) = (\overline{u}_N - u^d, \varphi) \quad \forall\, \varphi \in H_0^1(\Omega).$$

Analogous to Lemma 4 and Corollary 5 we obtain:

Lemma 7. *For the variables given by Theorem 6, there exists a constant $C > 0$ such that the estimates*

$$\begin{aligned}\|\overline{q}_N\| + \|\overline{u}_N\| + \|\overline{z}_{0,N}\| &\leq C(\|u^d\| + \|q_0\|),\\ |\overline{\mu}_N^+|_{\mathcal{M}} + |\overline{\mu}_N^-|_{\mathcal{M}} &\leq \overline{C}_\tau\end{aligned}$$

hold.

Corollary 8. *With the notation of Lemma 7, for any $s < \frac{d}{d-1}$, we have $\overline{q}_N \in W^{1,s}(\Omega)$, $\overline{u}_N, \overline{z}_{0,N} \in V \cap H^2(\Omega)$ and there exists a constant $C > 0$ such that*

$$\begin{aligned}\|\overline{u}_N\|_{2,2} + \|\overline{z}_{0,N}\|_{2,2} &\leq C(\|u^d\| + \|q_0\|),\\ \|\overline{q}_N\|_{1,s} &\leq C\overline{C}_\tau\end{aligned}$$

holds.

4. The Discrete Problem ($\mathbb{P}_h$) and its Properties

Consider a typical discretization $\mathcal{T}_h$ of Ω into triangular or tetrahedral elements K fulfilling the usual conformity and shape regularity conditions, cf., [2]. The discretization parameter h denotes the maximum element size, i.e., $h := \max h_K$, where h_K denotes the diameter of a element K. The standard conforming finite element space is given by

$$V_h := \{v \in C(\overline{\Omega}) \colon\ v_{|K} \in \mathcal{P}_1(K) \ \text{ for } K \in \mathcal{T}_h\}.$$

Here, $\mathcal{P}_1(K)$ denotes the space of linear polynomials. For a given control $q \in Q$, the discrete version of the state equation is then given by

$$(\nabla u_h, \nabla \varphi_h) = (q, \varphi_h) \quad \forall \varphi_h \in V_h. \tag{4.1}$$

The discrete state equation defines an operator $S_h \colon Q \to V_h$. We arrive at the following discrete problem

$$(\mathbb{P}_h) \qquad \begin{aligned}&\text{Minimize } J(q_h, u_h)\\ &\qquad \text{s.t. } (\nabla u_h, \nabla \varphi_h) = (q_h, \varphi_h) \quad \forall \varphi_h \in V_h,\\ &\qquad\qquad u_i^- \leq u_h(x_i) \leq u_i^+ \quad i \in \mathcal{B}_N,\end{aligned}$$

which we can equivalently express in reduced form as

$$(\mathbb{P}_h) \qquad \begin{aligned}&\text{Minimize } j_h(q) := J(q, S_h(q))\\ &\qquad \text{s.t. } u_i^- \leq S_h(q)(x_i) \leq u_i^+ \quad i \in \mathcal{B}_N.\end{aligned}$$

Note that we follow the variational discretization concept introduced by Hinze [9], i.e., we do not discretize the control explicitly. We point out that since we do not consider control constraints, this is equivalent to a full discretization of the optimal control problem using P_1 finite elements. This yields the following definition of discrete feasible controls:

$$Q^h_{\text{feas}} = \{q \in L^2(\Omega) \,|\, u_i^- \leq (S_h q)(x_i) \leq u_i^+ \quad \forall i \in \mathcal{B}_N\}.$$

Recall that while the number of constraints may or may not depend on h, it is still clear by standard error estimates for the state equation, i.e.,

$$\|S - S_h\|_{\mathcal{L}(L^2(\Omega), L^\infty(\Omega))} \leq c h^{2-d/2},$$

that for h small enough the Slater point q_τ is also a Slater point for the discrete problem, w.l.o.g. using the same value τ. This asserts the existence of a feasible point for $(\mathbb{P}_h)$ and hence a unique solution of $(\mathbb{P}_h)$, and, moreover, the existence of discrete Lagrange multipliers. For ease of presentation, we will not consider the superscript N in the notation for the discrete optimal variables, although the problem may depend on the set $\{x_i \,|\, i \in \mathcal{B}_N\}$.

Corollary 9. *For h small enough, there exists a unique optimal control $\bar{q}_h$ with associated optimal state $\bar{u}_h = S_h(\bar{q}_h)$ to the discretized problem $(\mathbb{P}_h)$.*

Theorem 10. *For h small enough, a control $\overline{q}_h \in Q^h_{feas}$ with associated state $\overline{u}_h = S_h(\overline{q}_h)$ is an optimal solution of Problem $(\mathbb{P}_h)$ if and only if there exist nonnegative Lagrange multipliers $\overline{\mu}_h^+, \overline{\mu}_h^- \in \mathcal{M}$ as well as an adjoint state $\overline{z}_h \in V_h$, such that the discrete adjoint equation*

$$(\nabla\varphi, \nabla\overline{z}_h) = (\varphi, \overline{u}_h - u^d) + \langle \overline{\mu}_h^+ - \overline{\mu}_{N,h}^-, \varphi\rangle \quad \forall\, \varphi \in V_h, \tag{4.2}$$

the discrete gradient equation

$$\alpha\overline{q}_h + \overline{z}_h = 0, \tag{4.3}$$

and the complementary slackness conditions

$$\langle \overline{\mu}_h^+, \overline{u}_h - u^+\rangle = \langle \overline{\mu}_h^-, u^- - \overline{u}_h\rangle = 0 \tag{4.4}$$

are fulfilled.

