What follows is my Master's thesis written in conclusion of my studies in Mathematics (Bonn). It's a work on Probability Theory and Mathematical Physics, where I try to offer a nice road in the modern research topic called "Liouville Quantum Gravity".

It does not contain any original theorem, but rather tries to provide an unified document where the reader can find a structured linear approach, instead of navigating between multiple (sometimes "obscure") references.

I did my best to preserve a rigorous style, clearness and coherence. I hope that if the reader is looking for a single undergraduate-to-graduate reference for studying LQG, he/she can feel to be in the right place.

The work is publicly and physically available in the library of University of Bonn (Mathematics Department) and here offered in the form of a single pdf file. It is not "published" in the sense of scientific research (e.g. in the form of paper).

Thank you and enjoy the journey.

Liouville Quantum Gravity and KPZ relations

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Introduction - Why Liouville Quantum Gravity?

In modern Physics a common problem is given by the need of integrating over large function spaces, requiring so appropriate definitions of measures. In string theory and quantum gravity, one possible case is represented by the object "$e^{-S_{\mathcal{L}}(U)}dU$", where dU is the "uniform Lebesgue measure", and the exponential weight $S_{\mathcal{L}}$ the "Liouville Action".

Unfortunately, the former is far from being well-defined if the space is infinite-dimensional (a common scenario), while Liouville Action might have a quite complicated form, depending e.g. on the "underlying metric of the space" or on the "central charge". Frequently, people try to proceed heuristically, but unfortunately there are circumstances where even this approach doesn't suffice.

In the first part of this master's thesis, we study a case in which a full mathematical treatment for a related problem is available. We usually work with a subset $D \subseteq \mathbb{C}$, but the theory can be extended to Riemann Surfaces thanks to a general well behavior under conformal change of coordinates. As described above, the first problem will be essentially to define a measure of the form "$e^{-S_{\mathcal{L}}(U)}dU$" on a space that extends $C_c^\infty(D)$ (smooth and compactly supported real functions with domain D). It will be possible because under this specific setting, the paper [2] shows that after changing coordinates, Liouville Action can be rewritten as a (Dirichlet) scalar product. Consequently, we can see the resulting object as "$e^{-<U,U>}dU$" similar to an attempt of defining a Gaussian variable on this infinite-dimensional space. Indeed, despite the notion of infinite-dimensional Lebesgue measure doesn't exist (and so the formal symbol "dU" alone is mathematically meaningless), infinite-dimensional Gaussians like in this case are possible (so being the full symbol "$e^{-S_{\mathcal{L}}(U)}dU$" meaningful).

Roughly speaking, the constructed measure is called *Gaussian Free Field* (abbreviated GFF) and represents the first important notion of this work. Gaussian Free Fields are preliminary for studying the central topic of this thesis - Liouville Quantum Gravity (abbreviated LQG) - whose origin intersects with Statistical Physics. Interacting particle systems are commonly modeled via 2D-(random) graphs. By making them "thinner and thinner", one could try to guess what happens in the continuous counterparts, where graphs are now "replaced" by Riemann Surfaces.

If M is the manifold chosen to work with (assume it to be smooth and simply connected), the Riemann Uniformization Theorem claims that it is conformally equivalent to the unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$, or the complex sphere $\mathbb{C} \cup \{\infty\}$. An equivalent version for expressing the theorem is to say that M can be parametrized by points $z = x + iy$ belonging to one of the three previous spaces, in a way that the metric takes the form $e^{\lambda(z)}(dx^2 + dy^2)$ for a real-valued function λ (isothermal coordinates). Notions like area, length and curvature can be easily expressed in the new form (again, showed in [2]). In such a setting, we have now a candidate way for defining a "random surface", by replacing the deterministic λ with a random variable h. Indeed, by choosing h to be distributed like a GFF, we obtain a measure μ called Liouville Quantum Gravity or Liouville measure.

This construction is not straightforward. We start by approximations h_ϵ and associated measures μ_ϵ easy for working with, showing that they finally converge to a measure defined to be μ.

We are then interested in some basic geometric properties. What happens to a set under the influence of LQG? The quantitative relation between the "dimensions" of it according to original Lebesgue measure, and a "dimension" according to LQG is called "KPZ relation". It was discovered in the late 80's by Knizhnik, Polyakov and Zamolodchikov but only recently completely formalized. Furthermore, essentially the "same" KPZ relation seems to appear in various different contexts of modern Statistical Physics, for reasons not yet completely understood (see [4] for a good overview).

Finally, I'd like to spend few non-technical words about my writing. I strongly hope to have been sufficiently clear, rigorous and precise on every aspect, aware that occasionally some particularly long computations have been omitted, preferring a direct use of the listed references. I chosen on purpose a compact style and I hope the reading experience might be a pleasure. I'd like to express my gratitude to the reader for his/her time and attention.

Chapter 1

Gaussian Free Fields

1.1 Conformal mappings

Every time we have objects and maps, it is important to understand which properties remain invariant up to appropriate change of coordinates. For instance, in differential geometry it is the practice to study objects up to diffeomorphisms, in elementary topology up to continuity or homotopy, in Riemannian geometry up to isometry, etc...

Our objects will be "invariant" up to conformal mappings. The inverted commas are needed because sometimes it is necessary to add some extra terms, but always without deep changes in the underlying structure. We assume the reader to be familiar with undergraduate complex analysis, and we recall the main useful theorems and definitions. For revising or studying more, the author recommends the classical book by Rudin [6]. The first step is to recall what a conformal function is.

Definition 1.1.1 (Conformal mapping)**.** Let D, $D' \subseteq \mathbb{C}$. If a function $f : D \to D'$ is bijective, holomorphic and admits an holomorphic inverse, then it is said to be *biholomorphic*, or *conformal.*

Sometimes the notion of conformal mapping is defined in a slightly different way. The references help in solving possible misunderstandings, but for our purposes we just rely on the definition above and avoid further characterizations.

Before proceeding, it's better to fix some notation. The default user probability triple is understood to be $(\Omega, \mathcal{A}, \mathbb{P})$, as frequently denoted in many textbooks. The set of natural numbers $\mathbb{N}$ is intended to start from 1, the real and imaginary part of a complex number z are referred respectively with $\Re(z)$, $\Im(z)$, or sometimes with $\operatorname{Re} z$ and $\operatorname{Im} z$. We use the letter $\mathbb{D}$ for the open unitary disk $\{z \in \mathbb{C} : \|z\|<1\}$, and the symbol $\mathbb{H}$ for the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. With $B_\epsilon(z)$ we mean the open ball of radius ϵ centered in z, while the "circle of radius ϵ and center z" is intended to be $\partial B_\epsilon(z)$, i.e. the set $\{y \in \mathbb{C} : |y - z| = \epsilon\}$. Finally, when we say a subset $D \subseteq \mathbb{C}$ to be proper, we mean $D \neq \mathbb{C}$.

A remarkable fact to underline, is the existence of a conformal map between the unitary disk and the half plane, despite the former being bounded while the latter not.

It is a very good example for illustrating the strength of conformal invariance.

Proposition 1.1.1 (Moebius Map)**.** The map $\phi : \mathbb{H} \to \mathbb{D}$ defined as $z \mapsto \frac{i-z}{i+z}$ is conformal.

Proof. For every $z \in \mathbb{H}$ the function $\phi(z) = \frac{i-z}{i+z}$ is well defined, and since $\Im(z) > 0$ we have $|\phi(z)| < 1$. It is straightforward to check that is it holomorphic. If we define $\psi : z \mapsto i\frac{1-z}{1+z}$, it holds $\Im(\psi(z)) > 0$ every time $z \in \mathbb{D}$. The new map $\psi : \mathbb{D} \to \mathbb{H}$ is actually well-defined and holomorphic too, and finally the fact that $\psi \circ \phi = Id = \phi \circ \psi$ concludes the proof. □

The function in the proposition is usually called Moebius Map and it belongs to a larger family of maps called *linear fractional transformations*. During his PhD thesis, Riemann discovered the following milestone of complex analysis.

Theorem 1.1.1 (Riemann Mapping Theorem)**.** Let $D \subsetneq \mathbb{C}$ be proper and simply connected. For each $z \in D$, there exists exactly one conformal map $f_z : D \to \mathbb{D}$ such that $f_z(z) = 0$ and $f_z'(z) \in \mathbb{R}_+$.

Proof. The proof is very long and involves an appropriate preliminary work. The reader is invited in consulting [6]. □

Note that, in the previous theorem, the existence of a generic conformal map is already in itself something not obvious. An immediate consequence is the opportunity of extending the notion of radius, typical of balls, to more generic complex subsets.

Definition 1.1.2 (Conformal Radius)**.** Let $D \subseteq \mathbb{C}$ proper and simply connected. For each $z \in D$, the quantity $R(z; D) \doteq \frac{1}{f_z'(z)}$ is called *conformal radius* of D from the point z. The function f_z is intended to be the one from the Mapping Theorem above.

The following lemma will help for discovering further properties.

Lemma 1.1.1 (Conformal radius in practice)**.** Let $D \subseteq \mathbb{C}$ be proper and simply connected. If $\phi : D \to \mathbb{D}$ is a generic conformal mapping with $\phi(z) = 0$, then $|\phi'(z)| = f_z'(z)$ (where again, f_z is the map in the Riemann Mapping Theorem). As a consequence, $R(z; D) = |\phi'(z)|^{-1}$.

Proof. The map ϕ is conformal, consequently $\phi'(z) \in \mathbb{C} \setminus \{0\}$. Being a non-zero complex number, it can be rotated until to reach the positive real axis. In other words, there exists $\theta \in \mathbb{R}$ such that the rotation map $r : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto ze^{i\theta}$ realize $\phi'(z)e^{i\theta} \in \mathbb{R}_+$. The map r can be restricted on the disk D giving a conformal mapping $\mathbb{D} \to \mathbb{D}$ that sends 0 in 0, consequently the composition $r \circ \phi : D \to \mathbb{D}$ is still conformal and sends z in 0. The derivative of this map in z is now positive, being $(r \circ \phi)'(z) = e^{i\theta}\phi'(z)$. Due to the uniqueness part in the Riemann Mapping Theorem, this composition map must coincide with f_z' and the claim follows by taking the modules. □

The previous lemma clarifies the alternative definition of conformal radius in the notes [3], and prepares the ground for the next proposition. A very important point is that, indeed, conformal radius behave regularly under conformal changes:

Proposition 1.1.2 (Conformal change of radius)**.** Let D, D' two proper simply connected subsets of $\mathbb{C}$. Suppose to have $\phi : D \to D'$ conformal. Then for every $z \in D$, $R(\phi(z); D') = |\phi'(z)|R(z; D)$

Proof. Let $\psi_D : D \to \mathbb{D}$ be a conformal mapping such that $\psi_D(z) = 0$. Define the function $\psi_{D'} : D' \to \mathbb{D}$ by $\psi_D \circ \phi^{-1}$, still conformal by construction (composition of conformal maps). Note that $\psi_{D'}(\phi(z)) = 0$, consequently (lemma above) it can be used for computing the conformal radius. By using the chain rule for complex derivatives, one obtains

$$R(\phi(z); D') = \frac{1}{\psi'_{D'}(\phi(z))} = \frac{|\phi'(z)|}{|\psi'_D(z)|} = |\phi'(z)|R(z; D)$$

which concludes the proof. □

For simplicity, from now on we will always work with subsets $D \subseteq \mathbb{C}$ required to be bounded, simply connected, open and with smooth boundary. We prefer to call them **standard** subsets, in a way to avoid long descriptions in every proposition. The reason for these property will be better explained during the work. Roughly speaking, being open and simply connected ensure a conformal mapping to the unitary disk, boundedness will guarantee normalization for some (probability) measures, and finally the smooth boundary will be required for the use of Gauss-Green formula.

We would like to end this section with an informal discussion about different possible alternative settings, possibly beneficial for the big picture. It is possible to extend the Riemann Mapping Theorem on Riemann Surfaces, where it takes the name of Uniformization Theorem, claiming that every simply connected Riemann Surface is conformally equivalent to the open disk, the complex plane or the Riemann Sphere. We will not study nor use this fact in details, but a rough idea that motivates its relation with Liouville Quantum Gravity (LQG) is given in the introductory section. Note that in this generalization, we have a more complicated geometry (a manifold), but the dimension remains unchanged. One could ask what happens if we for example keep the geometry euclidean, but instead we increase the dimensions, e.g. by working in $\mathbb{C}^n$. Unfortunately, the Riemann Mapping theorem does not hold into this setting, partially because for dimensions higher that two there is a theorem from Liouville that puts strong limits on the amount of available conformal functions. The author thinks that it is probably a good first reason for which LQG manages to be studied on simply connected Riemann Surfaces, but there are still obstacles for other general settings.

1.2 Test functions, distributions and Dirichlet energy

Let $D \subseteq \mathbb{C}$ be open, bounded. We consider the usual space of test functions $C_c^\infty(D) = \{f : D \to \mathbb{R} : f \text{ smooth and compactly supported}\}$. This space can be equipped with (at least) two nice topologies. The first is related to the notion of *distribution*, while the second to the one of *Dirichlet Energy*.

According to the first topology (the construction and more details are available in [5]), a sequence of test functions $\{\phi_n\}_{n\in\mathbb{N}}$ is said to converge to another test function

ϕ iff $\partial^\alpha \phi_n \to \partial^\alpha \phi$ uniformly for all possible multi-index α, with $supp\{\phi_n\} \subseteq K \subseteq D$ commonly contained in a compact set K. Elements belonging to the dual space $D'(D) = \{f : D \to \mathbb{R} : f \text{ linear, continuous w.r.t this topology}\}$ are called *distributions* (not to be confused with the homonym concept in probability theory). When we have a sequence of distributions $\{\psi_n\}_{n\in\mathbb{N}} \subseteq D'(D)$, $\psi \in D'(D)$ such that $\psi_n(\phi) \to \psi(\phi)$, $\forall \phi \in C_c^\infty(D)$ we say that ψ_n converge to ψ in the sense of distributions (note that the last convergence is on reals). Sometimes we indicate with the symbol H^{-1} or $H^{-1}(D)$, rather than $D'(D)$, the space of distributions over D endowed with the convergence above.

Many important tools from Mathematical Analysis are extended from functions to distributions, but for reasons of time we could only have a very concise overview on strictly needed results. First of all, the map $L^1_{loc}(D) \to D'(D)$, $f \mapsto \int f\cdot$ is a linear injective embedding. The resulting functional is intended to act like $C_c^\infty(D) \ni \phi \mapsto \int_D f\phi \in \mathbb{R}$ (Lebesgue integration). Such a map is usually not surjective, and suggests how some distributions are induced by measures like the pointwise evaluation obtained via Dirac integration ($f(x) \mapsto \int_D f(x)\delta_y(dx) = f(y)$). When ϕ is a distribution and $\exists f \in L^1_{loc}(D)$ s.t. $\phi(\cdot) = \int f\cdot$, we say that f represents ϕ.

Derivation is carried on in this setting, too. If $\psi \in D'(D)$, then $\partial_x\psi$ is the new distribution defined as $\partial_x\psi(\phi) = -\psi(\partial_x\phi)$ (compare the minus sign to the integration by parts formula). When f is a function regularly enough to be seen as a distribution (e.g. $f \in L^1_{loc}(D)$ - we usually use the symbol "f" both for the function and the represented distribution) its distributional derivative is commonly called *weak* derivative. When f admits an ordinary derivative too, commonly called *strong*, they usually coincide (in the sense that the strong derivative represents the weak one, for instance it happens for test functions, so in our setting there is no risk of ambiguity).

The sum of two distributions is defined in the elementary way, and so objects like the Laplacian are completely meaningful. Distributions whose Laplacian vanishes are called harmonics, and they play a strong role in our theory due to important properties, summed up in the following theorem (see [3]):

Theorem 1.2.1 (Weil's lemma)**.** Let ψ_n be a sequence of harmonic distribution converging to ψ. Then ψ must be harmonic too. Furthermore, every harmonic distribution ϕ is automatically a smooth function, which is also harmonic in the strong sense (i.e. it is represented by a $\phi \in C^\infty(D)$ s.t. $\Delta\phi = 0$).

