On Infinite-Dimensional Integration in Weighted Hilbert Spaces

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Introduction

Finite-dimensional integration:

- $f(x_1, \ldots, x_s) = \sum_{u \subseteq \{1,\ldots,s\}} f_u(\mathbf{x}_u)$, $\mathbf{x}_u = (x_j)_{j \in u}$
- $$\|f\| = \sum_{u \subseteq \{1,\ldots,s\}} \gamma_u^{-1/2} \|f_u\|$$
- $\to$ *reproducing kernel Hilbert spaces*

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- $\rightarrow$ *reproducing kernel Hilbert spaces*

Infinite-dimensional integration:

- $f(\mathbf{x}) = \sum_u f_u(\mathbf{x}_u)$, finite subsets $u \subseteq \mathbb{N}$

- How to define integration for infinitely many variables?

Basic assumptions, domain

(A1) $k \neq 0$ reproducing kernel on $D \times D$ where $D \neq \emptyset$

(A2) $$H(k) \cap H(1) = \{0\}$$

(A3) $(\gamma_u)_u$ family of non-negative weights such that

$$\sum_u \gamma_u \, m^{|u|} < \infty, \qquad m := \inf_{x \in D} k(x, x)$$

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The domain for functions of infinitely-many variables:

$$\mathfrak{X} = \left\{ \mathbf{x} \in D^{\mathbb{N}} : \sum_u \gamma_u \prod_{j \in u} k(x_j, x_j) < \infty \right\} \qquad \text{(A3)} \Leftrightarrow \mathfrak{X} \neq \emptyset$$

The kernel

Notation: $1:s = \{1,\ldots,s\}$

Define reproducing kernels on $\mathfrak{X} \times \mathfrak{X}$ by

$$k_u(\mathbf{x},\mathbf{y}) := \prod_{j\in u} k(x_j,y_j)$$

$$K_{1:s}(\mathbf{x},\mathbf{y}) := \sum_{u\subseteq 1:s} \gamma_u k_u(\mathbf{x},\mathbf{y})$$

$$K(\mathbf{x},\mathbf{y}) := \sum_{u} \gamma_u k_u(\mathbf{x},\mathbf{y}) = \lim_{s\to\infty} K_{1:s}(\mathbf{x},\mathbf{y})$$

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The function spaces

- **Proposition** *Wasilkowski, Woźniakowski (2004)*
  Every $f \in H(K_{1:s})$ has a unique decomposition

$$f = \sum_{u \subseteq 1:s} f_u, \quad f_u \in H(\gamma_u k_u).$$

  Moreover,

$$\|f\|_{K_{1:s}} = \sum_{u \subseteq 1:s} \gamma_u^{-1/2} \|f_u\|_{k_u}.$$

- proof based on inductive argument

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  Moreover,
  $$\|f\|_{K_{1:s}} = \sum_{u \subseteq 1:s} \gamma_u^{-1/2} \|f_u\|_{k_u} .$$
- proof based on inductive argument
- **Proposition** *Gnewuch, Mayer, Ritter (2012)* Given (A1) - (A3):
  $$H(K) = \bigoplus_u H(\gamma_u k_u).$$

Further basic assumptions

In addition to (A1) - (A3):

(A4) ρ a probability measure on D

(A5) k measurable and $\rho(\{x \in D : k(x,x) = 0\}) = 0$

(A6) $H(k) \subseteq L_1(\rho)$

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(A6) $H(k) \subseteq L_1(\rho)$

$H(k) \subseteq L_1(\rho)$ holds true if e.g. k square-root integrable along the diagonal

$$\int_D \sqrt{k(x,x)}\rho(\mathrm{d}x) < \infty.$$

- $\left\{\begin{array}{c} H(k) \subseteq L_1(\rho), \\ \text{closed graph theorem} \end{array}\right\} \Longrightarrow J : H(k) \to L_1(\rho)$ bounded

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- representer $h \in H(k)$:

$$\int_D g \, \mathrm{d}\rho = \langle g, h \rangle_k$$

$$\|h\|_k^2 = \int_D \int_D k(x,y)\rho(\mathrm{d}x)\rho(\mathrm{d}y)$$

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- if $k \geq 0$ then $\|J\| = \|h\|_k$.
  (see e.g. *Hinrichs (2010)*, *Gnewuch, Mayer, Ritter (2012)*)

Integration on $H(k_u)$?