Analogous to the continuous case, for any $i \in \mathcal{B}_N$, we have the associated discrete Green's function $z_{i,h} \in V_h$ given by

$$(\nabla\varphi, \nabla z_{i,h}) = \varphi(x_i) \quad \forall\; \varphi \in V_h. \tag{4.5}$$

With this we can split the adjoint state as $\overline{z}_h = \overline{z}_{0,h} + \sum_{i\in\mathcal{B}_N}(\overline{\mu}_{h,i}^+ - \overline{\mu}_{h,i}^-)z_{i,h}$ with $\mu_{h,i}^\pm \in \mathbb{R}_+$ where $\mu_h^\pm = \sum_{i\in\mathcal{B}_N}\mu_{h,i}^\pm\delta_{x_i}$ and $\overline{z}_{0,h} \in V_h$ solving

$$(\nabla\varphi_h, \nabla\overline{z}_{0,h}) = (\overline{u}_h - u^d, \varphi_h) \quad \forall\; \varphi_h \in V_h.$$

Lemma 11. *For the variables given by Theorem 10, there exists a constant $C > 0$ such that the bounds*

$$\begin{aligned}\|\overline{q}_h\| + \|\overline{u}_h\| + \|\overline{z}_{0,h}\| &\le C(\|u^d\| + \|q_0\|),\\ |\overline{\mu}_h^+|_{l^1} + |\overline{\mu}_h^-|_{l^1} &\le \overline{C}_\tau\end{aligned}$$

hold. Here, w.l.o.g., the constant $\overline{C}_\mu$ given by Lemma 4 can be used.

Proof. This is analogous to Lemma 4. □

5. Auxiliary Results

In preparation of our error estimates, we collect some auxiliary results. Let us first point out that by standard finite element estimates, we know with $s < \frac{d}{d-1}$, there is some $\varepsilon > 0$ such that

$$\begin{aligned}\|S - S_h\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} &\le ch^2,\\ \|S - S_h\|_{\mathcal{L}(L^2(\Omega), L^\infty(\Omega))} &\le ch^{2-d/2},\\ \|S - S_h\|_{\mathcal{L}(W^{1,s}(\Omega), L^\infty(\Omega_0))} &\le ch^{2-\varepsilon}.\end{aligned}$$

We refer to [18] for the finite element error on S in $\mathcal{L}(W^{1,s}, L^\infty)$. The estimates in $\mathcal{L}(L^2, L^2)$ and $\mathcal{L}(L^2, L^\infty)$ are standard finite element error estimates. We also collect a few standard estimates for the finite element error for the PDE solution, as they are provided in [12]. The only update is the explicit dependence on the Slater point since we will need it in our subsequent analysis.

Lemma 12. *Denote by $u(q)$ and $u_h(q)$ the solutions to the state equation and its discretization* (4.1) *with given, fixed, right-hand-side q. Then, for the optimal controls $\overline{q}_N$ and $\overline{q}_h$ to $(\mathbb{P}^N)$ and $(\mathbb{P}_h)$ it holds*

$$|\overline{u}_N(x_i) - u_h(\overline{q}_N)(x_i)| \le C\overline{C}_\tau h^{4-d}(|\ln h| + 1)^{7-2d},$$
$$|u(\overline{q}_h)(x_i) - \overline{u}_h(x_i)| \le C\overline{C}_\tau h^{4-d}(|\ln h| + 1)^{7-2d},$$

where $(\overline{q}_N, \overline{u}_N)$ solve $(\mathbb{P}^N)$ and $(\overline{q}_h, \overline{u}_h)$ solve $(\mathbb{P}_h)$.

Proof. The proof can be found in [12, Corollary 1], with the obvious modifications due to the bounds on the multipliers given in Lemma 7 and Lemma 11. □

Lemma 13. *Let $(\overline{q}_N, \overline{u}_N)$ solve Problem $(\mathbb{P}^N)$ and $(\overline{q}_h, \overline{u}_h)$ solve Problem $(\mathbb{P}_h)$. Then there exists a constant $C > 0$ such that*

$$\|\overline{u}_N - \overline{u}_h\| \le C\|\overline{q}_N - \overline{q}_h\| + Ch^2(\|u^d\| + \|q_0\|),$$
$$\|\nabla(\overline{u}_N - \overline{u}_h)\| \le C\|\overline{q}_N - \overline{q}_h\| + Ch(\|u^d\| + \|q_0\|)$$

holds

Proof. This is analogous to [12, Lemma 7]. □

Lemma 14. *Let $(\overline{q}_N, \overline{u}_N)$ solve Problem $(\mathbb{P}^N)$ and $(\overline{q}_h, \overline{u}_h)$ solve Problem $(\mathbb{P}_h)$. Further, we define $\overline{z}_{0,N}$, $\overline{z}_{0,h}$, z_i, and $z_{i,h}$ as in the discussion after Theorems 6 and 10. Then it holds*

$$\|\overline{z}_{0,N} - \overline{z}_{0,h}\| \le C\|\overline{q}_N - \overline{q}_h\| + Ch^2(\|u^d\| + \|q_0\|),$$
$$\|\nabla(\overline{z}_{0,N} - \overline{z}_{0,h})\| \le C\|\overline{q}_N - \overline{q}_h\| + Ch(\|u^d\| + \|q_0\|),$$
$$\|z_i - z_{i,h}\| \le Ch^{2-\frac{d}{2}}.$$

Proof. This is analogous to [12, Lemma 8 and Lemma 9]. □

6. An Error Estimate between $(\mathbb{P}^N)$ and $(\mathbb{P}_h)$

6.1. **The Dual Problem.** We remind the reader, that for the optimal controls $\overline{q}_N$ and $\overline{q}_h$ to Problems $(\mathbb{P}^N)$ and $(\mathbb{P}_h)$, respectively, and any $\varepsilon > 0$ there exists $C > 0$ such that

$$\|\overline{q}_N - \overline{q}_h\| \le C\overline{C}_\tau h^{1-\varepsilon},$$

compare, for instance [8, 16].