The other topology is induced by putting a scalar product on $C_c^\infty(D)$ called Dirichlet product, defined as $(f_1, f_2)_\nabla := \frac{1}{2\pi}\int_D \nabla f_2 \nabla f_2 dx$. Here the norm of a function is called *Dirichlet energy.* Note that (routine change of variables) if $g : D \to D'$ is a conformal mapping, then $\int_{D'} \nabla(f_1 \circ g^{-1})\nabla(f_2 \circ g^{-1}) = \int_D \nabla f_1 \nabla f_2$ (conformal invariance of Dirichlet Product in dimension 2).

A key property that we'll heavily use is a corollary of the Stokes theorem (see [5]):

Theorem 1.2.2 (Gauss-Green formula)**.** Let $\Omega \in \mathbb{C}$ be open and bounded, with smooth boundary. If $u, v \in C^2(D) \supseteq C_c^\infty(D)$ and at least one of them has compact support, then $\int_\Omega \nabla u \nabla v dx = -\int_\Omega u\Delta v dx$

As already said, from now on we assume D to have smooth boundary (for using Gauss-Green), denote with $H_s(D)$ the space of test functions when considered equipped with Dirichlet product, and define $H(D)$ to be its Hilbert completion. It is known [3] that $H(D)$ coincides with the Sobolev space $H_0^1(D) = \{f \in L^2(D) : \nabla f \in L^2(D)\}$ (where the gradient is intended to be weak). Furthermore, the Gauss-Green formula can be extended on this space with exactly the same assumptions and properties as before, except that $u \in H(D)$ and $v \in C_c^2(D)$ (see [3]).

1.3 Finite dimensional Gaussian random variables

In this section we revise some properties of multidimensional Gaussians on a *finite* dimensional real vector space V with scalar product $(\cdot,\cdot)$. Define μ_V to be the probability measure $e^{-\frac{(v,v)}{2}} Z^{-1} dv$, where dv is the Lebesgue measure on V and Z a normalizing constant (it always exists). Then μ_V is called *standard Gaussian measure* on V.

Theorem 1.3.1 (Characterization of Gaussians)**.** Let $v : \Omega \to V$ be a Lebesgue measurable random variable on $(V, (\cdot,\cdot))$. Then the following are equivalent:

i v has law μ_V (i.e. v is a standard Gaussian random variable on V);

ii v has the same law as $\sum_{j=1}^{d} \alpha_j v_j$ where $v_1, .., v_d$ is a (deterministic) orthonormal basis for V and the α_j are i.i.d. real Gaussian random variables with mean zero and variance one;

iii The characteristic function of v is given by $\mathbb{E}[e^{i(v,t)}] = e^{-\frac{1}{2}\|t\|^2}$ for all $t \in V$;

iv For each fixed $w \in V$, the inner product (v, w) is a real Gaussian variable with zero mean and variance (w, w).

Proof. This is a standard result that can be found in [1]. □

Another important fact is given by the Girsanov transform, used in various situations:

Theorem 1.3.2 (Girsanov's theorem for the finite dimensional case)**.** Let X be a gaussian vector in $(V, (\cdot,\cdot))$ with mean μ and covariance matrix C, under the probability measure $\mathbb{P}$. Let $u \in V$ and define a new probability measure $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{(X,u)}}{\tilde{Z}}$ where $\tilde{Z}$ is the normalization constant. Then under $\mathbb{Q}$, X is still a Gaussian vector but now with covariance matrix C and mean $\mu + Cu$. In particular, $X \sim N(0, 1)$ under $\mathbb{P}$ if and only if $X - u \sim N(0, 1)$ under $\mathbb{Q}$.

Proof. Again, this is a standard result that for instance can be found in [3]. □

Finally, we'd like to underline another property about one dimensional Gaussians useful later:

Proposition 1.3.1 (Exponential gaussians have exponented means)**.** Let N be a 1-dimensional Gaussian random variable with mean a and variance b. Then $\mathbb{E}(e^N) = e^{a+\frac{b}{2}}$.

Proof. This is again a common easy result, so we see no problems in referring directly to [2]. $\square$

Now that we have a clear idea about how to work with Gaussians on finite dimensional spaces, it is spontaneous to think about a possible counterpart in an infinite dimensional setting, motivated by the Introductory section. In other words, we wonder if it would be possible to define a notion of "gaussianity" for random variables with values in more general Hilbert spaces.

Proposition 1.3.2 (Hilbert spaces detect every rotation)**.** If H is an infinite dimensional separable Hilbert space and μ a rotational invariant measure on H, then μ must be the constant zero-measure.

Proof. Let $\{e_n\}_{n\in N}$ be an orthonormal basis for the space. Define the balls $B_n \doteq \{x \in H : \|x - e_n\| < \frac{1}{2}\}$ for every $n \in \mathbb{N}$. By hypothesis of rotational invariance they must have all the same measure. On the other hand, they are disjoint, consequently $\sum_{n=0}^{\infty} \mu(B_n) = \mu(\cup B_n) \leq \mu(H) = 1$. The only possible solution is therefore to have $\mu(B_n) = 0$, and so by doing a rescaling we find $\mu\{x \in H : \|x - h\| \leq \frac{\|h\|}{2}\} = 0$ for all $h \in H$, leading to $\mu = 0$ since every open set can be written as a countable union of balls of the previous type. $\square$

Since being rotational invariant is a key point we require on a measure if we hope to obtain something close to a "gaussian", the previous proposition answer negatively to the question above. Consequently, intuitively speaking, there is *no* way of defining a normal distribution on an infinite dimensional space, but a possible compromise is shown in the following section.

1.4 Construction of Gaussian Free Fields

Let H be an infinite-dimensional separable Hilbert space with scalar product $(\cdot, \cdot)$ and norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$. For a finite dimensional space $E \subseteq H$, we denote with μ_E the standard gaussian measure on it (see section 1.3). We introduce the notion of *measurable* norm, but we mainly rely on characterization theorems rather than on the direct definition (same strategy as [1]).

Definition 1.4.1. A norm $|\cdot|$ on H is said to be *measurable*, if $\forall \epsilon > 0$, $\exists E_\epsilon \subseteq H$ a finite dimensional subspace, such that $\forall E$ finite dimensional subspace, $E \perp E_\epsilon \Rightarrow \mu_E\{x \in E : |x| > \epsilon\} < \epsilon$.

A measurable norm $|\cdot|$ can actually differ from the original one $\|\cdot\|$. Let B be the Banach completion of H according to a measurable norm $|\cdot|$ and denote its dual by $B' = \{f : B \to \mathbb{R} \text{ s.t. } f \text{ is linear and continuous w.r.t } |\cdot|\}$. Let $\mathcal{B}$ be the smallest σ-algebra on B that makes all functionals in B' measurable. In his paper ([7]) Gross showed that every element of B' is necessarily continuous w.r.t the originally Hilbert space norm $\|\cdot\|$ too, and consequently (Riesz theorem) for each $f \in B'$ there is a unique $h_f \in H$ such that $(h_f, h) = f(h)$. With this identification in mind, we can write $B' \subseteq H \subseteq B$, and intend (f, h) to be (h_f, h) whenever $f \in B'$ and $h \in H$. But we introduce a new notation more: when $f \in B'$ and $b \in B$ (instead as before, where $b \in H \subseteq B$), the symbol (f, b) will mean $f(b)$.

Summing up, when $f \in B'$ the expression $(f, \cdot)$ is meaningful for every element in $b \in B$, coherent with the previous convention where $b \in H \subseteq B$. The notation is understood to be symmetric, fully justifying the case "(h, f)" (with $h \in B$, $f \in B'$) studied later (and this notation will be actually extended to $f \in H(D) \supseteq B'$).

Let $E \subseteq B'$ be a *finite*-dimensional subspace with H-orthonormal basis $e_1, ..., e_k$, $\phi_E : B \to E$ the map defined via $b \mapsto \sum_{i=1}^{k} (e_i, b) e_i$. Thanks to the notation just seen, the map ϕ_E is nothing but an extension to B of the ordinary projection from H to E. All the ideas here find a place in the following key theorem discovered by Gross:

Theorem 1.4.1 (Gross)**.** If $|\cdot|$ is measurable, then there exists a *unique* probability measure $\mathbb{G}$ on $(B, \mathcal{B})$ for which $\mathbb{G}(\phi_E^{-1} S) = \mu_E(S)$ on each finite-dimensional subspace E of B' and each Lebesgue measurable subset $S \subseteq E$.

Proof. It requires a specific paper. The reader is invited in consulting [7] or [1]. □

A very strong consequence obtained by considering one dimensional subspaces is given below.

Theorem 1.4.2 (GFF: Uniqueness)**.** Let $(H, \|\cdot\|)$ be a separable Hilbert space with scalar product $(\cdot, \cdot)$ and measurable norm $|\cdot|$. Then $\exists! \mathbb{G}$ probability measure on $(B, \mathcal{B})$ such that, whenever $(\Omega, A, \mathbb{P})$ is our starting probability space and $h : \Omega \to B$ is a B-valued random variable:

1 if $Law(h) = \mathbb{G}$, then $(*)$ $\forall f \in B'$, the random variable $(h, f) : \Omega \to \mathbb{R}$ defined as $\omega \mapsto (h(\omega), f)$ is a 1-dimensional real Gaussian with mean zero and variance (f, f);

2 if h has the property $(*)$ described above, then $Law(h) = \mathbb{G}$.

Proof. Part (1). Let $h : (\Omega, \mathcal{A}, \mathbb{P}) \to (B, \mathcal{B}, \mathbb{G})$ be a random variable with law $\mathbb{G}$. Let $f \in B'$ and $E = span\{f\}$. The map $\phi_E : (\cdot, \frac{f}{\|f\|}) \frac{f}{\|f\|} : B \to E$ is a projection as in the reasoning above (the basis needs to be normalized). Furthermore, $(h, \frac{f}{\|f\|}) \frac{f}{\|f\|}$ is nothing but the composition $\phi_E \circ h$, consequently $\mathbb{P}(h^{-1} \circ \phi_E^{-1}(A)) = \mathbb{G}(\phi_E^{-1}(A)) = \mu_E(A)$ by using the hypothesis Law(h)=$\mathbb{G}$. It implies that $(h, f) \frac{f}{\|f\|^2}$ is a standard Gaussian variable on E, so $((h, f) \frac{f}{\|f\|^2}, f) = (h, f)$ is a one-dimensional real Gaussian of zero mean and variance (f, f) by using property (iv) of the previous section.

Part(2). The reasoning is essentially the same. Let $h : (\Omega, \mathcal{A}, \mathbb{P}) \to (B, \mathcal{B}, \cdot)$ such that $(h, f) \sim N(0, (f, f))$ for every $f \in B'$. Let $E \subseteq B'$ a finite dimensional subspace with orthonormal basis $f_1, ..., f_n$. We want to show that $h^{-1} \circ \phi_E^{-1}(S) = \mu_E(S)$ for each measurable subset S, but it is straightforward: $h^{-1} \circ \phi_E^{-1} = (\phi_E \circ h)^{-1}$ and the latter is a standard Gaussian measure on E since $\phi_E \circ h = \sum_i^n (h, f_i) f_i$ (use characterization (ii) showed in the previous section). □

In other words, the theorem states that given a triple $(H, B, |\cdot|)$ with $|\cdot|$ measurable, then there exists a unique measure $\mathbb{G}$ that behaves like a Gaussian on every finite dimensional subspace of $B' \subseteq H \subseteq B$, and that every B-valued random variable whose law follows the same property, has to be distributed as $\mathbb{G}$. Note that this construction, trivial in finite dimension, is essentially a generalization of the content of the section before. Constructing the measure on B rather than directly on H can be seen as a nice alternative way for defining Gaussians over infinite dimensional spaces avoiding the obstacles previously described. The quadruple $(H, B, \mathbb{G}, |\cdot|)$ is conceptually so rich to deserve a name, and is called *Abstract Wiener Space*. The usual Wiener Space can actually be obtained by following the description in [1], fully justifying this name.

From now on, we focus on the case $H = H(D)$, where $H(D)$ is the separable Hilbert space introduced before (section 1.2). First, we need to know if a measurable norm on this space (always) exists. The paper [1] gives a positive answer, by using a construction based on Laplacian's eigenvalues. Consequently, we have the base quadruple $(H(D), B, \mathbb{G}, |\cdot|)$ by using the theorem above. We would like to go a further, trying to find (existence and) an explicit expression for the random variable h. This goal is achieved by "summing an infinite amount of Gaussians". As first step, we check that the sum is actually possible:

Proposition 1.4.1 (GFF: existence (1))**.** Let $\{\alpha_i\}_{i\in\mathbb{N}}$ be i.i.d. $\sim N(0,1)$. Let $\{f_i\}_{i\in\mathbb{N}}$ be any orthonormal basis of $H(D)$. The, the infinite sum $\hat{h} \doteq \sum_{i\in\mathbb{N}} \alpha_i f_i$ does *not* converge in $H(D)$, but it does in the larger space $B \supseteq H(D)$ w.r.t the measurable norm.

Proof. This is one of the main purpose of the paper [1]. □

On the other hand, $\hat{h}$ is actually a random variable satisfying property $(*)$ defined in the main theorem above:

Proposition 1.4.2 (GFF: existence (2))**.** For each $f \in B'$, the random variable $(\hat{h}, f)_\nabla$ is a one-dimensional real Gaussian with variance $(f, f)_\nabla$.

Proof. Since the convergence holds for every orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$ we choose one where $f_k = \frac{f}{\|f\|}$, for a fixed k, possible since $f \in B' \subseteq H(D)$. Then, $(h, f)_\nabla = f(h)$ (that's exactly the notation introduced at the beginning of this section, h is a random variable in B), but $f(h) = f(\sum_{j\in\mathbb{N}} \alpha_j f_j)$, where the limit it taken w.r.t the measurable norm. Since $f \in B'$, it is a linear functional $B \to \mathbb{R}$ *continuous* w.r.t such a norm, the previous expression is equal to $\sum_{j\in\mathbb{N}} \alpha_j (f, f_j)_\nabla = \alpha_k \sqrt{(f,f)_\nabla} \sim N(0, (f,f)_\nabla)$, which concludes the proof. □

In other words, $h \doteq \hat{h}$, whose existence is now guaranteed, is a B-valued random variable satisfying the properties listed in the theorem above. Consequently, $Law(h) = \mathbb{G}$.

Definition 1.4.2 (Gaussian Free Field)**.** Let $(H(D), B, \mathbb{G}, |\cdot|)$ the quadruple defined above. The random variable h is called *Gaussian Free Field.* Sometimes it will abbreviated with GFF.

The probability measure $\mathbb{G}$ will be frequently indicated by "dh" and can be interpreted like an infinite dimensional Gaussian measure on B. Later we'll see how in this concrete setting B can be identified with the space of distributions $H^{-1}(D)$, justifying the alternative definition of Gaussian free fields as "the choice of a random distribution".

Finally, we remark (immediate corollary of the main theorem) the solid **covariance structure** that will play a *central* role for the whole work: for each $f_1, f_2 \in B'$, $Cov[(h, f_1)_\nabla, (h, f_2)_\nabla] = (f_1, f_2)_\nabla$.

The reader is now invited in comparing this results with the first half of the paragraph "Introduction", section . If few words (see [2] for more details), one of the proposed goal was to give a meaning to the heuristic object "$e^{S_\mathcal{L}(U)}dU$" interpreted as an attempt to describe an infinite-Gaussian measure on the space $H(D), (\cdot,\cdot)_\nabla$. The work done in this section proposes a full mathematical treatment by setting "$e^{S_\mathcal{L}(U)}dU = dh$", trying to offer a possible solution.