- For finite $u \subseteq \mathbb{N}$ consider

$$h_u(\mathbf{x}) = \prod_{j \in u} h(x_j), \quad \mathbf{x} \in D^u.$$

- $h_u \in H(k_u)$, $\|h_u\|_{k_u} = \|h\|_k^{|u|}$.

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- Straight-forward calculation: for $f \in \operatorname{span}\left\{k_u(\cdot, \mathbf{x}) : \mathbf{x} \in D^u\right\}$

$$\int_{D^u} f \, \mathrm{d}\rho^u = \langle f, h_u \rangle_{k_u}$$

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- What about the closure? I.e. is

$$H(k_u) \subseteq L_1(\rho^u)?$$

Continuous embedding $H(k_u) \subseteq L_1(\rho)$

Proposition (*Gnewuch, Mayer, Ritter (2012)*)

Suppose $u \neq \emptyset$. Then $H(k_u) \subseteq L_1(\rho^u)$ and

$$\|h\|_k^{|u|} \leq \|J_u\| \leq \left(\sqrt{\pi/2}\|J\|\right)^{|u|}.$$

Furthermore,

$$\int_{D^u} f \, \mathrm{d}\rho^u = \langle f, h_u \rangle_{k_u}, \quad f \in H(k_u).$$

- Proof: Little Grothendieck Theorem,change of measure argument; cf. *Hinrichs (2010)*.

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- $k \geq 0 \implies \|J_u\| = \|J\|^{|u|} = \|h\|_k^{|u|}$.

Infinite-dimensional integration

- Put $\mu = \rho^{\mathbb{N}}$.
- Perspective 1: Integration with respect to μ,

$$I_1(f) = \int_{\mathfrak{X}} f(\mathbf{x})\mu(\mathrm{d}\mathbf{x})$$

if $\mu(\mathfrak{X}) = 1$.

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- Perspective 2: Integral as limit of representers. If

$$h^* = \sum_u \gamma_u h_u$$

  exists in $H(K)$, define

$$I_2(f) = \langle f, h^* \rangle_K .$$

  See *Kuo, Sloan, Wasilkowski, Woźniakowski (2010), Plaskota, Wasilkowski (2011)*.

Integration with respect to the product measure

- For $f = \sum_u f_u \in H(K)$ let
$$f^{(s)} : D^{\mathbb{N}} \to \mathbb{R}, \quad f^{(s)}(\mathbf{y}) = \sum_{u \subseteq 1:s} f_u(y_u).$$

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- **Definition:** We say

$$H(K) \hookrightarrow L_1(\mu)$$

if $H(K) \subseteq L_1(\mu)$ and

$$T : H(K) \to L_1(\mu), \quad Tf = \lim_{s \to \infty} f^{(s)}$$

defines a bounded linear mapping.

Integration with respect to the product measure

Lemma

If $\mu(\mathfrak{X}) = 1$ *and* $H(K) \subseteq L_1(\mu)$ *then* $H(K) \hookrightarrow L_1(\mu)$ *and*

$$\int_{D^{\mathbb{N}}} Tf \,\mathrm{d}\mu = \int_{\mathfrak{X}} f \,\mathrm{d}\mu|_{\mathfrak{X}}.$$

- $\mu(\mathfrak{X}) = 1$ and $H(K) \subseteq L_1(\mu)$ are satisfied if e.g.

$$\sum_u \gamma_u \left(\int_D k(x,x) \,\mathrm{d}\rho(x) \right)^{|u|} < \infty.$$

Proposition (*Gnewuch, Mayer, Ritter (2012)*)

Let $f = \sum_u f_u \in H(K)$, $f_u \in H(\gamma_u k_u)$.