From this we conclude that the respective optimal states $\overline{u}_N$ and $\overline{u}_h$ to $(\mathbb{P}^N)$ and $(\mathbb{P}_h)$ converge in $L^\infty(\Omega)$, since standard L^∞ estimates for the discretization of the PDE, see, e.g., [17], together with Lipschitz continuity of the solution operator $S\colon L^2(\Omega) \to L^\infty(\Omega)$, guarantee

$$\|\overline{u}_N - \overline{u}_h\|_\infty \le C\overline{C}_\tau h^{1-\varepsilon}.$$

In particular, for

$$h \le \Big(\frac{2\tau}{C\overline{C}_\tau}\Big)^{1/(1-\varepsilon)},$$

we conclude that

$$\{i \in \mathcal{B}_N \,|\, \bar{\mu}^+_{i,N} > 0 \text{ or } \bar{\mu}^+_{i,h} > 0\} \cap \{i \in \mathcal{B}_N \,|\, \bar{\mu}^-_{i,N} > 0 \text{ or } \bar{\mu}^-_{i,h} > 0\} = \emptyset,$$

i.e., the discretization does not lead to points in which the state switches from the upper to the lower bound or vice versa.

Before we come to the proof of the main result of this paper, we need to state an appropriate dual problem. The resulting dual is similar to the one considered in [12] with some modifications due to the consideration of inequality constraints and the possible increase of the number of constraints as $h \to 0$.

To this end, we define the following sets:

$$\begin{aligned}
\mathcal{A}^+_+ &= \{i \in \mathcal{B}_N \,|\, (\bar{\mu}^+_h - \bar{\mu}^+_N)_i > 0\},\\
\mathcal{A}^+_- &= \{i \in \mathcal{B}_N \,|\, (\bar{\mu}^+_h - \bar{\mu}^+_N)_i < 0\},\\
\mathcal{A}^-_+ &= \{i \in \mathcal{B}_N \,|\, (\bar{\mu}^-_N - \bar{\mu}^-_h)_i > 0\},\\
\mathcal{A}^-_- &= \{i \in \mathcal{B}_N \,|\, (\bar{\mu}^-_N - \bar{\mu}^-_h)_i < 0\}.
\end{aligned}$$

By this construction it follows:

$$\begin{aligned}
i \in \mathcal{A}^+_+ &\Rightarrow \overline{u}_h(x_i) = u_i^+,\\
i \in \mathcal{A}^+_- &\Rightarrow \overline{u}_N(x_i) = u_i^+,\\
i \in \mathcal{A}^-_+ &\Rightarrow \overline{u}_N(x_i) = u_i^-,\\
i \in \mathcal{A}^-_- &\Rightarrow \overline{u}_h(x_i) = u_i^-.
\end{aligned} \tag{6.1}$$

From our above arguments, we conclude that all four sets are pairwise disjoint once h is sufficiently fine.

Now, let

$$\tilde{u}^d = \frac{\overline{u}_N - \overline{u}_h}{\|\overline{u}_N - \overline{u_h}\|}$$

and pick $\theta \geq 0$. To shorten notation, we introduce the sets

$$\mathcal{D}^+ = \mathcal{A}^-_+ \cup \mathcal{A}^+_+, \qquad \mathcal{D}^- = \mathcal{A}^+_- \cup \mathcal{A}^-_-.$$

By definition of these sets, we observe the relations

$$\begin{aligned}
\bar{u}_N - \bar{u}_h \leq 0 &\qquad \text{on } \mathcal{D}^+,\\
\bar{u}_N - \bar{u}_h \geq 0 &\qquad \text{on } \mathcal{D}^-.
\end{aligned} \tag{6.2}$$

With this we can consider the following dual problem to find $(\tilde{q}, \tilde{u})$ solving

$$\begin{aligned}
&\text{Minimize } \frac{1}{2}\|\tilde{u} - \tilde{u}^d\|^2 + \frac{\alpha}{2}\|\tilde{q}\|^2\\
&\text{s.t. } (\nabla \tilde{u}, \nabla \varphi) = (\tilde{q}, \varphi) \quad \forall \varphi \in V,\\
&\tilde{u}(x_i) \leq -\theta \quad i \in \mathcal{D}^+,\\
&\tilde{u}(x_i) \geq \theta \quad i \in \mathcal{D}^-.
\end{aligned} \tag{$\mathbb{P}_D$}$$

Lemma 15. *There exists a Slater point q^θ_τ for Problem $(\mathbb{P}_D)$.*

Proof. This is a direct consequence of the fact that by construction we know that $\operatorname{dist}(\mathcal{D}^+, \mathcal{D}^-) > \varepsilon$, where $\varepsilon = \varepsilon(h) > 0$ depends on the distance of the points x_i. □

In particular, we assert that there exist Lagrange multipliers $\tilde\mu^+, \tilde\mu^- \in \mathcal{M}$ associated to the upper bound on $\mathcal{D}^+$ and the lower bound on $\mathcal{D}^-$ and an adjoint state $\tilde z \in W_0^{1,s}(\Omega)$ for $s < \frac{d}{d-1}$ such that

$$
\begin{aligned}
(\nabla \tilde u, \nabla\varphi) &= (\tilde q, \varphi) && \forall \varphi \in V,\\
(\nabla\varphi, \nabla\tilde z) &= (\tilde u - \tilde u^d, \varphi)\\
&\quad + \langle \varphi, \tilde\mu^+\rangle - \langle\varphi, \tilde\mu^-\rangle && \forall\varphi \in W_0^{1,s'}(\Omega),\\
\alpha\tilde q + \tilde z &= 0,\\
\tilde\mu_i^+ &= 0 && \forall i \in \mathcal{B}_N \setminus \mathcal{D}^+,\\
\tilde\mu_i^- &= 0 && \forall i \in \mathcal{B}_N \setminus \mathcal{D}^-,\\
\langle \tilde u + \theta, \tilde\mu^+\rangle &= 0,\\
\langle \theta - \tilde u, \tilde\mu^-\rangle &= 0,\\
\tilde\mu^+, \tilde\mu^- &\ge 0,\\
(\theta + \tilde u)(x_i) &\le 0 && \forall i \in \mathcal{D}^+,\\
(\theta - \tilde u)(x_i) &\le 0 && \forall i \in \mathcal{D}^-.
\end{aligned}
\tag{6.3}
$$

Again, we can split $\tilde z = \tilde z_0 + \sum_{i\in\mathcal{B}_N}(\tilde\mu_i^+ - \tilde\mu_i^-)z_i$.