1.5 Random distributions and conformal invariance

In the previous section we gave a complete construction of the object h, from which the symbols $(h, f)_\nabla$ are defined every time $f \in B' \subseteq H(D)$. Unfortunately, it actually doesn't suffice for our purposes, since in "practice" it's very common to have $f \in H(D)$. Here we describe a solution to this problem.

As seen, one can think of h as a the infinite sum $\sum_{i=1}^\infty \alpha_i f_i$, where α_i are i.i.d. as $N(0,1)$ and $\{f_i\}_{i\in\mathbb{N}}$ is an orthonormal basis for $H(D)$, having in mind the convergence in the larger space B. On the other hand, observe that if $f \in H(D)$, then $f = \sum_{i=1}^\infty \beta_i f_i$. Consequently, we literally *define* the symbol "$(h, f)_\nabla$" as $\sum_{i=1}^\infty \alpha_i\beta_i$. It is completely legal since the partial sums converges almost surely due to the same principle as before, see [1] for more details. Furthermore, under this construction the covariance structure is still preserved, i.e. $Cov[(h, f_1)_\nabla, (h, f_2)_\nabla] = (f_1, f_2)_\nabla$ (by ordinary computation and the use of Parseval's identity).

In other words, although "ignoring the full construction of h", we have a concrete collection of random variables, namely $(h, f)_\nabla$ for each $f \in H(D)$, suitable for our problem and retro-compatible with the GFF definition in the section before. With these properties in mind, we'll move now to another very important fact.

For every $\rho \in C_c^\infty(D)$, we'll see a way for defining the function "$-\Delta^{-1}\rho$" belonging to $H(D)$. The author suggests to temporarily accept this property in order to proceed towards the main point of this section, postponing its construction later (section 1.7). We introduce a new important notation/definition: for every $\rho \in C_c^\infty(D)$, define $(h, \rho) \doteq 2\pi(h, -\Delta^{-1}\rho)_\nabla$ (now meaningful since $\Delta\rho \in H(D)$) and we show why Gaussian Free Fields are sometimes called "random distributions":

Proposition 1.5.1 (GFFs are random distributions)**.** Almost surely for every $\omega \in \Omega$, the map $(h(\omega), \cdot) : C_c^\infty(D) \to \mathbb{R}$ is a distribution (in the sense of Analysis, see section 1.2).

Recall that the truncated series $h^N \doteq \sum_{i=1}^N \alpha_i f_i$ converges almost surely in the larger space $B \supseteq H$. This property is preserved in the new setting: indeed, such a series converges in the sense of distribution, too. Define $(h^N, \rho) \doteq 2\pi(h^N, -\Delta^{-1}\rho)_\nabla = -2\pi \sum_{i=1}^N \alpha_i \int_D f_i \rho dx$, where the last equality holds since the basis f_i can be chosen with elements of $H_s(D) = C_c^\infty(D)$ allowing the use of the Gauss-Green formula (section 1.2).

Proposition 1.5.2 (Partial sums converge in distribution)**.** The truncated series h^N converges almost surely in the space of distributions. More precisely, almost surely for each $\omega \in \Omega$, for each $\rho \in C_c^\infty(D)$, $(h^N(\omega), \rho) \to (h(\omega), \rho)$.

Both the proofs can be found in [1] and [3]. What done is nothing but showing that under our setting (i.e. the choice $H = H(D)$), we can identify $B = H^{-1}(D)$ (in case, the paper [1] offers a complete formal proof in every detail), implying that we can interpret h as a random variable on $H^{-1}(D)$. Recall (section 1.2) that with the symbol "$H^{-1}(D)$" we intend the space of distributions over D.

We conclude by commenting how Gaussian Free Fields behave under change of coordinates. If f_n is an orthonormal basis of $H(D)$, $\phi : D \to \tilde{D}$ a conformal map, then $f_n \circ \phi^{-1}$ defines an orthonormal basis for $H(\tilde{D})$ (conformal invariance of Dirichlet product). As a consequence, the series $\sum_{i \in \mathbb{N}} \alpha_i (f_i \circ \phi^{-1})$ must converge to a Gaussian Free Fields in the space $\tilde{B} \supseteq H(\tilde{D})$, call it $h_{\phi^{-1}}$. On the other hand, if h is a distribution on D it is usual to define $h \circ \phi^{-1}$, the pullback distribution on $\tilde{D}$ as $(h \circ \phi^{-1}, \tilde{\rho}) \mapsto (h, \tilde{\rho} \circ \phi |\phi'|^2)$. Here $\tilde{\rho} \in C_c^\infty(\tilde{D})$ and the square term (requested by definition) is intuitively justified as follows: when the distribution admits an integral representation, the added term counterbalances the square jacobian arising due the change of variables. The following proposition ensures that $h_{\phi^{-1}} = h \circ \phi^{-1}$ avoiding so any risk of ambiguity.

Proposition 1.5.3 (GFFs are conformally invariant)**.** Let $\phi : D \to \tilde{D}$ be a conformal mapping. Then $h_{\phi^{-1}} = h \circ \phi^{-1}$.

Proof. Let $\tilde{\rho} \in C_c^\infty(\tilde{D})$. By using directly the definitions, we have (almost surely):
$(h_{\phi^{-1}}, \tilde{\rho}) = \lim_{N\to\infty}(h^N_{\phi^{-1}}, \tilde{\rho}) = -2\pi \lim_{N\to\infty} \sum_{i=1}^N \alpha_i (f_i \circ \phi^{-1}, \tilde{\rho}) =$
$= -2\pi \lim_{N\to\infty} \sum_{i=1}^N \alpha_i \int_{\tilde{D}} (f_i \circ \phi^{-1})(\tilde{x}) \rho(\tilde{x}) d\tilde{x} =$
$= -2\pi \lim_{N\to\infty} \sum_{i=1}^N \alpha_i \int_D f_i(x)(\tilde{\rho} \circ \phi)(x) |\phi'(x)|^2 dx =$
$= -2\pi \lim_{N\to\infty} \sum_{i=1}^N \alpha_i (f_i, \tilde{\rho} \circ \phi |\phi'|^2) =$
$= \lim_{N\to\infty}(h^N, \tilde{\rho} \circ \phi |\phi'|^2) = (h, \tilde{\rho} \circ \phi |\phi'|^2) = (h \circ \phi^{-1}, \tilde{\rho})$
The "key" point is the change of variables $\tilde{x} = \phi(x)$ that introduces the required derivative term. □

1.6 The spatial Markov property

In this section we study a "spatial Markov property". The name is suggested because if the usual Markov property is about independence from the past, here we have an independence result between far areas in the space (see corollary).

Theorem 1.6.1 (Spatial Markov Property)**.** Let $U \subseteq D$ be an open subset with smooth boundary, and h a Gaussian Free Field on D. Then $h = h_0 + \phi$ where:

i h_0 is a GFF on U (and zero outside U);

ii $\forall \omega \in \Omega$, $\phi(\omega)$ is a distribution on D, harmonic when restricted to U;

iii h_0 and ϕ are independent random variables.

Remark 1.6.1. Before proceeding with the proof, it's better to remark some concepts:

1 the intended sum is between two random distributions, i.e.

$$a.s. \forall \omega \in \Omega, \forall \rho \in C_c^\infty(D), (h(\omega), \rho) = (h_0(\omega), \rho) + (\phi(\omega), \rho)$$

where $\phi(\omega)$ is a distribution on D and so $(\phi(\omega), \rho)$ means its evaluation on the function ρ; $(h_0(\omega), \rho)$ is constant zero on elements ρ outside $C_c^\infty(U) \subseteq C_c^\infty(D)$;

2 almost surely for each ω, $\phi(\omega)$ is a distribution on D *harmonic* when restricted on U. By Weyl's lemma, this restriction is then a full harmonic smooth function (see section 1.2);

3 possible intuitive meaning: in order to know a GFF on a domain, we can estimate it on a smaller region and then perturb the results with an appropriate harmonic function.

Proof of the theorem. The key point is the Hilbert space decomposition $H(D) = H(U) \oplus Harm(U)$, where the latter set refers to smooth functions on D that are harmonic on $U \subseteq D$.

Indeed, the two spaces are orthogonal by the Gauss-Green formula, so it suffices to verify that their sum spans the full space $H(D)$. Since $H(U)$ is complete by construction, it is a closed subspace of $H(D)$. Let $f \in H(D)$ and denote by f_0 its orthogonal projection onto $H(U)$. The function $\phi = f - f_0$ is actually harmonic, as the following reasoning proves. By construction (projection) ϕ is orthogonal to $H(U)$, and so for every test function $\psi \in C_c^\infty(U) \subseteq H(U)$ we have $0 = (\phi, \psi)_\nabla = -\int_D (\Delta\phi)\psi = -\int_U (\Delta\phi)\psi$. The first equality is orthogonality, the second the Gauss-Green formula and the third uses the support of ψ. The obtained equation proves $\Delta\phi = 0$ in U in the sense of distribution. By Weyl's lemma ϕ is a smooth function harmonic in the usual sense, leading to the conclusion $\phi \in Harm(U)$.

Now the the Hilbert decomposition is done, we come back to the original claim. Let h be an instance of the GFF on D, $\{f_n^0\}$ an orthonormal basis for $H(U)$ and $\{\phi_n\}$ an orthonormal basis for $Harm(U)$. Let X_n and Y_n be i.i.d $N(0,1)$ and try to define $h_0 = \sum_{n=1}^\infty X_n f_n^0$, $\phi = \sum_{n=1}^\infty Y_n \phi_n$. In the following lines with the word "converges", we mean "almost surely, converges in the sense of distributions" (section 1.2). The sum $h_0 + \phi$ is nothing but a truncated series of the form h^N, so it converges to h. On the other hand h_0 converges too, precisely to a GFF on $H(U)$ (and by construction is zero outside U), so the same kind of convergence must happen for the second term. In other words, almost surely for each $\omega \in \Omega$, the sum $\sum_{n=1}^{\infty} Y_n(\omega)\phi_n$ converges in the sense of distribution. But every term of this series is a smooth function in D harmonic on U, so we can identify it with a distribution on D harmonic in U, and consequently the same holds for every partial sum. Furthermore, being harmonicity preserved under convergence (section 1.2), the sum limit is still a distribution on D harmonic on U, concluding with property (ii).

Finally, independence (iii) directly follows from the σ-algebras splitting, consequence of the Hilbert space decomposition. □

Corollary 1.6.1 (Disjoint means indipendent)**.** Let $A, C \subseteq D$ open, bounded, disjoint and with smooth boundaries. Let h be a GFF on D, denote with h_X its restriction on the subdomain X. Then $h_A \perp\!\!\!\perp h_B$.

Proof. $h = h_A + \phi$ and $h_A \perp\!\!\!\perp \phi$, implying in particular $h_A \perp\!\!\!\perp \phi_{|B}$. Taking the restriction on the first expression, we obtain $h_B = h_{A_{|B}} + \phi_{|B} = 0 + \phi_{|B}$, but $\phi_{|B} \perp\!\!\!\perp h_A$ and so the claim. □

1.7 Green Functions

Let $D \subseteq \mathbb{C}$ be an open set, B_t a Brownian Motion starting inside D with exit time $\tau_D =$ $\inf\{t \in \mathbb{R} | B_t \notin D\}$. When $A \subseteq D$ is a measurable subset of non-zero Lebesgue measure, the quantity $\mathbb{E}_x[\int_0^{\tau} \mathbb{1}_A(B_t)dt]$ is the average time that a Brownian Motion starting in $x \in D$ spends in A before leaving D. By using Fubini, it is the same as $\int_0^{\tau} p_t(x, A)dt$ where p_t is the Brownian Motion transition function. Roughly speaking, Green Functions deal with the case when A is replaced by a singleton: the previous formal equality fails (because every singleton measures zero according to Lebesgue), but it's still beneficial for our intuition.

Definition 1.7.1 (Green Functions)**.** Let $D \subseteq \mathbb{C}$ open, $y \in D$. The *Green Function on the domain* D is the map $(D \times D) \setminus \Delta \to \mathbb{R}$, defined as $G(x, y) := G_D(x, y) :=$ $\pi \int_0^{\tau} p_t(x, y)dt$, where τ_D is the exit time from D and $p_t(x, y) = (2\pi t)^{-1} exp(-|x-y|^2/2t)$ is the Brownian Motion transition function.

In the definition, the diagonal needs to be excluded, since we would have "$G(x, x) = \infty$". For convenience, we say that G is *finite* when $G(x, y) < \infty$ every time $x \neq y$. By properties of Brownian Motion we have that G_D is finite every time D is bounded. On the other hand, to have a finite Green Function on an unbounded domain is a possible scenario, as shown by $G_{\mathbb{H}}(x, y) = \frac{log|x-\hat{y}|}{log|x-y|}$ (direct computation possible by using the reflection principle of Brownian Motion on the half plane). Green Functions behaves coherently w.r.t. conformal mappings. Before the main claim, we need an easy lemma for preparing the ground.

Lemma 1.7.1 (Green Functions as distributions)**.** For each $x \in D$, the map $G(x, \cdot) \in L^1_{loc}(D \setminus \{x\})$.

Proof. Let $A \subseteq D \setminus \{x\}$ be bounded. Then $\int_A G(x,y)dy = \pi \int_0^\tau \int_A p_t(x,y)dydt = \pi \int_0^\tau p_t(x,A)dt = \mathbb{E}_x[\int_0^\tau \mathbb{1}_A(B_t)dt] < \infty$ since A is bounded. □

The previous lemma justifies the distributional interpretation in the incoming proposition.

Proposition 1.7.1 (Green Functions are conformally invariant)**.** Let $T : D' \to D$ be conformal. Then for every $(x,y) \in (D \times D) \setminus \Delta$, one has $G_{T(D)}(T(x),T(y)) = G_D(x,y)$

Proof. Thanks to the previous lemma it suffices to check the equality in the distributional sense. Let $\phi \in C_c^\infty(D)$ be a test function and (notation) let $x' = T(x)$. Then (Fubini)

$$\int_{D'} G_{D'}(x',y')\phi(y')dy' = \mathbb{E}_{x'}[\int_0^{\tau'} \phi(B'_{t'})dt']$$

where $B'_{t'}$ is a Brownian motion (in D') starting from x', and τ' its exit time from D'. Recall now that for an holomorphic function $f(z) = (f(x_0), f(y_0))$ written in $\mathbb{R}^2$ coordinates, one has $detJ(x_0,y_0) = |f'(z_0)|^2$; it implies that the change of variable $y' = T(y)$ on the left side gives:

$$\int_{D'} G_{D'}(x',y')\phi(y')dy' = \int_D G_{D'}(T(x),T(y))\phi(T(y))|T'(y)|^2 dy$$

We want now to rewrite the right hand of the previous formula with variables in D too. This purpose is reachable by using the Dublin-Schwartz formula which guarantees the existence of a Brownian Motion $\hat{B}_t$ in D, starting in $x' = T(x)$, such that $B'_t = T(\hat{B}_{F(t)})$ where $F : t \mapsto \int_0^t \frac{1}{|T'(T^{-1}(B'_s))|^2} ds$. Consequently we have:

$$\mathbb{E}_{x'}[\int_0^{\tau'} \phi(B'_t)dt'] = \mathbb{E}_{x'}[\int_0^{\tau'} \phi(T(\hat{B}_{F(t)})dt]$$

Note that if $\hat{\tau} = inf\{t \in \mathbb{R}^+ : \hat{B}_t \notin D\}$, then $F^{-1}(\tau') = \hat{\tau}$. By setting $s = F(t)$ we have $dt = |T'(B_s)|^2 ds$ and so the last expectation equals:

$$\mathbb{E}_x[\int_0^\tau \phi(T(\hat{B}_s))|T'(\hat{B}_s)|^2 ds = \int_D G_D(x,y)\phi(T(y))|T'(y)|^2 dy$$

In other words we proved:

$$\int_D G_{D'}(T(x),T(y))\phi(T(y))|T'(y)|^2 dy = \int_D G_D(x,y)\phi(T(y))|T'(y)|^2 dy$$

and so the claim follows thanks to the arbitrary choice of $\phi \in C_c^\infty(D)$ and symmetry of G. □

Summing up: we know the explicit form of G over the half plan, which is conformally mapped into the unitary disk by Moebius map (section 1.1). Furthermore, the unitary disk is conformal to any proper simply connected set of $\mathbb{C}$ (Riemann mapping theorem). As a consequence, by composing all the maps and considering the conformal invariance just proved, we are able to compute G for all the cases in our interest. A complete analytic description is not necessarily, the following approximations will suffice.