- *If* $\sum_u \gamma_u \|h\|_k^{2|u|} < \infty$ *then* $h^* = \sum_u \gamma_u h_u \in H(K)$ *and*

$$\langle f, h^* \rangle_K = \sum_u \int_{D^u} f_u(\mathbf{x}_u)\rho^u(\mathrm{d}\mathbf{x}_u).$$

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- $$\sum_u \gamma_u \|J_u\|^2 < \infty \ \Rightarrow \ H(K) \hookrightarrow L_1(\mu) \ \Rightarrow \ \sum_u \gamma_u \|h\|_k^{2|u|} < \infty.$$

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- $$\sum_u \gamma_u \|J_u\|^2 < \infty \Rightarrow H(K) \hookrightarrow L_1(\mu) \Rightarrow \sum_u \gamma_u \|h\|_k^{2|u|} < \infty.$$

- *If* $H(K) \hookrightarrow L_1(\mu)$ *then*

$$\langle f, h^* \rangle = \int_{D^{\mathbb{N}}} Tf(\mathbf{x}) \mu(\mathrm{d}\mathbf{x}).$$

Example 1: Non-negative Kernel

- ρ the uniform distribution on $D = [0, 1]$
- $k(x, y) = \min(x, y)$
- Fix $\alpha > 0$; $\gamma_{1:s} = \alpha^s$; $\gamma_u = 0$ otherwise.

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$$\mu(\mathfrak{X}) = 1 \Leftrightarrow \alpha < \exp(1).$$

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- For $\alpha \in \big[\exp(1), 3\big)$, $H(K) \hookrightarrow L_1(\mu)$ holds true, but $\mu(\mathfrak{X}) = 0$.

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\{\ldots\} &\Leftrightarrow \alpha < 3, \\
\mu(\mathfrak{X}) = 1 &\Leftrightarrow \alpha < \exp(1), \\
\sum_u \gamma_u \left(\int_D k(x, x)\, d\rho(x) \right)^{|u|} < \infty &\Leftrightarrow \alpha < 2, \\
\mathfrak{X} = D^{\mathbb{N}} &\Leftrightarrow \alpha < 1.
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- $H(K) \hookrightarrow L_1(\mu) \not\Rightarrow \sum_u \gamma_u \left(\int_D k(x,x)\, d\rho(x)\right)^{|u|} < \infty$

Example 2: ANOVA-type kernel

- ρ uniform distribution on $D = [-1, 1]$
- $k(x, y) = x \cdot y$
- γ_u as before

$$\sum_u \gamma_u \|h\|_k^{2|u|} < \infty \Leftrightarrow \alpha > 0 \quad \text{(trivial)},$$

$$\alpha < 4 \Leftrightarrow \sum_u \gamma_u \|J_u\|^2 < \infty \Rightarrow H(K) \hookrightarrow L_1(\mu) \Rightarrow \alpha \leq 4,$$

$$\mu(\mathfrak{X}) = 1 \Leftrightarrow \alpha < \exp(2),$$

$$\sum_u \gamma_u \left(\int_D k(x, x) \, d\rho(x) \right)^{|u|} < \infty \Leftrightarrow \alpha < 3,$$

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$$\mu(\mathfrak{X}) = 1 \Leftrightarrow \alpha < \exp(2),$$

For $\alpha \in (4, \exp(2))$

- $\langle f, h^* \rangle_K$ well-defined and $\mu(\mathfrak{X}) = 1$.
- $H(K) \hookrightarrow L_1(\mu)$ not satisfied.

Summary, open questions

- Sometimes the number

$$I_2(f) = \langle f, h^* \rangle = \lim_{s\to\infty} \sum_{u\subseteq 1:s} \int_{D^u} f_u \, \mathrm{d}\rho^u$$

may exist, but can not be interpreted as

$$I_1(f) = \int_{D^{\mathbb{N}}} Tf \, \mathrm{d}\mu.$$

- $I_1(f)$ can make sense non-trivially even if $\mu(\mathfrak{X}) = 0$.

Open questions:

- How to characterize $H(k) \subseteq L_1(\rho)$?
- A necessary condition for $H(K) \hookrightarrow L_1(\mu)$?