Lemma 16. *For the solution of* $(\mathbb{P}_D)$ *and the associated variables by* (6.3) *there exists a constant* $C > 0$ *such that there holds:*

$$
\begin{aligned}
\|\tilde q\| + \|\tilde u\| + \|\tilde z_0\| &\le C(\|\tilde u^d\| + \|q_0^\theta\|) =: \widetilde C_0,\\
|\tilde\mu^+|_{\mathcal{M}} + |\tilde\mu^-|_{\mathcal{M}} &\le \frac{C}{\tau}(\|\tilde u^d\| + \|q_0^\theta\| + \|q_\tau^\theta\|)(\|\tilde u^d\| + \|q_0^\theta\|)\\
&=: \widetilde C_\tau.
\end{aligned}
$$

Without loss of generality, we assume $1 \le \widetilde C_0 \le \widetilde C_\tau$.

Proof. The proof is analog to Lemma 4. □

The essential difficulty at this point is that, in contrast to $\overline{C}_\tau$, the multiplier bound $\widetilde C_\tau$ for the dual problem might degenerate since $\|q_\tau^\theta\|$ might go to infinity if the distance between the points x_i tends to zero. Hence, $\widetilde C_0 \le \widetilde C_\tau$ need to be tracked explicitly.

6.2. **The Main Result.** We have now collected all needed auxiliary details to come to the proof of the main result. Before that we restate well known estimates for the control variable tracking the constant $\widetilde C_\tau$ explicitly.

Theorem 17. *Let* $\overline{q}_N$ *and* $\overline{q}_h$ *be the solutions to* $(\mathbb{P}^N)$ *and* $(\mathbb{P}_h)$. *Then there exists a constant* $C = c\overline{C}_\tau > 0$ *such that*

$$\|\overline{q}_N - \overline{q}_h\| \le C h^{2-\frac{d}{2}}(|\ln(h)| + 1)^{\frac{7}{2}-d}$$

holds.

Proof. The proof closely resembles the proofs of [8, Theorem 3.6] and [12, Theorem 4].

Using the optimality conditions (3.2) we obtain

$$
\begin{aligned}
(u(\overline{q}_h)-\overline{u}_N,\overline{u}_N-u^d) &= (\nabla\overline{z},\nabla(u(\overline{q}_h)-\overline{u}_N)) - \sum_{i\in\mathcal{B}_N}(\overline{\mu}^+_{i,N}-\overline{\mu}^-_{i,N})(u(\overline{q}_h)-\overline{u}_N)(x_i)\\
&= (\overline{z},\overline{q}_h-\overline{q}_N) - \sum_{i\in\mathcal{B}_N}(\overline{\mu}^+_{i,N}-\overline{\mu}^-_{i,N})(u(\overline{q}_h)-\overline{u}_N)(x_i)\\
&= -\alpha(\overline{q}_N,\overline{q}_h-\overline{q}_N) - \sum_{i\in\mathcal{B}_N}(\overline{\mu}^+_{i,N}-\overline{\mu}^-_{i,N})(u(\overline{q}_h)-\overline{u}_N)(x_i).
\end{aligned}
$$

For the difference in the functional values, we obtain from the above calculation

$$
\begin{aligned}
j(\overline{q}_h)-j(\overline{q}_N) &= \frac{1}{2}\|u(\overline{q}_h)-\overline{u}_N\|^2+\frac{\alpha}{2}\|\overline{q}_N-\overline{q}_h\|^2\\
&\quad+(u(\overline{q}_h)-\overline{u}_N,\overline{u}_N-u^d)+\alpha(\overline{q}_N,\overline{q}_h-\overline{q}_N)\\
&= \frac{1}{2}\|u(\overline{q}_h)-\overline{u}_N\|^2+\frac{\alpha}{2}\|\overline{q}_N-\overline{q}_h\|^2\\
&\quad-\sum_{i\in\mathcal{B}_N}(\overline{\mu}^+_{i,N}-\overline{\mu}^-_{i,N})(u(\overline{q}_h)-\overline{u}_N)(x_i).
\end{aligned}\tag{6.4}
$$

By an analogous calculation, we obtain

$$
\begin{aligned}
j_h(\overline{q}_N)-j_h(\overline{q}_h) &= \frac{1}{2}\|\overline{u}_h-u_h(\overline{q}_N)\|^2+\frac{\alpha}{2}\|\overline{q}_N-\overline{q}_h\|^2\\
&\quad-\sum_{i\in\mathcal{B}_N}(\overline{\mu}^+_{i,h}-\overline{\mu}^-_{i,h})(\overline{u}_h-u_h(\overline{q}_N))(x_i).
\end{aligned}\tag{6.5}
$$

Summation of the equations (6.4) and (6.5) yields

$$
\begin{aligned}
\alpha\|\overline{q}_N-\overline{q}_h\|^2 &\le |j(\overline{q}_N)-j_h(\overline{q}_N)|+|j(\overline{q}_h)-j_h(\overline{q}_h)|\\
&\quad+\sum_{i\in\mathcal{B}_N}(\overline{\mu}^+_{i,N}-\overline{\mu}^-_{i,N})(u(\overline{q}_h)-\overline{u}_N)(x_i)\\
&\quad+\sum_{i\in\mathcal{B}_N}(\overline{\mu}^+_{i,h}-\overline{\mu}^-_{i,h})(\overline{u}_h-u_h(\overline{q}_N))(x_i).
\end{aligned}\tag{6.6}
$$