Proposition 1.7.2 (Green Functions: analytic properties)**.** For each $x \in D$, the following holds:

1 $G(x, \cdot)$ is harmonic in $D \backslash \{x\}$ and, as a distribution, $\Delta G(x, \cdot) = -2\pi\delta_x(\cdot)$;

2 $G(x, y) = -\log|x - y| - \hat{G}_x(y)$ where, when $x \in D$ is fixed, $\hat{G}_x(y)$ is the harmonic extension to $y \in D$ of the map $-\log|x - y|$, function of $y \in \partial D$;

3 $G(x, y) = -\log|x - y| + \log R(x; D) + o(1)$ as $y \to x$;

4 $G(x, y) = -\log|x - y| + O(1)$ as $y \to x$;

5 For each $x \in D$, $-\hat{G}_x(x) = \log R(x; D)$;

Proof. (1): Let $\phi \in C_c^\infty(D)$. Note that ϕ vanishes on ∂D and that $\int_D G(x, y)\Delta\phi(y)dy = \pi\mathbb{E}_x[\int_0^\tau \Delta\phi(B_t)dt]$ by using Fubini (τ is the exiting time from D). On the other hand, *Itô formula* implies:

$$\phi(B_\tau) - \phi(B_0) = \int_0^\tau \nabla\phi(s) \cdot dB_s + \frac{1}{2}\int_0^\tau \Delta\phi(B_t)dt$$

By taking the expectation and recalling that the mid term is a martingale, one obtains:

$$2\pi(\phi(B_\tau) - \phi(B_0)) = \pi\mathbb{E}_x[\int_0^\tau \Delta\phi(B_t)dt] = G(x, y)$$

and finally, since ϕ vanishes on ∂D and $B_0 = x$ the claim follows. To be more precise, we deduce that $\Delta G(x, \cdot) = -2\pi\delta_x(\cdot)$ in the distributional sense. Consequently, $G(x, \cdot)$ is an harmonic distribution on $D \setminus \{x\}$, and so a smooth harmonic function thanks to the Weyl's principle (section 1.2).

(2): Note that the Green function G can be extended on the boundary ∂D, where it assumes the constant value 0 (a Brownian motion starting outside D, is already outside D at time 0). We proved harmonicity in part (1), consequently both the y-functions $G(x, y)$ and $-\log|x - y| - \hat{G}_x(y)$ solve the same Dirichlet problem and so they must coincide by uniqueness theorems. The author has actually no direct experience in PDEs theory, he apologies for that and suggests to consult [3] for more precise details.

(3): Let's start by proving the claim for the disk $\mathbb{D}$. By using the explicit form of the conformal Radius for the half plane $\mathbb{H}$ (section 1.1) and conformal invariance, one directly computes $G_{\mathbb{D}}(0, z) = -\log|z|$ making the statement trivial. Let now be $x, y \in \mathbb{D}$, and $T : \mathbb{D} \to \mathbb{D}$ a conformal mapping such that $T(0) = x$ and $T(z) = y$. $G_{\mathbb{D}}(x, y) = G_{\mathbb{D}}(T(0), T(z)) = G_{\mathbb{D}}(0, z) = -\log|z|$. Then by adding and subtracting the same quantity we have:

$G_{\mathbb{D}}(x, y) = -\log|x - y| + \log R(x; \mathbb{D}) + \log|T(0) - T(z)| - \log R(x; \mathbb{D}) - \log|z|$.

It is clear that $y \to x$ iff $T(y) \to T(x)$ (continuity of T and its inverse), and observe how the second part in the previous estimation can be rewritten as:

$$\log|\frac{T(0) - T(z)}{z}| - \log|\frac{1}{R(x; \mathbb{D})}| = \log|\frac{T(0) - T(z)}{z}| - \log|\frac{R(0; \mathbb{D})}{R(x; \mathbb{D})}| = \log|\frac{T(0) - T(z)}{z}| - \log|T'(0)|$$

by using the conformal property of the radius (again, proved in section 1.1). The previous quantity is then an $o(1)$ as $y \to x$, since it converges to the differences of the same derivative. The result is now true for the unitary disk, and can be easily extended for any proper simply connected domain D by using the same strategy. Let's see how.

Let $a, b \in D$ and $T : \mathbb{D} \to D$ a conformal mapping. Call $a = T(x)$, $b = T(y)$. $G_D(a,b) = G_{\mathbb{D}}(x,y) = -\log|x-y| + logR(x;\mathbb{D}) + o(1)$, and again by adding and subtracting the same quantity:

$$G_D(a,b) = -\log|a-b| + \log R(a;D) + \log|\frac{T(x)-T(y)}{x-y}| + \log|\frac{R(x;\mathbb{D})}{R(T(x);D)}|$$

The last two members are an $o(1)$ for $a \to b$ since they converge to $\log|T'(x)| - \log|T'(x)| = 0$, and the claim follows.

(4): direct consequence of (3);

(5): By part (2) and (3) we have $G(x,y) = -\log|x-y| - \hat{G}_x(y)$ and $G(x,y) = -\log|x-y| + \log R(x;D) + o(1)$ as $y \to x$. We want to show: $-\hat{G}_x(x) = \log R(x;D)$. By using Taylor expansion, write $\hat{G}_x(y) = \hat{G}_x(x) + o(1)$ as $y \to x$ and substitute in the first expression. Finally, by taking the differences of the two members we have $0 = \hat{G}_x(x) - \log R(x;D) + o(1)$ as $y \to x$ and so the claim follows by taking the limit. □

1.8 Green Functions and Gaussian Free Fields

Green Functions enjoy a good intuitive meaning (as seen), and at the same time can be used for computing covariances of Gaussian Free Fields (as we'll prove).

Proposition 1.8.1 (Inverse Laplacian)**.** The inverse Laplacian operator $\Delta^{-1} : C_c^\infty(D) \to H(D)$ acting as $f \mapsto \Delta^{-1} f$, where $\Delta^{-1} f$ is the map $x \mapsto -\frac{1}{2\pi}\int_D G(x,y)f(y)dy$ is well-defined. The notation is coherent with intuition, since we have $\Delta\Delta^{-1} f = \Delta^{-1}\Delta f = f$ every time $f \in C_c^\infty(D)$.

Proof - sketch. We need only check that $\Delta^{-1} f \in H(D)$, the second part directly follows from harmonicity of G in the sense of distributions. We will not give a complete proof here, rather we describe the underlying idea. As shown in the appropriate resources ([3],[1]), the inverse Laplacian can be equivalently constructed in way based instead on some eigenfunction powers of the classical Laplacian operator. One then checks that new operator acts from test functions to $H_0^1(D)$, where now this set is a Sobolev space whose definition is based on such eigenfunctions too. Finally, one proves that $H_0^1(D)$ is isomorphic to $H(D)$, leading to the final result. □

The previous property implies injectivity for Δ^{-1}, but not surjectivity which is actually not true. Indeed, in the equation $h = \Delta^{-1} f$, $h \in H(D)$, one might be tempted to use Laplacian on both sides, but it is generally not possible since this operator is not defined on $H(D)$, being this space constructed taking into account only weak gradients and not second derivatives (e.g. see [3]). We usually call $-\rho_i$ the inverse Laplacian of a test function f_i, whenever there is no risk of ambiguity. According to this new notation, when $f \in C_c^\infty(D)$, the symbol (h, f) is then equal to $2\pi(h, \rho)_\nabla$ (see section 1.5). Finally, we define the gamma map for test functions f_1 and f_2, as $\Gamma(f_1, f_2) := \int_{D^2} G(x, y) f_1(x) f_2(x) dx dy$. The symbol $\Gamma(f)$ will be a shortcut for $\Gamma(f, f)$.

Proposition 1.8.2 (Green Functions compute covariances)**.** If $f_1, f_2 \in C_c^\infty(D)$, then

$$Cov[(h, f_1), (h, f_2)] = \Gamma(f_1, f_2)$$

Proof. By construction $Cov[(h, f_1), (h, f_2)] = (2\pi)^2 Cov[(h, \rho_1)_\nabla, (h, \rho_2)_\nabla] = (2\pi)^2 (\rho_1, \rho_2)_\nabla$ so the only real fact to check is that $(2\pi)^2(\rho_1, \rho_2)_\nabla = \Gamma(f_1, f_2)$. By direct computation:

$$\Gamma(f_1, f_2) = \int_{D^2} f_1(x) G(x, y) f_2(y) dy dx = \int_D f_1(x) [\int_D G(x, y) f_2(y) dy] dx =$$

$$= -\int_D f_1(x) [\int_D G(x, y) \Delta\rho_2(y) dy] dx = 2\pi \int_D f_1(x) \rho_2(x) dx = -2\pi \int_D \Delta\rho_1(x) \rho_2(x) dx =$$

$= (2\pi)^2 (\rho_1, \rho_2)_\nabla$ where every step directly uses the respective definitions, the distributional equality $\Delta G(x, \cdot) = -2\pi\delta_x(\cdot)$ and finally the Gauss-Green formula. □

The following theorem allows a coherent alternative definition of the notion of GFF.

Theorem 1.8.1 (GFF as a process indexed by test functions)**.** A GFF h is the unique stochastic process $(h_\rho)_{\rho \in C_c^\infty(D)}$ such that for every choice of $\rho_1, \ldots, \rho_n$, the random vector $(h_{\rho_1}, \ldots, h_{\rho_n})$ is centered and Gaussian with covariance structure $Cov[h_{\rho_i}, h_{\rho_j}] = \Gamma(\rho_i, \rho_j)$.

Proof. Thanks to Kolmogorov's theorem we need to check only on finite-dimensional marginals. Since we deal with an hypothetical Gaussian process, coherence between sub-marginals is automatic and the result becomes equivalent to the positive definition of the covariance matrix (symmetry is directly inherited by $G(x, y) = G(y, x)$). But this condition is easy to check, while $\sum_{i,j} \lambda_i \lambda_j \Gamma(\rho_i, \rho_j) = \Gamma(\sum_i \lambda_i \rho_i, \sum_j \lambda_j \rho_j) \geq 0$ since $\Gamma(\rho, \rho) \geq 0$ being the Dirichlet energy (i.e. a norm) of a certain function as proved above. □

The previous proposition is nice since it allows to forget for a moment the full structure we studied before, in favor of a more abstract viewpoint. It suggests how Gamma maps and Green Functions can be another way for building a GFF, and so trying to extend them might be a way for discovering more general definitions. In the remaining section, since the checking of every property was particularly long and strictly specific for this context, we chosen to directly refer to [3] and illustrate here a brief but hopefully clear description of the winning ideas.

Let $\mathcal{M}_+$ be the set of (non-negative) measures ρ with compact support in D such that $\int_{D^2} G(x, y) \rho(dx) \rho(dy) < \infty$. Then, define $\mathcal{M}$ as the collection of signed measures $\rho = \rho_+ - \rho_-$, $\rho_\pm \in M_+$. Note that M includes all the test functions, by splitting them into the positive and negative part, and letting each of them induce the weighted measure $A \mapsto \int_A f_\pm dx$. The function Γ can be now be extended on $\rho_1, \rho_2 \in \mathcal{M}$ via

$\Gamma(\rho_1, \rho_2) = \int_{D^2} G(x,y)\rho_1(dx)\rho_2(dy)$ keeping coherence with the previous definition. The same happens for the inverse Laplacian operator, acting now from $\mathcal{M} \to H(D)$ and defined as $\rho \mapsto \Delta^{-1}\rho(\cdot) \doteq -\frac{1}{2\pi}\int_D G(\cdot, y)\rho(dy)$. The same reasoning as before is repeated for the new scenario: to integrate against measures is valid by coherently defining $(h, \rho) \doteq 2\pi(h, -\Delta^{-1}\rho)_\nabla$ for $\rho \in \mathcal{M}$. By using a density argument we obtain our most general definition of Gaussian Free Field:

Theorem 1.8.2 (GFF: most general definition)**.** There exists a unique stochastic process $(h_\rho)_{\rho\in\mathcal{M}}$ such that for every choice of $\rho_1, \dots, \rho_n$, the vector $(h_{\rho_1}, \dots, h_{\rho_n})$ is a centered Gaussian with covariance structure $Cov[h_{\rho_i}, h_{\rho_j}] = \Gamma(\rho_i, \rho_j)$.

It is a good moment for summing up the various GFF constructions. We start from the space $H(D)$ with an orthonormal basis $\{f_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}} \sim N(0,1)$ i.i.d. The sequence $h \doteq \sum_{n\in\mathbb{N}} f_i\alpha_i$ converges a.s. in a larger space B, so the GFF h is a random variable in B (and B can be identified with the space of distributions $H^{-1} = H^{-1}(D)$). The law of h behaves like an infinite-dimensional Gaussian, in particular it has a *covariance structure* that holds for every element of $B' \subseteq H(D)$. For our purposes, this is not enough, and we would like to work with the larger space $H(D) \supseteq B'$. We reach this goal by summing orthonormal basis coefficients, as seen in section 1.5, making now meaningful the symbol $(h, f)_\nabla$ for $f \in H(D)$. Since the Inverse Laplacian brings test functions to $H(D)$, integration against test functions is valid by defining $(h, f) \doteq 2\pi(h, -\Delta^{-1}f)_\nabla$. Finally, the integration domain can be extended to the most general class $\mathcal{M}$, composed by Greenian measures. Every step is done *preserving* the covariance structure. This section also describes how Kolmogorov's theorem and the structure characterization of Gaussian processes are sufficient for describing the existence of Gaussian Free Fields, omitting all the steps before. The two viewpoints are very important and the paper [3] confirms their full compatibility in more details. The author hopes that the efforts for describing the first method can be beneficial and interpreted as a *concrete* way for understanding what is truly happening beyond formalism.

Recall now the construction of the truncated series $h^N = \sum_{i=1}^N \alpha_i f_i$, where $\{f_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $H(D)$ (without loss of generality it can be composed by test functions only). For a measure ρ (not necessarily in $\mathcal{M}$) we define $(h^N, \rho) = \sum_{i=1}^N \int_D f_i(x)\rho(dx)$. Many properties similar to the test function case done in section 1.5 hold here in a similar fashion.

Proposition 1.8.3. Let D be a Greenian domain, h^N the truncated series as above. Then, for any $\rho \in \mathcal{M}$, we have $(h^N, \rho) \to (h, \rho)$ in probability and in $L^2(\mathbb{P})$[1], as $N \to \infty$.

Proof. This is again an approximation argument for which we directly refer to [3]. □

Proposition 1.8.4. Suppose that $\rho_k \in \mathcal{M}$ is a sequence of measures weakly converging to ρ (with ρ not necessarily in $\mathcal{M}$). Then, for each $\omega \in \Omega$, $(h^N(\omega), \rho_k) \to (h^N(\omega), \rho)$ as $k \to \infty$.

[1]Recall that $\mathbb{P}$ is intended to be the probability measure in the user's triple, as clarified in section 1.1.