To continue, we note that, due to the nonnegativity of $\bar{\mu}^+_N$, complementary slackness (3.4), and feasibility of $\overline{u}_h$, it holds

$$
\begin{aligned}
\sum_{i\in\mathcal{B}_N}\overline{\mu}^+_{i,N}(u(\overline{q}_h)-\overline{u}_N)(x_i) &= \sum_{i\in\mathcal{B}_N}\overline{\mu}^+_{i,N}(u(\overline{q}_h)-u^++u^+-\overline{u}_N)(x_i)\\
&= \sum_{i\in\mathcal{B}_N}\overline{\mu}^+_{i,N}(u(\overline{q}_h)-u^+)(x_i)\\
&\le \sum_{i\in\mathcal{B}_N}\overline{\mu}^+_{i,N}(u(\overline{q}_h)-\overline{u}_h)(x_i)\\
&\le |\mu^+_N|_{l^1}\max_{i\in\mathcal{B}_N}|u(\overline{q}_h)-\overline{u}_h)(x_i)|.
\end{aligned}
$$

By Lemma 12 and Lemma 7, we conclude

$$
\sum_{i\in\mathcal{B}_N}\overline{\mu}^+_{i,N}(u(\overline{q}_h)-\overline{u}_N)(x_i)\le C\overline{C}_\tau^2h^{4-d}(|\ln h|+1)^{7-2d}.\tag{6.7}
$$

Analog calculations for the remaining multiplier terms, using Lemma 12 and Lemma 11 for the second sum in (6.6), give

$$\alpha\|\overline{q}_N-\overline{q}_h\|^2 \le |j(\overline{q}_N)-j_h(\overline{q}_N)|+|j(\overline{q}_h)-j_h(\overline{q}_h)| + C\widetilde{C}_\tau^2 h^{4-d}(|\ln h|+1)^{7-2d}. \tag{6.8}$$

Further, a straightforward calculation for j yields

$$\begin{aligned} 2|j(\overline{q}_N)-j_h(\overline{q}_N)| &= \|\overline{u}_N-u^d\|^2-\|u_h(\overline{q}_N)-u^d\|^2 \\ &= (\overline{u}_N-u_h(\overline{q}_N),\overline{u}_N-u^d+u_h(\overline{q}_N)-u^d) \\ &\le \|\overline{u}_N-u_h(\overline{q}_N)\|(\|\overline{u}_N\|+\|u_h(\overline{q}_N)\|+2\|u^d\|) \\ &\le c\overline{C}_0h^2. \end{aligned}$$

Similarly, we obtain

$$|j(\overline{q}_h)-j_h(\overline{q}_h)| \le c\overline{C}_0h^2.$$

Combining this with (6.8) gives the assertion. □

Theorem 18. *Let $\tilde{q}$ be the solution to $(\mathbb{P}_D)$ and define*

$$\tilde{q}_h := I_h\tilde{z}_0+\sum_{i\in\mathcal{B}_N}(\tilde{\mu}_i^+-\tilde{\mu}_i^-)z_{i,h},$$

with $z_{i,h}$ as in (4.5). Then it holds

$$\|\tilde{q}-\tilde{q}_h\| \le C\widetilde{C}_\tau h^{2-\frac{d}{2}}(|\ln(h)|+1)^{\frac{7}{2}-d}.$$

Proof. The proof is similar to the one for the primal problem in Theorem 17. The only difference occurs in estimates (6.7) and (6.8), where it is sufficient to point out that the properties of the interpolant I_h guarantee that $\tilde{q}_h$ fulfills the estimates from Lemma 12. □

Theorem 19. *There is a constant $C>0$, independent of θ, N, h such that for the solutions $\overline{u}_N$ and $\overline{u}_h$ to $(\mathbb{P}^N)$ and $(\mathbb{P}_h)$ and the corresponding Lagrange multipliers $\overline{\mu}_N^\pm$, $\overline{\mu}_h^\pm$ given by Theorems 6 and 10 it holds*

$$\|\overline{u}_N-\overline{u}_h\|+\theta|\overline{\mu}_N^+-\overline{\mu}_h^+|_{\mathcal{M}}+\theta|\overline{\mu}_N^--\overline{\mu}_h^-|_{\mathcal{M}} \le C\widetilde{C}_\tau h^{4-d}(|\ln(h)|+1)^{7-2d}$$

with the constant $\widetilde{C}_\tau$ as in Lemma 16.

Proof. We abbreviate the errors as follows

$$\begin{aligned} e^u &= \overline{u}_N-\overline{u}_h \in V\cap W^{1,s'}(\Omega), \\ e^q &= \overline{q}_N-\overline{q}_h \in Q, \\ e^z &= \overline{z}_N-\overline{z}_h \in W_0^{1,s}(\Omega), \\ e^{\mu^+} &= \overline{\mu}_N^+-\overline{\mu}_h^+ \in \mathcal{M}, \\ e^{\mu^-} &= \overline{\mu}_N^--\overline{\mu}_h^- \in \mathcal{M}. \end{aligned}$$

From the second equation in the optimality conditions (6.3) of the dual problem, we deduce, by definition of $\tilde{u}^d$,

$$\begin{aligned} \|e^u\| &= (\tilde{u}^d,e^u) \\ &= (\tilde{u},e^u)+\langle e^u,\tilde{\mu}^+\rangle-\langle e^u,\tilde{\mu}^-\rangle-(\nabla e^u,\nabla\tilde{z}). \end{aligned}$$

Further, by definition of $\mathcal{A}_+^\pm$ and $\mathcal{A}_-^\pm$, we assert,

$$|e^{\mu^+}|_{\mathcal{M}} = \sum_{i\in\mathcal{A}_+^+} -e_i^{\mu^+} + \sum_{i\in\mathcal{A}_-^+} e_i^{\mu^+},$$
$$|e^{\mu^-}|_{\mathcal{M}} = \sum_{i\in\mathcal{A}_+^-} e_i^{\mu^-} + \sum_{i\in\mathcal{A}_-^-} -e_i^{\mu^-}.$$

Further, by the sign and complementarity conditions in (6.3), it follows

$$\begin{aligned}
0 \le &\sum_{i\in\mathcal{A}_+^+} (\theta+\tilde{u})(x_i)e_i^{\mu^+} - \sum_{i\in\mathcal{A}_+^-} (\theta+\tilde{u})(x_i)e_i^{\mu^-}\\
&- \sum_{i\in\mathcal{A}_-^+} (\theta-\tilde{u})(x_i)e_i^{\mu^+} + \sum_{i\in\mathcal{A}_-^-} (\theta-\tilde{u})(x_i)e_i^{\mu^-}\\
= &-\theta|e^{\mu^+}|_{\mathcal{M}} + \sum_{i\in\mathcal{A}_+^+} \tilde{u}(x_i)e_i^{\mu^+} + \sum_{i\in\mathcal{A}_-^+} \tilde{u}(x_i)e_i^{\mu^+}\\
&- \theta|e^{\mu^-}|_{\mathcal{M}} - \sum_{i\in\mathcal{A}_+^-} \tilde{u}(x_i)e_i^{\mu^-} - \sum_{i\in\mathcal{A}_-^-} \tilde{u}(x_i)e_i^{\mu^-}.
\end{aligned}$$

Combination of the previous calculations with the first and third equation in (6.3) yields

$$\begin{aligned}
\|e^u\| + \theta|e^{\mu^+}|_{\mathcal{M}} + \theta|e^{\mu^-}|_{\mathcal{M}} \le &(\tilde{u}, e^u) + \langle e^u, \tilde{\mu}^+\rangle - \langle e^u, \tilde{\mu}^-\rangle - (\nabla e^u, \nabla\tilde{z})\\
&+ \sum_{i\in\mathcal{A}_+^+\cup A_-^+} \tilde{u}(x_i)e_i^{\mu^+} - \sum_{i\in\mathcal{A}_+^-\cup A_-^-} \tilde{u}(x_i)e_i^{\mu^-}\\
&+ (\tilde{q}, e^z) - (\nabla\tilde{u}, \nabla e^z) + \alpha(e^q, \tilde{q}) + (e^q, \tilde{z}).
\end{aligned}$$

Reordering the terms according to the equations in the necessary optimality conditions for the primal problem in Theorem 6 and its discretization in Theorem 10 gives

$$\begin{aligned}
\|e^u\| + \theta|e^{\mu^+}|_{\mathcal{M}} + \theta|e^{\mu^-}|_{\mathcal{M}} \le &(e^q, \tilde{z}) - (\nabla e^u, \nabla\tilde{z})\\
&+ (\tilde{u}, e^u) + \sum_{i\in\mathcal{A}_+^+\cup A_-^+} \tilde{u}(x_i)e_i^{\mu^+} - \sum_{i\in\mathcal{A}_+^-\cup A_-^-} \tilde{u}(x_i)e_i^{\mu^-} - (\nabla\tilde{u}, \nabla e^z)\\
&+ \alpha(e^q, \tilde{q}) + (\tilde{q}, e^z)\\
&+ \langle e^u, \tilde{\mu}^+\rangle - \langle e^u, \tilde{\mu}^-\rangle\\
\le &(e^q, \tilde{z}) - (\nabla e^u, \nabla\tilde{z})\\
&+ (\tilde{u}, e^u) + \sum_{i\in\mathcal{A}_+^+\cup A_-^+} \tilde{u}(x_i)e_i^{\mu^+} - \sum_{i\in\mathcal{A}_+^-\cup A_-^-} \tilde{u}(x_i)e_i^{\mu^-} - (\nabla\tilde{u}, \nabla e^z)\\
&+ \alpha(e^q, \tilde{q}) + (\tilde{q}, e^z).
\end{aligned}$$

The last inequality follows from the sign of $\tilde{\mu}^\pm$, see (6.2), together with the implied sign of e^u by the construction of $\mathcal{A}_\pm^\pm$, see (6.1). Now, by subtracting the continuous and discrete state equation, the adjoint equation (3.2) and (4.2) and the gradient equations (3.3) and (4.3), we see that we can add an arbitrary discrete function on the right in each of the three lines. To this end, let $\tilde q_h$ be chosen as in Theorem 18 and define $\tilde u_h := S_h(\tilde q_h)$. Then, consider

$$\tilde e^u = \tilde u - \tilde u_h, \quad \tilde e^q = \tilde q - \tilde q_h.$$

For the approximation of $\tilde z$, we define

$$\tilde z_h = \tilde z_{0,h} + \sum_{i\in\mathcal{B}} (\tilde\mu_i^+ - \tilde\mu_i^-) z_{i,h} \in V_h,$$

where $\tilde z_{0,h}$ solves

$$(\nabla\varphi, \nabla\tilde z_{0,h}) = (\tilde u_h - \tilde u^d, \varphi) \quad \forall\varphi\in V_h,$$

and define

$$\tilde e^z = \tilde z - \tilde z_h.$$

Thus we obtain

$$\begin{aligned}\|e^u\| + \theta|e^{\mu^+}|_{\mathcal{M}} + \theta|e^{\mu^-}|_{\mathcal{M}} \le{}& (e^q, \tilde e^z) + (\tilde e^u, e^u) + \alpha(e^q, \tilde e^q) + (\tilde e^q, e^z)\\ &- (\nabla\tilde e^u, \nabla e^z) - (\nabla e^u, \nabla\tilde e^z)\\ &+ \sum_{i\in\mathcal{A}_+^+\cup A_-^+} \tilde e^u(x_i) e_i^{\mu^+} - \sum_{i\in\mathcal{A}_+^-\cup A_-^-} \tilde e^u(x_i) e_i^{\mu^-}.\end{aligned} \tag{6.9}$$

Now, due to the gradient equation (3.3) and its analogs for the discrete and dual problem, and the stability of the solution operator, we see that

$$(\tilde e^u, e^u) + \alpha(e^q, \tilde e^q) + (\tilde e^q, e^z) \le C\|e^q\|\|\tilde e^q\|.$$

From Theorems 17 and 18 we conclude,

$$\begin{aligned}(\tilde e^u, e^u) &+ \alpha(e^q, \tilde e^q) + (\tilde e^q, e^z)\\ &\le C\widetilde C_\tau h^{4-d}(|\ln(h)|+1)^{7-2d}\end{aligned} \tag{6.10}$$