Proof. By definition $(h^N(\omega), \rho_k) = \sum_{i=1}^N \alpha(\omega) \int_D f_i(x)\rho_k(dx)$, but each integral converges to $\int_D f_i(x)\rho(dx)$ since $f_i \in C_c^\infty(D)$ and by the definition of weakly converging measures. □

One of the main goal we want to reach, is the construction of a measure (Liouville Quantum Gravity) that depends on the "pointwise evaluation" of a GFF h. Altought the expression $h(z)$ for $z \in \mathbb{C}$ is meaningless, an attempt to formalize it could be to consider the quantity (h, δ_z), i.e. the GFF evaluated in the Dirac delta in z. Unfortunately, it is not allowed because $\delta_z \notin \mathcal{M}$, since the related Green Function would be infinite thanks to the divergence on the diagonal. A way for avoiding this problem is given by replacing Dirac deltas with uniform measures ρ_ϵ^z on circles $\partial B_\epsilon(z)$, then thinking to $h(z)$ as a sort of limit for $\epsilon \to 0$. The purpose of the following section will be to clarify more about that technique and prove various related properties.

1.9 Circle averages and definition of $h_\epsilon(z)$

Now we define and study the object $h_\epsilon(z)$ whose intuitive description has been given before.

Definition 1.9.1 (Circle Averages)**.** If h is a Gaussian Free field on D, and ρ_ϵ^z the normalized uniform measure on the circle $\partial B_\epsilon(z) \subseteq D$, we define the *circle average of* h as $h_\epsilon(z) \doteq (h, \rho_\epsilon^z)$.

Proposition 1.9.1 (Variance and existence)**.** For every $z \in D$, $\epsilon > 0$ s.t. $\partial B_\epsilon \subseteq D$, $\rho_\epsilon^z \in \mathcal{M}$. In particular, the just defined quantity $h_\epsilon(z) = (h, \rho_\epsilon^z)$ is well-defined. Furthermore, $\Gamma(\rho_\epsilon^z) = Var[h_\epsilon(z)] = -\log\epsilon + \log R(x; D)$ [2].

Proof. It is sufficient to prove $\Gamma(\rho_\epsilon^z) = -\log\epsilon + \log R(x; D)$, since it directly implies $\rho \in \mathcal{M}$ by construction, while $\mathrm{Var} h_\epsilon(z) = \Gamma(\rho_\epsilon^z)$ holds automatically thanks to the covariance description given in the previous section. Note that

$$\Gamma(\rho_\epsilon^z) = \int_{D^2} G(x,y)\rho_\epsilon^z(dx)\rho_\epsilon^z(dy) = \int_D \rho_\epsilon^z(dx)(\int_D G(x,y)\rho_\epsilon^z(dy))$$

Since in the second integral x is fixed, $G(x, \cdot)$ is harmonic in the integration variable, and that integral is exactly equal to $G(x, z)$ by using the mean-value property for harmonic functions. We can consequently continue this equality with:

$$\int_D \rho_\epsilon^z(dx)(\int_D G(x,y)\rho_\epsilon^z(dy)) = \int_D G(x,z)\rho_\epsilon^z(dx) = \int_D -\log|x-z| + \log R(z; D)\rho_\epsilon^z(dx)$$

thanks to property (iv) in section 1.7, where the last quantity is equal to $-\log\epsilon + \log R(z; D)$. □

[2] We requested $d(z, \partial D) > \epsilon$, but it can be done without relevant consequences: in the whole work ϵ will be always very small and/or made converging to 0.

With the risk of being repetitive, we remark again how the pointwise evaluation of circle averages is now meaningful, so we could think as "$h(z)$" (an object required for the Liouville Quantum Gravity, we'll see later) as a "limit" for $\epsilon \to 0$ of $h_\epsilon(z)$, in a sense to be specified.

Since $\rho_\epsilon^z \to \delta_z$ weakly, by using proposition 1.8.4 in the previous section, one deduces $h_{\epsilon,z}^N \to h^N(z)$ punctually (where we mean $h_{\epsilon,z}^N \doteq (h^N, \rho_\epsilon^z)$). Since it holds for each $z \in D$ and radius small enough, we sometimes refer to this convergence by the symbol "$h_\epsilon^N \to h^N$" as $\epsilon \to 0$.

By using the same idea for the variance computation done in the first proposition, it is possible to reveal more about the covariance structure.

Proposition 1.9.2 (Covariance of circle averages)**.** Let $z \in D$, $\epsilon_1, \epsilon_2 > 0$ such that both $B_{\epsilon_1}(z) \subseteq D$ and $B_{\epsilon_2}(z) \subseteq D$. Then, $Cov[h_{\epsilon_1}(z), h_{\epsilon_2}(z)] = -\log(\min(\epsilon_1, \epsilon_2)) + R(z; D)$.

Proof. Recall that $Cov[h_{\epsilon_1}(z), h_{\epsilon_2}(z)] = \int_{D\times D} \rho_{\epsilon_1,z}(dx) G(x,y) \rho_{\epsilon_2,z}(dy)$. Without loss of generality, assume $\epsilon_2 < \epsilon_1$ (if not, use symmetry of G for reversing the integral). We focus on the term $G(x,y)\rho_{\epsilon_2,z}(dy)$. Since x always lies on a larger circle, $G(x,\cdot)$ has no singularity for y in the circle of radius ϵ_2, consequently the mean value property for harmonic functions can be used and the full integral reduces to $\int_D G(x,z)\rho_{\epsilon_1,z}(dx) = -\log\epsilon_1 + R(z;D)$ by using the same computation as in the proposition above. □

At this point, for a fixed $z \in D$, we know the process $h_\epsilon(z)$ to be Gaussian, with mean zero and covariance structure as before. But there is actually much more:

Proposition 1.9.3 (Circle averages admits a continuous version)**.** The process $h_\epsilon(z)$ has a modification which is almost surely locally η-Hoelder continuous in the pair $(z,\epsilon) \in \mathbb{C} \times (0,\infty)$ for every $\eta < 1/2$.

Proof. We refer to [2], where the technique is to use Kolmogorov regularity theorem. □

Proposition 1.9.4 (Circle averages are Brownian Motions)**.** Let $t_0^z \doteq inf\{t : B_{e^{-t}}(z) \subseteq D\}$, and let $\mathcal{V}_t \doteq h_{e^{-t}}(z)$. If $z \in D$ is fixed, then the process $V_t \doteq \mathcal{V}_{t_0^z+t} - \mathcal{V}_{t_0^z}$ is a standard Brownian Motion in t.

Proof. The process $\mathcal{V}_{t_0^z+t}$ is continuous and Gaussian, with covariances equal to:

$$Cov[\mathcal{V}_{t_0^z+t}, \mathcal{V}_{t_0^z+s}] = Cov[h^z_{e^{-t_0^z+t}}, h^z_{e^{-t_0^z+s}}] = min(t,s) + t_0^z + \log R(x; D)$$

thanks to the explicit computations done before. Consequently, by using a characterization theorem, it is a (non-standard) Brownian Motion. On the other hand, since $\mathrm{Var}\mathcal{V}_{t_0^z} = t_0^z + \log R(z; D)$, the "corrected" process $V_t \doteq \mathcal{V}_{t_0^z+t} - \mathcal{V}_{t_0^z}$ is now a standard Brownian Motion in t. □

This is the last result of this first master's thesis chapter, where the goal was to give a description of this mathematical object called Gaussian Free Field. It will be used in the upcoming chapter in order to construct a measure called Liouville Quantum Gravity, the main topic of this work.

Chapter 2

Liouville Quantum Gravity

2.1 A quick overview

The author hopes that a small break for summing up the previous results in favor of the big picture could be beneficial. In few words, the goals of the previous parts can be seen as to:

1 introduce and define the notion of Gaussian Free Field, named h, with full mathematical rigour. At the same time, try to give an interpretation as an infinite dimensional Gaussian and a random distribution, with connections to a specific problem in modern Physics;

2 investigate some concretely useful approximation property, for example via the objects h^N or h_ϵ. Not only they will play a central role in many proofs, but they can be used for numerical simulations, giving a practical tool in case of need;

3 study the well-behavior of h under conformal mappings.

Our progresses can be represented via the following conceptual "commutative diagrams":

$$\begin{array}{ccc} h^N & \xrightarrow{N\to\infty} & h \\ {\scriptstyle \epsilon\to 0}\uparrow & & {\scriptstyle \epsilon\to 0}\uparrow \\ h^N_\epsilon & \xrightarrow{N\to\infty} & h_\epsilon \end{array}$$

$$\begin{array}{ccc} D & \xrightarrow{\quad GFF \quad} & h_D \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle pullback-of-a-distribution} \\ \tilde{D} & \xrightarrow{GFF} & h_{\tilde{D}} = h_{\phi^{-1}} = h \circ \phi^{-1} \end{array}$$

They author hopes that they might help to clarify the general structure we developed. Not everything in the pictures is completely formal and literally right. It is done on purpose. When the reader is able to have clear ideas about the meaning of the two diagrams, he can surely move to the next part without any problem. In order to give an example, recall that there is no "true" limit $h_\epsilon \to h$, but the writing should help in keeping in mind how we'll think about the pointwise evaluation "$h(z)$" as a "limit" of the circle averages. All the required details will be given later, the goal for now is only to have a reasonable good overview.

We can now express in a similar way what we intend to do in the next step. Briefly speaking, we introduce a new object $\mu = \mu_h$ (a measure associated to the Gaussian Free Field h) called "Liouville Measure" or "Liouville Quantum Gravity" (abbreviated LQG), for which diagram (1) is still "conceptually valid", but unfortunately the symmetry of (2) is "broken", altought still "recoverable" by adding a small extra term. This phenomenon will be called "conformal covariance" (rather than "invariance"), and needs not to be confused with the homonym in Category Theory.

$$\begin{array}{ccc} \mu^N & \xrightarrow{N\to\infty} & \mu \\ {\scriptstyle \epsilon\to 0}\uparrow & & \uparrow{\scriptstyle \epsilon\to 0} \\ \mu_\epsilon^N & \xrightarrow{N\to\infty} & \mu_\epsilon \end{array}$$

$$\begin{array}{ccc} D & \xrightarrow{LQG} & \mu_D \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle pullback-of-a-measure} \\ \tilde{D} & \xrightarrow{LQG} & \mu_{\tilde{D}} \neq \mu_D \circ \phi \end{array}$$

Again, both the diagrams must be taken with a grain of salt, but the reader is encouraged in having a look to them again after having read the new upcoming section.

2.2 Definition of μ as limit of μ_ϵ

The goal of this section is to define a random probability measure μ called "Liouville Quantum Gravity" or "Liouville measure" on a set D with standard hypothesis (see section 1.1), by using an approximation argument. There is no difficulty in defining the family of random measures $\{\mu_\epsilon\}_{\epsilon>0}$ where $\mu_\epsilon(dz) \doteq \epsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\epsilon(z)} dz$, absolutely continuous w.r.t. Lebesgue. The constant $\gamma \in [0, 2)$ is fixed and for $\gamma = 0$ we reduce to the Lebesgue case. Boundedness of D guarantees that the measures can be normalized. For sake of clarification, randomness is intended as: $\omega \mapsto \mu_\epsilon(dz)(\omega) = \epsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\epsilon(z)(\omega)} dz$. We are interested in showing a convergence result for this family of random variables, to a limit random measure $\mu(dz)$. Let $\epsilon > 0$ and $\delta \doteq \frac{\epsilon}{2}$. In order to make notations shorter, define $\sigma(dz) \doteq R(z; D)^{\frac{\gamma^2}{2}} dz$, an useful measure ($R$ is the conformal radius function, section 1.1). Define also $\hat{h}_\epsilon \doteq \gamma h_\epsilon(z) - (\frac{\gamma^2}{2}) Var[h_\epsilon(z)]$ and, for an arbitrary $A \subseteq D$ measurable, let $I_\epsilon \doteq \mu_\epsilon(A)$.

Theorem 2.2.1 (LQG convergence for the case $\gamma < \sqrt{2}$). Assume $\gamma < \sqrt{2}$. Under the conditions stated above, we have $\mathbb{E}[(I_\epsilon - I_\delta)^2] \leq C\epsilon^{2-\gamma^2}$ for a constant C. Being so I_ϵ a Cauchy sequence in L^2, it converges in probability too. In particular, among $\epsilon = 2^k$ such a convergence happens almost surely. In other words, for each measurable $A \subseteq D$ the sequence of real-valued random variables $\{\mu_\epsilon(A)\}_{\epsilon>0}$ converges almost surely among the dyadic sequence.

Proof. By Fubini we have

$$\mathbb{E}[(I_\epsilon - I_\delta)^2] = \int_A \mathbb{E}[(e^{\hat{h}_\epsilon(x)} - e^{\hat{h}_\delta(x)})(e^{\hat{h}_\epsilon(y)} - e^{\hat{h}_\delta(y)})]\sigma(dx)\sigma(dy)$$

We try to simplify this integral. On the set $\{(x,y) \in D : |x-y| > 2\epsilon\}$, the objects "$h_\epsilon(x) - h_\delta(x)$" and "$h_\epsilon(y) - h_\delta(y)$" are independent by the Markov Property (corollary in section 1.6). By decomposing

$$(e^{\hat{h}_\epsilon(x)} - e^{\hat{h}_\delta(x)})(e^{\hat{h}_\epsilon(y)} - e^{\hat{h}_\delta(y)}) = (1 - e^{\hat{h}_\delta(x)+\hat{h}_\epsilon(x)})(1 - e^{\hat{h}_\delta(y)+\hat{h}_\epsilon(y)})e^{\hat{h}_\epsilon(x)+\hat{h}_\epsilon(y)}$$

we detect more: the three factors are indeed measurable respectively w.r.t. $\mathcal{F}_{B_\epsilon(x)}$, $\mathcal{F}_{B_\epsilon(y)}$ and $D \setminus (\mathcal{F}_{B_\epsilon(x)} \cup \mathcal{F}_{B_\epsilon(y)})$ (by the Markov property again). But since they are three disjoint σ-algebras, the expectation factorizes. Furthermore, by the martingale property we have $\mathbb{E}[e^{\hat{h}_\delta(x)-\hat{h}_\epsilon(x)}] = 1$ (same for y), which makes the whole product vanishing. In other words:

$$\mathbb{E}[(I_\epsilon - I_\delta)^2] = \int_{|x-y|\leq 2\epsilon} \mathbb{E}[(e^{\hat{h}_\epsilon(x)} - e^{\hat{h}_\delta(x)})(e^{\hat{h}_\epsilon(y)} - e^{\hat{h}_\delta(y)})\sigma(dx)\sigma(dy)$$

$$\leq \int_{|x-y|\leq 2\epsilon} \sqrt{\mathbb{E}[(e^{\hat{h}_\epsilon(x)} - e^{\hat{h}_\delta(x)})^2)\mathbb{E}[(e^{\hat{h}_\epsilon(y)} - e^{\hat{h}_\delta(y)})^2)}\sigma(dx)\sigma(dy) =$$

$$K \int_{|x-y|\leq 2\epsilon} \sqrt{\mathbb{E}[e^{2\hat{h}_\epsilon(x)}]\mathbb{E}[e^{2\hat{h}_\epsilon(y)}]}\sigma(dx)\sigma(dy)$$

by using firstly Cauchy-Schwartz, and then observing that now the expectations involve only functions of Gaussian variables. Continuing the computation using that fact leads to the final concluding result $\mathbb{E}[(I_\epsilon - I_\delta)^2] \leq C\epsilon^{\gamma^2-2}$. □

The second part of this section is devoted to the proof of the same convergence in the remaining case $\gamma \in [\sqrt{2}, 2)$. The main idea is not to stop the domain's reduction on the set $|x-y| \leq 2\epsilon$, rather try to continue. We will see later that the only "relevant" points are the "thick points", so the idea, roughly speaking, is to remove all the others. Let's convert these ideas into Mathematics. For each $\alpha > 0$ and $z \in D$ we define the "good event" $G_\epsilon^\alpha(z) \doteq \{\omega \in \Omega : h_\epsilon(z)(\omega) \leq \alpha \log(\frac{1}{\epsilon})\}$, denoting then with $\hat{G}_\epsilon^\alpha(z)$ its complement. Later we'll comment more about that property (section on "thick points"), but for the moment observe that:

Lemma 2.2.1 (Good events happen frequently)**.** If $\alpha > \gamma$, then $\mathbb{E}[e^{\hat{h}_\epsilon(z)}\mathbb{1}_{G_\epsilon^\alpha(z)}] \geq 1 - p(\epsilon)$, where p is s.t. $p(\epsilon) \to 0$ when $\epsilon \to 0$. The map p might depend on α, and the decay happens polynomially. The same estimation holds if we substitute "$\hat{h}_\epsilon(z)$" with "$\hat{h}_{\frac{\epsilon}{2}}(z)$" (preserving ϵ in all the other terms).

Proof. If $\tilde{\mathbb{P}}$ is the probability measure with Radon-Nikodym derivative $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{\hat{h}_\epsilon(z)}$ (possible since $e^{\hat{h}_\epsilon(z)}$ is the Brownian exponential martingale), then $\mathbb{E}[e^{\hat{h}_\epsilon(z)}\mathbb{1}_{G_\epsilon^\alpha(z)}] = \tilde{\mathbb{P}}(G_\epsilon^\alpha(z)) = 1 - \tilde{\mathbb{P}}(\hat{G}_\epsilon^\alpha(z))$. By the change of variables $\epsilon = e^{-t}$, if $X_s \doteq h_{e^{-s}}(z)$ is the circle-averages Brownian Motion under $\mathbb{P}$, we know that under $\tilde{\mathbb{P}}$, is $X_s - s\gamma$ to preserve such a law (classic Girsanov's theorem). The event $\hat{G}_\epsilon^\alpha(z)$ becomes $\{\omega \in \Omega : X_s \leq \alpha s\} = \{\omega \in \Omega : X_s - s\gamma \leq (\alpha - \gamma)s\} = \{\omega \in \Omega : \tilde{B}_t \leq kt\}$ where $\tilde{B}_t$ is a Brownian motion under $\tilde{\mathbb{P}}$ and $k > 0$. But $\tilde{\mathbb{P}}(\tilde{B}_t \leq kt)$ is known to decay to zero exponentially as $t \to \infty$, consequently it goes to zero polynomially as $\epsilon \to 0$. Finally, if we replace $h_\epsilon(z)$ with $h_{\frac{\epsilon}{2}}(z)$ we obtain the same kind of estimation (but with $t + \log 2$ rather than t) which does not influence the asymptotic behavior. □

The following quantities will have a key role in the final proof:

Definition 2.2.1 (J-integrals)**.** Define:

$$J_\epsilon^\alpha \doteq \int_A e^{\hat{h}_\epsilon(z)}\mathbb{1}_{G_\epsilon^\alpha(z)}\sigma(dz)$$

$$J'^{\alpha}_{\frac{\epsilon}{2}} \doteq \int_A e^{\hat{h}_{\frac{\epsilon}{2}}(z)}\mathbb{1}_{G_\epsilon^\alpha(z)}\sigma(dz)$$

Corollary 2.2.1 (J-integrals approximate μ_ϵ well)**.** If $\alpha > \gamma$, we have $\lim_{\epsilon\to 0}\mathbb{E}[|I_\epsilon - J_\epsilon^\alpha|] = 0$ and $\lim_{\epsilon\to 0}\mathbb{E}[|I_{\frac{\epsilon}{2}} - J'^{\alpha}_{\frac{\epsilon}{2}}|] = 0$.

Proof. Direct computations:

$$\mathbb{E}[|I_\epsilon - J_\epsilon^\alpha|] = \mathbb{E}[\int_A e^{\hat{h}_\epsilon(z)}\mathbb{1}_{\hat{G}_\epsilon^\alpha(z)}\sigma(dz)] = \int_A \mathbb{E}[e^{\hat{h}_\epsilon(z)}\mathbb{1}_{\hat{G}_\epsilon^\alpha(z)}]\sigma(dz)] = \int_A \tilde{\mathbb{P}}(\hat{G}_\epsilon^\alpha(z))\sigma(dz) \leq \sigma(A)p(\epsilon)$$

Since exactly the same inequality holds for the other case, the proof is concluded. □

Proposition 2.2.1 (J-integrals converges in $L^2(\mathbb{P})$)**.** Let $\gamma \in [\sqrt{2}, 2)$. Then there exists an $\alpha > \gamma$ sufficiently close to γ, such that $\mathbb{E}[(J_\epsilon^\alpha - J'^{\alpha}_{\frac{\epsilon}{2}})^2] \leq \epsilon^r$ for some $r > 0$.

Proof. We follow *exactly* the same reasoning as in the proof before (Fubini → Spatial Markov Property → Expectation splitting → Martingale property). The difference comes in the final step that we voluntarily pointed out:

$$... \leq K' \int_{|x-y|\leq 2\epsilon} \sqrt{\mathbb{E}[e^{2\hat{h}_\epsilon(x)}\mathbb{1}_{G_\epsilon^\alpha(x)}]\mathbb{E}[e^{2\hat{h}_\epsilon(y)}\mathbb{1}_{G_\epsilon^\alpha(x)}]}\sigma(dx)\sigma(dy)$$

If in the previous proof the term to evaluate was $\mathbb{E}[e^{2\hat{h}_\epsilon(x)}]$ by using its gaussianity, here we have $\mathbb{E}[e^{2\hat{h}_\epsilon(x)}\mathbb{1}_{\hat{G}_\epsilon^\alpha(z)}]$ instead, for which can follow a similar strategy but there will be a drift coming from Girsanov's theorem responsible for the shifting of γ. If $X_s = h_{e^{-s}}(z)$ is the Brownian Motion in $\mathbb{P}$, then $X_s - 2\gamma s$ is a Brownian Motion in $\tilde{\mathbb{P}}$, leading to $\mathbb{E}[e^{2\hat{h}_\epsilon(x)}\mathbb{1}_{\hat{G}_\epsilon^\alpha(z)}] = \tilde{\mathbb{P}}(\tilde{B}_s \leq (\alpha - 2\gamma)s)$ decaying polynomially in $\epsilon = e^{-s}$. By reassessing these results in the original integral, one obtains ... $\leq \epsilon^{2-\gamma^2}\epsilon^{\frac{1}{2}(2\gamma-\alpha)^2}$ and so when $\gamma \in [\sqrt{2}, 2)$, by choosing α enough close to γ we obtain a positive exponent ϵ^r with $r > 0$ leading to the claim. $\square$

Corollary 2.2.2 (LQG convergence for the case $\gamma \in [\sqrt{2}, 2)$). Let $\gamma \in [\sqrt{2}, 2)$. Then $\{I_\epsilon\}_\epsilon$ is a Cauchy sequence in L^1. Consequently, it converges in probability and almost surely along the dyadic sequence $\epsilon = 2^k$.

Proof. By triangle inequality:

$$\mathbb{E}[|I_\epsilon - I_{\frac{\epsilon}{2}}|] \leq \mathbb{E}[|I_\epsilon - J_\epsilon^\alpha|] + \mathbb{E}[|J_\epsilon^\alpha - J'^{\alpha}_{\frac{\epsilon}{2}}|] + \mathbb{E}[|I_{\frac{\epsilon}{2}} - J'^{\alpha}_{\frac{\epsilon}{2}}|]$$

where each term converges to zero thanks to the results just proved. $\square$

Summing up, in this section we have successfully proved the almost surely weak convergence of the family of random measures μ_ϵ, depending on a fixed parameter $\gamma \in [0, 2)$. This is a key central result, on which is based the main notion of this master's thesis:

Definition 2.2.2 (Liouville Quantum Gravity). Under the conditions above, the random probability measure $\mu(dz)$ ω-almost surely defined as the weak limit of $\mu(dz)(\omega) \doteq \lim_{\epsilon\to 0}\mu_\epsilon(dz)(\omega)$ is called *Liouville Quantum Gravity* (abbreviated LQG).

In other words, almost surely for every $\omega \in \Omega$ we have an object $\mu(\omega)$ measuring Borelian subsets of the plane (null on $\emptyset$, σ-countable and positive), s.t. if $A \subseteq D$ is measurable, then $\mu(\omega)(A) = \lim_{\epsilon\to 0}\mu_\epsilon(A)(\omega)$ and s.t. $\mu(D) < \infty$ a.s. (so can be normalized obtaining a random probability measure). LQG is sometimes formally written as "$\mu(dz) = e^{\gamma h(z)}dz$", despite there is no "meaning" for the point evaluation of a GFF (compare to the end of section 1.8).

Sometimes Liouville Quantum Gravity will be written by the symbol "μ_h" rather than just "μ", when it is convenient to point out its dependency from the random Gaussian Free Field h.

2.3 Rooted probability measure

Let $(H^{-1}, \mathcal{B}, dh)$ be the probability space defined in section 1.4, where $H^{-1} = H^{-1}(D)$ is the space of distributions over D and a natural identification for $B \supseteq H(D)$ (section 1.5); $\mathcal{B}$ the σ-algebra generated by certain class of functionals; dh the law of a Gaussian Free Field $\tilde{h}$, i.e. an "infinite-dimensional Gaussian" over H^{-1}. Let now $(D, \mathcal{D}, dz)$ be the probability space where $D \subseteq \mathbb{C}$ is a standard subset (section 1.1), $\mathcal{D}$ the Borel σ-algebra and dz the normalized Lebesgue measure. Let $h \in H^{-1}$ (in this section $\tilde{h}$ is the GFF, and we think of $h \in H^{-1}$ as a realization $h = \tilde{h}(\omega)$) and $z \in D$. The function $H^{-1} \times D \to \mathbb{R}$, $(h,z) \mapsto \epsilon^{\frac{\gamma^2}{2}} e^{\gamma(h,\rho_z^\epsilon)}$ is measurable so it is legitimate to define the weighted measure $\Theta_\epsilon(dh,dz) \doteq Z_\epsilon^{-1}\epsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\epsilon(z)} dhdz$ on $H^{-1} \times D$ where $dhdz$ is the product measure and Z_ϵ^{-1} a normalizing constant (see below). The meaning of such a construction will be clearer later, meanwhile we investigate some basic features.

Recall from section 1.3 that since $\mathbb{E}[e^N] = e^{a+\frac{b}{2}}$ when $N \sim (a,b)$, then $\mathbb{E}[e^{\gamma h_\epsilon(z)}] = e^{Var[\gamma h_\epsilon(z)]/2} = (\frac{R(z;D)}{\epsilon})^{\frac{\gamma^2}{2}}$ thanks to the work done in section 1.9. Then by imposing $1 = \int_D \int_B Z_\epsilon^{-1}\epsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\epsilon(z)} dhdz = \int_D Z_\epsilon^{-1} R(z;D)^{\frac{\gamma^2}{2}} dz$ we immediately deduce $Z_\epsilon = \int_D R(z;D)^{\frac{\gamma^2}{2}} dz < \infty$. Since it doesn't depend on epsilon, we define $Z \doteq Z_\epsilon$. So our new probability measure can be explicitly formulated as:

$$\Theta_\epsilon(dhdz) = (\int_D R(z;D)^{\frac{\gamma^2}{2}} dz)^{-1} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\epsilon(z)} dhdz$$

The knowledge of its marginals and conditional distributions will be of great help in many various situations:

z-marg.:

$$\Theta_\epsilon^z(dz) = \int_{H^{-1}} \Theta_\epsilon(h,z)dh = Z^{-1} \int_{H^{-1}} e^{\gamma h_\epsilon(z)} \epsilon^{\frac{\gamma^2}{2}} dhdz = Z^{-1} R(z;D)^{\frac{\gamma^2}{2}} dz$$

h-marg:

$$\Theta_\epsilon^h(dh) = \int_D \Theta_\epsilon(h,z)dz = Z^{-1} \int_D e^{\gamma h_\epsilon(z)} \epsilon^{\frac{\gamma^2}{2}} dzdh = Z^{-1}\mu_\epsilon(D)dh$$

z-cond:

$$\Theta_\epsilon(dh|z) = \frac{\Theta_\epsilon(dhdz)}{\Theta_\epsilon^z(dz)} = R(z;D)^{-\frac{\gamma^2}{2}} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\epsilon(z)} dh = \mathbb{E}_h[e^{\gamma h_\epsilon(z)}]^{-1} e^{\gamma h_\epsilon(z)} dh = \frac{e^{\gamma h_\epsilon(z)}}{\int_{H^{-1}} e^{\gamma h_\epsilon(z)} dh} dh$$

h-cond:

$$\Theta_\epsilon(dz|h) = \frac{\Theta_\epsilon(dhdz)}{\Theta_\epsilon^h(dh)} = \frac{\mu_\epsilon(dz)}{\mu_\epsilon(D)}$$

It is from the decomposition $\Theta_\epsilon(dhdz) = \Theta_\epsilon(dh|z)\Theta_\epsilon(dz)$ that we deduce *very* important ideas:

Meaning The "sampling procedure" of a point (h,z) according to Θ_ϵ can be done in two steps. First, extract a point $z \in D$ by following $\Theta_\epsilon(dz)$, actually proportional to the Lebesgue measure. Then extract h, according to $\Theta_\epsilon(dh|z)$, which is basically a weighted infinite Gaussian. In other words, the functionality of this measure Θ_ϵ is now supported by an intuitive interpretation.

Convergence The first factor in the product above is $\Theta_\epsilon(dh|z) = \mathbb{E}_h[e^{\gamma h_\epsilon(z)}]^{-1} e^{\gamma h_\epsilon(z)} dh$ $(*)$, whose formula resembles Girsanov theorem in section 1.3. If it is true that the theorem holds for the finite dimensional case, it can actually be extended under this setting with scalar product given by $(\cdot,\cdot)_\nabla$. It means that if $\tilde{h}$ is our original GFF with law dh, then under the measure $(*)$, i.e. dh weighted by $e^{\gamma h_\epsilon(z)} = e^{-2\pi\gamma(h,\Delta^{-1}\rho_\epsilon^z)_\nabla}$, the law of $\tilde{h}$ becomes $\tilde{h} - \gamma 2\pi\Delta^{-1}\rho_\epsilon^z$. But this family of random variables converges in law, as $\epsilon \to 0$, to $\tilde{h} - \gamma 2\pi\Delta^{-1}\delta_z(\cdot) = \tilde{h} + \gamma G(z,\cdot)$, where $G(z,\cdot)$ is the usual Green function on D and where we used properties shown in section 1.7. Consequently, the measures $\Theta_\epsilon(dh|z)$ admits a weak $\epsilon \to 0$ limit to a final measure, call it $\Theta(dh|z)$, characterized by the just stated drifting property. On the other hand, dealing with the second factor of the product, observe how it is given by $\Theta_\epsilon^z(dz)$ which does not depend on ϵ, allowing to define $\Theta^z(dz) \doteq \Theta_\epsilon^z(dz)$ and trivially obtain the weak convergence $\Theta_\epsilon^z(dz) \to \Theta^z(dz)$.

As the title suggests, the goal of the current section is to define the following object:

Definition 2.3.1 (**Rooted Probability Measure**)**.** We call *Rooted probability measure* the probability measure on $H^{-1}(D) \times D$ defined as $\Theta(dh, dz) \doteq \Theta(dh|z)\Theta^z(dz)$.

By using the other probability decomposition $\Theta_\epsilon(dhdz) = \Theta_\epsilon(dz|h)\Theta_\epsilon^h(dh)$ it is possible to reach the same final result, since actually the measures $\Theta_\epsilon(dz|h) = \frac{\mu_\epsilon(dz)}{\mu_\epsilon(D)}$ weakly converge to $\frac{\mu(dz)}{\mu(D)}$, and the same behavior happens for $\Theta_\epsilon^h(dh) = Z^{-1}\mu_\epsilon(D)dh \xrightarrow{\epsilon \to 0} Z^{-1}\mu(D)dh$. There is therefore no problem in writing:

$$\Theta(dh, dz) = Z^{-1}\mu_h(dz)dh$$

by using uniqueness of the limit. This result is remarkable, since by comparing the two possible forms for Θ a very nice natural interpretation arises: to sample a GFF h and then extract a point from the determined Liouville Quantum Gravity, is the same of sampling a point according to (essentially) the Lebesgue measure and then extract a shifted GFF.