Now, we note that to estimate the term $(e^q, \tilde e^z)$ it is sufficient to bound

$$\begin{aligned}\|\tilde e^z\| &\le \|\tilde z_0 - \tilde z_{0,h}\| + \sum_{i\in\mathcal{B}_N} |\tilde\mu_i^+ - \tilde\mu_i^-|\|z_i - z_{i,h}\|\\ &\le C\|\tilde e^q\| + c\widetilde C_0 h^2 + C\widetilde C_\tau h^{2-\frac{d}{2}}\\ &\le C\widetilde C_\tau h^{2-\frac{d}{2}},\end{aligned}$$

where the last inequality follows from Lemma 14 applied to the dual problem and the bound on the l^1-norm of the multipliers in Lemma 7. Theorem 18 and Theorem 17 yield

$$(e^q, \tilde e^z) \le C\widetilde C_\tau h^{4-d}(|\ln(h)|+1)^{7-2d}. \tag{6.11}$$

To proceed, we recall that

$$e^z = \overline z_{0,N} - \overline z_{0,h} + \sum_{i\in\mathcal{B}_N}\Big((\overline\mu_i^+ - \overline\mu_i^-)z_i - (\overline\mu_{i,h}^+ - \overline\mu_{i,h}^-)z_{i,h}\Big).$$

We have, by Theorems 17 and 18 as well as Lemma 13 and 14, neglecting higher order terms in h,

$$\begin{aligned}(\nabla\tilde e^u, \nabla(\overline z_{0,N} - \overline z_{0,h})) &\le C\|\tilde q - \tilde q_h\|\|\overline u_N - \overline u_h\|\\ &\le C\|\tilde q - \tilde q_h\|\|\overline q_N - \overline q_h\|\\ &\le C\widetilde C_\tau h^{4-d}(|\ln(h)|+1)^{7-2d}.\end{aligned} \tag{6.12}$$

To continue, we calculate, for any $i \in \mathcal{B}_N$ noting that z_i is the Green's function for the point x_i,

$$
\begin{aligned}
(\nabla\tilde{e}^u, \overline{\mu}_i^+\nabla z_i - \overline{\mu}_{i,h}^+\nabla z_{i,h}) &= e_i^{\mu^+}(\nabla\tilde{e}^u, \nabla z_i) + \overline{\mu}_{i,h}^+(\nabla\tilde{e}^u, \nabla(z_i - z_{i,h})) \\
&= e_i^{\mu^+}(\tilde{u} - \tilde{u}_h)(x_i) + \overline{\mu}_{i,h}^+(\nabla\tilde{e}^u, \nabla(z_i - z_{i,h})).
\end{aligned} \tag{6.13}
$$

For the last summand, we use Galerkin orthogonality twice and get, using the properties of the Green's function and Lemma 12 applied to the dual problem,

$$
\begin{aligned}
|(\nabla\tilde{e}^u, \nabla(z_i - z_{i,h}))| &= |(\nabla(\tilde{u} - u_h(\tilde{q})), \nabla(z_i - z_{i,h}))| \\
&= |(\nabla(\tilde{u} - u_h(\tilde{q})), \nabla z_i)| \\
&= |(\tilde{u} - u_h(\tilde{q}))(x_i)| \\
&\le C\widetilde{C}_\tau h^{4-d}(|\ln h| + 1)^{7-2d}.
\end{aligned}
$$

Combining the last equation with (6.13), (6.12), we obtain, using analog calculations for the terms involving $\overline{\mu}_{i,N}^-$,

$$
\begin{aligned}
-(\nabla\tilde{e}^u, \nabla e^z) &\le C\widetilde{C}_\tau h^{4-d}(|\ln(h)| + 1)^{7-2d} \\
&\quad - \sum_{i\in\mathcal{B}_N} e_i^{\mu^+}(\tilde{u} - \tilde{u}_h)(x_i) + \sum_{i\in\mathcal{B}_N} e_i^{\mu^-}(\tilde{u} - \tilde{u}_h)(x_i) \\
&\quad + C\widetilde{C}_\tau h^{4-d}(|\ln h| + 1)^{7-2d} \sum_{i\in\mathcal{B}_N}(\overline{\mu}_{i,h}^+ + \overline{\mu}_{i,h}^-) \\
&\le C\widetilde{C}_\tau h^{4-d}(|\ln(h)| + 1)^{7-2d} \\
&\quad - \sum_{i\in\mathcal{A}_+^+\cup A_-^+} e_i^{\mu^+}(\tilde{u} - \tilde{u}_h)(x_i) + \sum_{i\in\mathcal{A}_+^-\cup A_-^-} e_i^{\mu^-}(\tilde{u} - \tilde{u}_h)(x_i),
\end{aligned} \tag{6.14}
$$

where, for the last inequality, we used the bounds on $|\overline{\mu}_h^\pm|_{l^1}$ from Lemma 11.

Noting that the last two sums are the same as those in (6.9) except for the opposite sign. We can combine (6.9), (6.10), (6.11), and (6.14) to obtain:

$$
\|e^u\| + \theta|e^{\mu^+}|_{\mathcal{M}} + \theta|e^{\mu^-}|_{\mathcal{M}} \le C\widetilde{C}_\tau h^{4-d}(|\ln(h)| + 1)^{7-2d} - (\nabla e^u, \nabla\tilde{e}^z). \tag{6.15}
$$

For the remaining term, we calculate, analog to $(\nabla\tilde{e}^u, \nabla e^z)$, as follows

$$
\begin{aligned}
(\nabla e^u, \nabla\tilde{e}^z) &\le C\widetilde{C}_\tau h^{4-d}(|\ln(h)| + 1)^{7-2d} \\
&\quad + \sum_{i\in\mathcal{B}_N}(\tilde{\mu}_i^+ - \tilde{\mu}_i^-)(\overline{u}_N - u_h(\overline{q}_N))(x_i).
\end{aligned}
$$

Hence, Lemma 12 and the l^1 bound on the dual multipliers in Lemma 16 prove the assertion. □

7. Implications of Theorems 19

In the following, we will consider two special cases in which we can show a bound on $\widetilde{C}_\tau$ independent of the variables h and N. From Lemma 16, we know that we have to construct a suitable dual Slater point q_τ^θ.