It is possible to reveal more about the constant Z, as a result of the following propositions.

Proposition 2.3.1 (An uniform integrability result)**.** Let $S \subseteq D$ measurable. Then the random variables $M_\epsilon = \mu_\epsilon(S)$ are uniformly integrable as $\epsilon \to 0$.

Proof. The proof makes use of the measure Θ (in the form here defined), but is rather long, very specific and technical. We prefer to directly refer to [2] (pages 24,25,26). □

Corollary 2.3.1 (**Expectation of Liouville Quantum Gravity**)**.** Let $S \subseteq D$ measurable. Then we have $\mathbb{E}_h[\mu(S)] = \int_S R(z;D)^{\frac{\gamma^2}{2}} \sigma(dz)$

Proof. From section 2.2 we know $\mu_\epsilon(S) \to \mu(S)$ in probability. Since we also have uniform integrability, we deduce convergence in L^1 (this is a standard theorem) and so the claim because $\mathbb{E}[\mu_\epsilon(S)] = \int_S R(z;D)^{\frac{\gamma^2}{2}} \sigma(dz)$ by use Fubini on the equality $\mathbb{E}[e^{\gamma h_\epsilon(z)}] = (\frac{R(z;D)}{\epsilon})^{\frac{\gamma^2}{2}}$ recalled at the beginning of the section 2.3 before. □

And here there is a very nice consequence: the corollary implies that the value of Z, that we recall to be the integral $\int_D R(z;D)^{\frac{\gamma^2}{2}}\sigma(dz)$, is actually *exactly* the gravity expectation $\mathbb{E}_h[\mu_h(D)]$. For avoiding any kind of doubt, the author would like to stress how $\mu(D)$ (the Liouville measure of the whole space) is *not* equal to 1: it is a.s. finite, but we normalize it explicitly only when advantageous.

Summing up, all the results cam be patched together in claiming: the Rooted probability measure is the measure on $H^{-1}\times D$ defined as

$$\Theta(dh,dz)=\mathbb{E}_h^{-1}[\mu_h(D)]\mu(dz)dh$$

having then the following decompositions (deduced with the help of the ϵ-approximation):

z-marg:

$$\Theta^z(dz)=\frac{R(z;D)^{\frac{\gamma^2}{2}}}{\mathbb{E}_h[\mu_h(D)]}dz$$

h-marg:

$$\Theta^h(dh)=\frac{\mu_h(D)}{\mathbb{E}_h[\mu_h(D)]}dh$$

z-cond:

$$\Theta(dh|z)=\frac{e^{\gamma h(z)}}{\int_{H^{-1}}e^{\gamma h(z)}dh}dh$$

h-cond:

$$\Theta(dz|h)=\frac{\mu(dz)}{\mu(D)}$$

In the previous formulas, some terms must be intended to be a formal writing (like the pointwise evaluation $h(z)$), but the work done makes everything completely mathematically meaningful and avoid any kind of ambiguity. Consequently, as a result of their severe construction, these formulas can now be used as the intuition suggests and we can peacefully proceed.

Let A_h be a random subset of D, i.e. a random variable such that a.s. for each $\omega\in\Omega$, $A_{h(\omega)}(\omega)\subseteq D$ is measurable (not important in which way this random variable is distributed, the only things we require is the a.s. mapping into measurable subsets). We will have a concrete example soon, but for the moment we focus on a preliminary useful lemma:

Lemma 2.3.1 (Mean proportionality)**.** $\Theta(\{(h,z):h\in H^{-1},z\in A_h\})=Z^{-1}\mathbb{E}_h[\mu_h(z\in A_h)]$.

Proof. For simplicity we still use the symbol Z and we proceed by using the formulas above: $\Theta(\{(h,z):h\in H^{-1},z\in A_h\})=\int_{H^{-1}}\Theta(\{(h,z):h\in H^{-1},z\in A_h\}|h)\Theta^h(dh)=$
$=\int_{H^{-1}}Z^{-1}\mu_h(A_h)dh=Z^{-1}\mathbb{E}_h[\mu_h(A_h)]$ as claimed. □

Finally, it's time to introduce and prove a statement concerning *thick points*, i.e. points around which circle averages behave in a controlled way. It's here that all the results done reveal a great power.

Definition 2.3.2 (Thick points)**.** Let $\alpha > 0$. A point $z \in D$ is said to be *α-thick* w.r.t. a GFF realization $h = \tilde{h}(\omega)$ iff $\lim_{\epsilon \to 0} \frac{\tilde{h}_\epsilon(z)(\omega)}{-\log \epsilon} = \alpha$.

We define various different subsets like $\mathrm{T}_\alpha(\omega) = \{z \in D : z \text{ is } \alpha\text{-thick w.r.t. } \tilde{h}(\omega)\}$ or $\mathrm{T}_\alpha^h = \{z \in D : z \text{ is } \alpha\text{-thick w.r.t. } h\}$ or also $\mathrm{T}_\alpha^z = \{h \in H^{-1} : z \text{ is } \alpha\text{-thick w.r.t. } h\}$ that will be used according the necessity. Recall $\gamma \in [0, 2)$ to be the constant in the LQG definition, and consider the subset $\{(h, z) : h \in H^{-1}, z \in \mathrm{T}_\gamma^h\} \subseteq H^{-1} \times D$.

Proposition 2.3.2. γ-thick points are Θ-everywhere: $\Theta(\{(h, z) : h \in H^{-1}, z \in \mathrm{T}_\gamma^h\}) = 1$.

Proof. We have that $\Theta(\{(h, z) : h \in H^{-1}, z \in \mathrm{T}_\gamma^h\}) = \Theta(\{(h, z) : z \in D, h \in \mathrm{T}_\gamma^z\}) = \int_D \Theta(\{(h, z) : z \in D, h \in \mathrm{T}_\gamma^z\}|z)\Theta^z(dz) = \int_D \Theta^z(dz) = 1$ since $\Theta(\{(h, z) : z \in D, h \in \mathrm{T}_\gamma^z\}|z) = 1$ thanks to the reasoning explained in the following lines.

Under the probability measure $\Theta(dh|z)$, if h is a realization of our original GFF, its law is updated into $h + \gamma G(z, \cdot)$ and so for concluding the proof, we have then to check the limit for this shifted law. By taking the scalar product, circle averages have now law $h_\epsilon(z) - 2\pi\gamma(G(z, \cdot), \Delta^{-1}\rho_\epsilon^z)_\nabla$, but by using exactly the same computation that we have already done in section 1.8, when we studied circle averages and $Var[h_\epsilon(z)]$, we know that the second part of the formula equals $\gamma(-\log \epsilon + R(z; D))$. In other words, under this probability measure our limit becomes

$$\lim_{\epsilon \to 0} \frac{h_\epsilon(z) - \gamma \log \epsilon + \gamma R(z; D)}{-\log \epsilon} = \lim_{\epsilon \to 0} \frac{h_\epsilon(z)}{-\log \epsilon} + \gamma = \lim_{t \to \infty} \frac{h_{e^{-t}}(z)}{t} + \gamma = \gamma$$

recalling that after the time change $\epsilon = e^{-t}$ circle averages behaves like a Brownian Motion (section 1.9). □

This results implies that LQG is actually supported by thick points.

Proposition 2.3.3 (LQG lives on think points)**.** γ-Liouville Quantum Gravity is supported by the set of γ-thick points, i.e. $\mu_{\tilde{h}(\omega)}(\mathrm{T}_\gamma(\omega)) = \mu_{\tilde{h}(\omega)}(D)$ a.s. for every $\omega \in \Omega$.

Proof. From lemma 2.3.1 and the theorem above, we obtain $1 = Z^{-1}\mathbb{E}_h[\mu_h(T_\gamma^h)]$ which implies $Z = \mathbb{E}_h[\mu_h(T_\gamma^h)]$. By using the explicit form of Z, we have $\mathbb{E}_h[\mu_h(D)] = \mathbb{E}_h[\mu_h(T_\gamma^h)]$ and so the claim by using positivity of both integrands. □

Proposition 2.3.4 (The set of think points is a fractal)**.** We have $dim(\mathrm{T}_\gamma) = (2 - \frac{\gamma^2}{2})$ a.s., where the dimension is intended to be Hausdorff. Furthermore, the set T_γ is empty for $\gamma > 2$.

Proof. This is a long work cited in [3].

In other words, we discovered how Liouville Quantum Gravity lives almost surely on a fractal-shaped space [1]. This surely happens for every $\gamma \in [0, 2)$, coherently with our construction. It is natural to ask what about other setting: the case $\gamma = 2$ is considered to be the *critical* one, while for higher values LQG is believed to collapse to the zero measure because there are no more thick points that can support the gravity.

2.4 Approximation of μ as limit of μ^N

As usual, we assume to have μ a Liouville Quantum Gravity on a standard[2] subset $D \subseteq \mathbb{C}$. The goal of this section is to prove how μ can be approximated with a new family of measures μ^N that we are going to introduce. This fact is remarkable because, while the convergence of μ_ϵ is useful mainly for proving the LQG existence, this alternative approach will help for understanding other properties, for instance how to deal with conformal changes of coordinates. Recall that we defined the measure $\sigma(dz) = R(z; D)^{\frac{\gamma^2}{2}} dz$, and that if X is a centered Gaussian, then we have $\mathbb{E}[e^{\gamma X - \frac{\gamma^2}{2} Var[X]}] = 1$ by an immediate application of what studied in section 1.3.

Definition 2.4.1 (the family $\{\mu^N\}_{N \in \mathbb{N}}$)**.** For each $N \in \mathbb{N}$, we define the random measure $\mu^N(dz)$ as:

$$\mu^N(S) \doteq \int_S \exp(\gamma h^N(z) - \frac{\gamma^2}{2} \mathrm{Var} h^N(z)) \sigma(dz)$$

for each Borel measurable set $S \subseteq D$. Due to positivity the integral always exists, and since D is bounded normalization into a probability measure is guaranteed.

Theorem 2.4.1 (Convergence $\mu^N \to \mu$)**.** The measure μ is the almost surely weak limit of μ^N, i.e. for each $S \subseteq D$ measurable, a.s. for each $\omega \in \Omega$, $\mu^N(S)(\omega) \to \mu(S)(\omega)$ as $N \to \infty$.

Proof. The proof of this theorem is subdivided into many steps.

Step0 : Definition of μ^*

Let $S \subseteq D$ a Borel measurable subset. Recall by definition that $h^N(z) = \sum_{i=1}^N \alpha_i f_i(z)$ where α_i are i.i.d $\sim N(0, 1)$ and $\{f_i\}_{i \in \mathbb{N}}$ is a basis for $H(D)$. Define the filtration $\mathcal{F}_N \doteq \sigma(\alpha_N)$. Then h^N is a martingale w.r.t $\mathcal{F}_N$, and the same holds for $e^{\gamma h^N - \frac{\gamma^2}{2} \mathrm{Var} h^N}$. The latter is usually called the *exponential martingale*, and the required properties are clear via short routine computations. If we now consider the integral $\mu^N(S) = \int_S e^{\gamma h^N(z) - \frac{\gamma^2}{2} \mathrm{Var} h^N(z)} \sigma(dz)$, it continues to be a martingale thanks to Fubini's theorem. Consequently since the family $\mu^N(S)$ is non-negative, it must admit an almost surely limit $N \to \infty$ as stated by the usual Martingale Convergence Theorem. We call this limit $\mu^*(S)$. Observe that, in other words what remains to prove is just the almost surely equality $\mu^*(S) = \mu(S)$.

[1] in the literature the word "fractal" seems to be referred to spaces with non-integer Hausdorff measure, differently from other formal definitions when further properties are usually required.

[2] more precisely, to be open, bounded and with smooth boundary is enough.

Step1 : $\mathrm{Var}h_\epsilon(z) = \mathrm{Var}h'_\epsilon(z) + \mathrm{Var}h^N_\epsilon(z)$

We define $h'_\epsilon(z) = h_\epsilon(z) - h^N_\epsilon(z)$. Intuitively speaking, $h'_\epsilon(z)$ is the usual series starting from the index $N+1$ rather than 0. We need to prove it well: let's define $h^{N<m\leq k}_\epsilon(z) \doteq \sum_{m>N}^k \alpha_i(f_i, \rho^z_\epsilon)$ for k enough large. Recall that we have the limit $h^k_\epsilon(z) \xrightarrow{k\to\infty} h_\epsilon(z)$ both a.s. *and* in $L^2(\mathbb{P})$ (it's the Martingale Convergence Theorem for the L^2 case). By rewriting $h^k_\epsilon(z) = h^N_\epsilon(z) + h^{N<m\leq k}_\epsilon(z)$ and taking the limit, we formally obtain $h'_\epsilon(z) = h_\epsilon(z) - h^N_\epsilon(z) = \lim_{k\to\infty} h^{N<m\leq k}_\epsilon(z)$ as intuitively expected. We are interested in knowing more about the variance. Starting from $h_\epsilon(z) = h^N_\epsilon(z) + h'_\epsilon(z)$ we have: $Var[h_\epsilon(z)] = Var[h^N_\epsilon(z) + h'_\epsilon(z)] = Var[h^N_\epsilon(z) + \lim_{k\to\infty} h^{N<m\leq k}_\epsilon(z)]$. Since the previous convergence happens also in L^2, and since we sum over a finite number of independent Gaussians (and so the variance "splits"), it's equal to $\lim_{k\to\infty}[Varh^N_\epsilon(z) + Varh^{N<m\leq k}_\epsilon(z)]$. Finally, this term is the same of $Var[h^N_\epsilon(z)] + \lim_{k\to\infty} Var[h^{N<m\leq k}_\epsilon(z)] = Var[h^N_\epsilon(z)] + Var[h'_\epsilon(z)]$ again by using the convergence in $L^2(\mathbb{P})$, concluding Step1.

Step2 : More information about $\mathbb{E}[\mu_\epsilon(S)|\mathcal{F}_N]$

By definition:

$$\mu_\epsilon(S) = \int_S e^{\gamma h_\epsilon(z) - \frac{\gamma^2}{2}\mathrm{Var}h_\epsilon(z)}\sigma(dz) = \int_S e^{\gamma h^N_\epsilon(z) - \frac{\gamma^2}{2}\mathrm{Var}h^N_\epsilon(z)} e^{\gamma h'_\epsilon - \frac{\gamma^2}{2}\mathrm{Var}h'_\epsilon(z)}\sigma(dz)$$

where the equality is justified by the splitting proved in the section before. Take now the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_N]$ on both sides. By using: Fubini (surely working due to the integrands positivity), the independence of h' w.r.t $\mathcal{F}_N$ and the measurability of h^N w.r.t. $\mathcal{F}_N$ (both immediate by construction) we obtain:

$$\mathbb{E}[\mu_\epsilon(S)|\mathcal{F}_N] = \int_S e^{\gamma h^N_\epsilon(z) - \frac{\gamma^2}{2}\mathrm{Var}h^N_\epsilon(z)} \mathbb{E}[e^{\gamma h'_\epsilon - \frac{\gamma^2}{2}\mathrm{Var}h'_\epsilon(z)}]\sigma(dz)$$

and the term under the expectation is equal to 1 thanks to the observation before definition 2.4.1. We can finally conclude this section with the equality:

$$\mathbb{E}[\mu_\epsilon(S)|\mathcal{F}_N] = \int_S e^{\gamma h^N_\epsilon(z) - \frac{\gamma^2}{2}\mathrm{Var}h^N_\epsilon(z)}\sigma(dz)$$

Step3 : Taking the limit for $\epsilon \to 0$.