7.1. **Finitely Many Constraints.** We start our discussion with the case that one is interested in the solution of $(\mathbb{P}^N)$ with a given fixed finite set $\mathcal{B}_N$. In this case $\mathcal{B}_N$ is independent of h. It is then intuitively clear that the constant $\widetilde{C}_\tau$ defined in Lemma 16 and used in Theorem 19 is bounded independently of h while the dependence on N is irrelevant since N is fixed. To make this precise, we have the following result:

Theorem 20. *Let $\mathcal{B}_N$ be a given, fixed, finite set. Then there exists a constant C depending on $\mathcal{B}_N$ but independent of h, such that for h sufficiently small the error estimate*

$$\|\overline{u}_N - \overline{u}_h\| + |\overline{\mu}_N^+ - \overline{\mu}_h^+|_{\mathcal{M}} + |\overline{\mu}_N^- - \overline{\mu}_h^-|_{\mathcal{M}} \le Ch^{4-d}(|\ln(h)| + 1)^{7-2d}$$

holds for the solution $\overline{u}_N$ of $(\mathbb{P}^N)$ and $\overline{u}_h$ of $(\mathbb{P}_h)$ with corresponding Lagrange-multipliers $\overline{\mu}_N^\pm$ and $\overline{\mu}_h^\pm$.

Proof. In view of the results of Theorem 19, the only thing left to be proven is that the constant $\widetilde{C}_\tau$ coming from Lemma 16 can be chosen independent of h.

This is indeed the case. For the construction of the dual Slater point q_1^1, i.e., $\theta = 1$ and $\tau = 1$, we observe that the finitely many points in $\mathcal{B}_N$ have positive distance. Hence there exists $\varepsilon > 0$ such that the balls $B_\varepsilon(x_i)$ for $i \in \mathcal{B}_N$ do not intersect. Consequently, for each i there exists a $C_0^\infty(\Omega)$ function u_i such that $u_i(x_i) \ge 2 = \theta + \tau$ and $u_i \equiv 0$ on $\Omega \setminus B_\varepsilon(x_i)$. An appropriate right-hand-side is defined as $q_i = -\Delta u_i$. Summing these functions q_i with appropriate signs gives the desired Slater point q_1^1 independent of h. $\square$

7.2. **Weakly Active Constrains.** Next, we consider the Problem $(\mathbb{P})$ in the special situation where the Lagrange multiplier $\overline{\mu}^\pm$ to the optimal solution $(\overline{q}, \overline{u})$ is identically zero. In particular, in this situation the optimal pair $(\overline{q}, \overline{u})$ is a solution of the problem $(\mathbb{P})$ without inequality constraints and consequently $(\overline{q}, \overline{u}) = (\overline{q}_N, \overline{u}_N)$ for any choice of $\mathcal{B}_N$. The difficulty in this situation is that while the state constraint is irrelevant for the continuous problem; it may still be strictly active for any discretized problem, i.e., exhibit non-zero multipliers $\overline{\mu}_h^\pm$. The standard discretization $(\mathbb{P}_h)$ for this problem would involve $\mathcal{B}_N$ to be the set of all vertices on the given mesh. In this case, we obtain the following error estimate:

Theorem 21. *Let $(\overline{q}, \overline{u})$ be a solution to $(\mathbb{P})$ with Lagrange-multiplier $\overline{\mu}^\pm \equiv 0$, and let $(\overline{q}_h, \overline{u}_h)$ be the solution of $(\mathbb{P}_h)$ with corresponding Lagrange-multipliers $\overline{\mu}_h^\pm$ for the set $\mathcal{B}_N$ given by all vertices of the mesh. Then there exists a constant C independent of h such that for h sufficiently small the error estimate*

$$\|\overline{u} - \overline{u}_h\| + |\overline{\mu}^+ - \overline{\mu}_h^+|_{\mathcal{M}} + |\overline{\mu}^- - \overline{\mu}_h^-|_{\mathcal{M}} \le Ch^{4-d}(|\ln(h)| + 1)^{7-2d}$$

holds.

Proof. By the preliminary discussion at the beginning of this section, we know that $\overline{\mu}_N^\pm \equiv 0$ hence by the sign condition on $\overline{\mu}_h^\pm$ we know that the sets $\mathcal{A}_-^+ = \mathcal{A}_+^- = \emptyset$.

As a consequence $\mathcal{D}^+ = \mathcal{A}_+^+$ and $\mathcal{D}^- = \mathcal{A}_-^-$, i.e., these sets are a subset of the sets where the upper and lower bounds are strictly active, respectively, for the discrete problem. Let $\mathcal{A}^+$ and $\mathcal{A}^-$ be the set of active upper and lower bounds for $\overline{u}$ then for any $\varepsilon > 0$ the estimate

$$\operatorname{dist}(\mathcal{A}_+^+, \mathcal{A}^+) \le \varepsilon \quad \text{and} \quad \operatorname{dist}(\mathcal{A}_-^-, \mathcal{A}^-) \le \varepsilon$$

holds for h sufficiently small by the known, suboptimal, convergence estimates. Again, as in the proof of Theorem 20, we can pick ε such that

$$\operatorname{dist}(\mathcal{A}^+, \mathcal{A}^-) > 3\varepsilon.$$

Noticing that $A^\pm$ have positive distance to $\partial\Omega$ independent of h, there exists a function $u \in C_0^\infty(\Omega)$ such that

$$u(x) = 2, \quad \text{if } \operatorname{dist}(x, \mathcal{A}^-) \le \varepsilon, \quad \text{and} \quad u(x) = -2, \quad \text{if } \operatorname{dist}(x, \mathcal{A}^+) \le \varepsilon$$

and the function $q = -\Delta u$ is a Slater point for the dual problem with $\tau = \theta = 1$ independent of N and h. □

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