We already know (proved at the end of the previous chapter), that $h^N_\epsilon(z) \to h^N(z)$ a.s. as $\epsilon \to 0$ (note that the limit is w.r.t ϵ, not N). We check now that such a convergence happens also in $L^2(\Omega)$. Indeed $|h^N_\epsilon(z)| = |\sum_{i=1}^N \alpha_i \int f_i(y)\rho^z_\epsilon(dy)| \leq K\sum_{i=1}^N |\alpha_i|$, where since each $f_i \in C^\infty_c(D)$ we can bound $|f_i| \leq K_i$ and choose $K \doteq \max K_i$. Finally, since $K\sum_{i=1}^N |\alpha_i|$ is in $L^2(\Omega)$ we conclude by using the dominated convergence theorem. In particular, what we deduce is that $\mathrm{Var}h^N_\epsilon(z) \to \mathrm{Var}h^N(z)$ as $\epsilon \to 0$, very important for concluding that:

$$e^{\gamma h^N_\epsilon(z) - \frac{\gamma^2}{2}\mathrm{Var}h^N_\epsilon(z)} \to e^{\gamma h^N(z) - \frac{\gamma^2}{2}\mathrm{Var}h^N(z)}$$

as $\epsilon \to 0$.

Step 4 : The convergence holds also in $L^1(S, \sigma(dz))$

In this section we show how the convergence in the previous line happens also in $L^1(S, \sigma(dz))$. Set $\phi_\epsilon \doteq e^{\gamma h_\epsilon^N(z) - \frac{\gamma^2}{2}\mathrm{Var} h_\epsilon^N(z)}$, and $\phi \doteq e^{\gamma h^N(z) - \frac{\gamma^2}{2}\mathrm{Var} h^N(z)}$. For this purpose we define $\mu_\epsilon^N(S) = \int_S \phi(z)\sigma(dz)$ (completely coherent with the notation before), consequently what we claim is that the family $\mu_\epsilon^N(S)$ a.s. converges to $\mu^N(S)$. But we already know a more general property, i.e. that $\mu_\epsilon(S) \to \mu(S)$ and so by adapting the previous reasoning (but truncating the series rather than considering the limit) we can obtain the needed property.

Step 5 : Use Fatou's lemma in our two different limits.

All the steps done lead a very remarkable property: $\lim_{\epsilon\to 0} \mathbb{E}[\mu_\epsilon(S)|\mathcal{F}_N] = \mu^N(S)$ (compare 2+4). By using Fatou on this approximation we see:

$$\lim_{\epsilon\to 0} \mathbb{E}[\mu_\epsilon(S)|\mathcal{F}_N] \geq \mathbb{E}[\lim_{\epsilon\to 0} \mu_\epsilon(S)|\mathcal{F}_N]$$

implying

$$\mu^N(S) \geq \mathbb{E}[\mu(S)|\mathcal{F}_N]$$

Taking now the limit $N \to \infty$ to both sides and by using $\mathcal{F}_\infty$ measurability of $\mu(S)$ (clear by construction), we deduce a first key inequality: $\mu^*(S) \geq \mu(S)$. Let's prove the other way around. The idea is to use Fatou again, but rather on the other (almost surely) limit $\mu^N(S) \to \mu^*(S)$. By definition

$$\mathbb{E}[\mu^*(S)] = \mathbb{E}[\lim_{N\to\infty} \mu^N(S)] \leq \lim_{N\to\infty} \mathbb{E}[\mu^N(S)] = \int_S R(z, D)^{\frac{\gamma^2}{2}} \sigma(dz)$$

A very crucial point is that he last quantity is *exactly the same* as $\mathbb{E}[\mu(S)]$ as shown in the section 2.3 and so, since $\mu^*(S) - \mu(S) \geq 0$, the only possibility is to have $\mu^*(S) = \mu(S)$ a.s.

A concrete useful consequence of this theorem is explained in the following corollary.

Corollary 2.4.1 (Conditional expectation of LQG)**.** For each $n \in \mathbb{N}$, $A \subseteq D$ measurable, we have $\mathbb{E}[\mu(A)|h^N] = \mu^N(A)$.

Proof. Since in the previous proof we checked $\mu^* = \mu$, it suffices to verify $\mathbb{E}[\mu^*(A)|h^N] = \mu^N(A)$. But the property $\mathbb{E}[\mu^*(A)|\mathcal{F}_N] = \mu^N(A)$ is a standard well-known fact in martingale theory, and since $\mathcal{F}_N = \sigma(h^N)$ the claim follows.

This corollary can be used for computing conditional expectation w.r.t. other random variables. Recall that we have $h^N = \sum_{i=1}^N \alpha_i f_i = \sum_{i=1}^N (h, f_i)_\nabla f_i$ by construction of h and continuity (see section 1.4). Let Y be a random variable and suppose we are interested in the random quantity $\mathbb{E}[\mu(A)|Y]$. This problem can be solved by finding (when possible) a $g_1 \in H(D)$ s.t. $Y = (h, g_1)_\nabla g_1$ and (choosing $N = 1$ and assuming WLOG g_1 to be normalized) using the just studied corollary: $\mathbb{E}[\mu(A)|Y] = \mathbb{E}[\mu(A)|(h, g_1)_\nabla g_1] = \mathbb{E}[\mu(A)|h^1] = \mu^1(A)$, where now $\mu^1(A) = \int_A e^{\gamma Y(z) - \frac{\gamma^2}{2} VarY(z)} \sigma(dz)$ is a quantity that can be explicitly computed.

2.5 Random surfaces and conformal covariance

In this section we study how LQG behaves w.r.t conformal change of coordinates, allowing finally a formal definition of *random surface*. Recall that we call a subset $D \subseteq \mathbb{C}$ to be a **standard set** if bounded, open, simply connected and with smooth boundary. We have already worked and introduced this notion (section 1.1), but we recall it here for sake of clarification[3].

Proposition 2.5.1 (conformal **co**variance of LQG)**.** Let $\tilde{D}, D \subseteq \mathbb{C}$ two standard subsets of the plane. Let h be a GFF on D, $\psi : \tilde{D} \to D$ a conformal map. Let $\tilde{h}$ be the (random) distribution on $\tilde{D}$ given by $\tilde{h} = h \circ \psi + Qlog|\psi'|$, where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$. Then, $\forall \tilde{A} \subseteq \tilde{D}$ measurable, $\mu_{\tilde{h}}(\tilde{A}) = \mu_h(\psi(\tilde{A}))$

Remark 2.5.1. Note how it is generally *not* true "$\mu_{h\circ\psi}(\tilde{A}) = \mu_h(\psi(\tilde{A}))$", a property that would have been called conformal **in**variance.

Remark 2.5.2. We know well that given a GFF, it is possible to construct a LQG. But what happens if the GFF is perturbed by a deterministic function, as in the case of $\tilde{h}$? There are definitely no problems at all, and the random measure $\mu_{\tilde{h}}$ is defined as the intuition suggests. Indeed, all the computation and properties studied before hold exactly in the same way.

Proof. Thanks to the previous section we know:

$$\mu_h(\psi(\tilde{A})) = \lim_{N\to\infty} \mu_h^N(\psi(\tilde{A}))$$

where:

$$\mu_h^N(\psi(\tilde{A})) = \int_{\psi(\tilde{A})} e^{\gamma h^N(z) - \frac{\gamma^2}{2}\mathrm{Var} h^N(z)} R(z;D)^{\frac{\gamma^2}{2}} dz$$

(for this proof it is better to write the measure $\sigma(dz) = R(z;D)^{\frac{\gamma^2}{2}} dz$ explicitly). Consider now the change of variables $\tilde{z} = \psi^{-1}(z)$. By using properties of the conformal radius (section 1.1) we see that the term $R(z;D)^{\frac{\gamma^2}{2}} dz$ becomes:

$$R(\tilde{z};\tilde{D})^{\frac{\gamma^2}{2}} |\gamma'(\tilde{z})|^{\frac{\gamma^2}{2}} |\gamma'(\tilde{z})|^2 d\tilde{z} = R(\tilde{z};\tilde{D})^{\frac{\gamma^2}{2}} |\gamma'(\tilde{z})|^{\gamma Q}$$

On the other hand, the exponential term is replaced by:

$$e^{\gamma h^N(\psi(\tilde{z})) - \frac{\gamma^2}{2}\mathrm{Var} h^N(\psi(\tilde{z}))} = e^{\gamma (h\circ\psi)^N(\tilde{z}) - \frac{\gamma^2}{2}\mathrm{Var}(h\circ\psi)^N(\tilde{z})}$$

by using the conformal change of variables for Gaussian Free Fields (section 1.5). Furthermore:

$$\mathrm{Var}(h\circ\psi)^N(\tilde{z}) = \mathrm{Var}((h\circ\psi)^N(\tilde{z}) + Qlog|\psi'(\tilde{z})|) = \mathrm{Var}\tilde{h}^N(\tilde{z})$$

[3] technically, not all the hypothesis are **truly** needed, but they help to avoid too long discussions.

where the former equality holds since the variance of a random variable does not change by adding a deterministic function, while the latter follows directly from the definition of $\tilde{h}$. By patching these results all together we obtain:

$$\mu_h^N(\psi(\tilde{A})) = \int_{\tilde{A}} e^{\gamma[(h\circ\psi+Qlog|\psi'|)(\tilde{z})]-\frac{\gamma^2}{2}\text{Var}[(h\circ\psi+Qlog|\psi'|)^N(\tilde{z})]} R(\tilde{z};\tilde{D})^{\frac{\gamma^2}{2}} d\tilde{z}$$

But the last term, directly by definition of $\tilde{h}$, is actually the same of:

$$\int_A e^{\gamma\tilde{h}^N(\tilde{z})+\frac{\gamma^2}{2}\text{Var}\tilde{h}^N(\tilde{z})} R(\tilde{z},\tilde{D})^{\frac{\gamma^2}{2}} d\tilde{z} = \mu_{\tilde{h}}^N(\tilde{A})$$

In other words, we have:

$$\mu_h^N(\psi(\tilde{A})) = \mu_{\tilde{h}}^N(\tilde{A})$$

for each $N \in \mathbb{N}$ and so the claim follows by taking the limit $N \to \infty$. □

We consider now all the pairs (D, h) where $D \subseteq \mathbb{C}$ is standard and h is a GFF on D. On this set, we consider the relation: $(D_1, h_1) \sim (D_2, h_2)$ iff $\exists f : D_1 \to D_2$ conformal such that $h_1 = h_2 \circ f + Qlog|f'|$.

Proposition 2.5.2. The previous is actually an equivalence relation.

Proof. For *reflexivity*, it's enough to choose $f = Id$. Concerning *symmetry*, it can be checked by using f^{-1}. Indeed, if $h_1 = h_2 \circ f + Qlog|f'|$, then $h_1 \circ f^{-1} = h_2 - Qlog|(f^{-1})'|$ by using the derivative of the inverse function. Finally, *transitivity* is proved, as the intuition suggests, via the usual composition. Indeed, if $(D_1, h_1) \sim (D_2, h_2)$ via f, and $(D_2, h_2) \sim (D_3, h_3)$ via g, then $h_1 = h_2 \circ f + Qlog|f'|$ and $h_2 = h_3 \circ g + Qlog|g'|$. Consequently $h_1 = h_3 \circ g \circ f + Qlog|g'(f)| + Qlog|f'| = h_3 \circ g \circ f + Qlog|(g \circ f)'|$. □

Definition 2.5.1 (Random Surface)**.** A couple (D, f) considered up to the previous equivalence class is called a *random surface*.

Directly by construction and by using the conformal covariance just proved, it is clear that on every random surface the notion of Liouville Quantum Gravity is well-defined. According to [3], very interesting examples arise by "zooming" near appropriate sampled points, as in the case of *quantum cones* or *quantum wedges*, actually not included in this master's thesis.

The work done here concludes the second chapter, where the author hopes to have given a good overview of Liouville Quantum Gravity and related concepts like thick points, conformal covariance and the notion of random surface. Let X be a subset of D. A very natural question that arises is the following one: what is the "difference" between X seen from the euclidean metric, and X from the LQG-viewpoint? How does X change under the influence of gravity? In the next chapter we are going to investigate a *geometric* property concerning this problem and offer an answer known as "KPZ relation".

Chapter 3

KPZ relation

3.1 Expected quantum areas

In order to reach the final KPZ relation, we would like to increase our knowledge about "quantum balls" (the definition will be given later), but in particular to know more about expected quantum areas of euclidean balls will be of great help.

Let's fix $z \in D$ and $0 < \epsilon < \epsilon_0$ s.t. $B_{\epsilon_0}(z) \subseteq D$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. We want to efficiently compute $\mathbb{E}[\mu_h(B_\epsilon(z))|h_\epsilon(z) - h_{\epsilon_0}(z)]$ and we try to reach this goal by using the strategy explained at the end of section 2.4. Therefore we need a good function as g_1, and for this purpose define/recall $\xi^z_\epsilon(y) \doteq -2\pi\Delta^{-1}\rho^z_\epsilon(y)$. By doing explicit computations *exactly* in the same style of the variance estimation, section 1.9 we obtain $\|\xi^z_\epsilon\|^2_\nabla = (\xi^z_\epsilon, \xi^z_\epsilon)_\nabla = \xi^z_\epsilon(z)$ and $(\xi^z_\epsilon, \xi^z_{\epsilon_0})_\nabla = \xi^z_{\epsilon_o}(z)$ from which one can deduces $\left\|\xi^z_\epsilon - \xi^z_{\epsilon_0}\right\|^2 = -\log(\epsilon/\epsilon_0)$. The interest on this arises from the choice $g_1(y) \doteq \frac{\xi^z_\epsilon(y) - \xi^z_{\epsilon_0}(y)}{\left\|\xi^z_\epsilon - \xi^z_{\epsilon_0}\right\|^2}$, which implies:

$$h^1(y) = (h, g_1)_\nabla g_1(y) = (h_\epsilon(z) - h_{\epsilon_0}(z))\frac{(\xi^z_\epsilon - \xi^z_{\epsilon_0})(y)}{-\log \epsilon/\epsilon_0}$$

and

$$Var h^1(y) = \frac{(\xi^z_\epsilon - \xi^z_{\epsilon_0})^2(y)}{-\log(\epsilon/\epsilon_0)}$$

from the fact that $Var[h_\epsilon(z) - h_{\epsilon_0}(z)] = -\log(\epsilon/\epsilon_0)$.

Since the final measured set is $B_\epsilon(z)$, we can safely assume to work with points y inside such a ball, so both the formulas strongly simplify into $h^1(y) = h_\epsilon(z) - h_{\epsilon_0}(z)$ and $Var h^1(y) = -\log(\epsilon/\epsilon_0)$ thanks to spatial properties of $\xi^z_\epsilon(y) - \xi^z_{\epsilon_0}(y)$ for the case $y \in B_\epsilon(z)$. Indeed, we know that $\xi^z_\epsilon(y) - \xi^z_{\epsilon_0}(y) = ...$ exactly 0 when $\epsilon_0 \leq |y - z|$, the quantity $-\log \frac{|y-z|}{\epsilon_0}$ for $\epsilon \leq |y - z| \leq \epsilon_0$ and finally $-\log \epsilon/\epsilon_0$ for the case $0 \leq |y - z| \leq \epsilon$ (as seen, it's essentially the mean value theorem for harmonic functions). By computing $\mu^1(dy)$ as described in section 2.5, for our case $y \in B_\epsilon(z)$ we obtain

$$\mu^1(dy) = \mu_0(dy)(\frac{\epsilon}{\epsilon_0})^{\frac{\gamma^2}{2}} e^{\gamma(h_\epsilon(z) - h_{\epsilon_0}(z))}$$

where $\mu_0(dy) \doteq [R(y; D)]^{\frac{\gamma^2}{2}} dy$ (not to be confused with μ^0, symbol that we'll usually use for the Lebesgue measure). In order to make the notation still more compact we introduce $[R_\epsilon(z; D)]^{\frac{\gamma^2}{2}} \doteq \frac{\mu^0(B_\epsilon(z))}{\mu_0(B_\epsilon(z))} \doteq \frac{1}{\pi\epsilon^2} \int_{B_\epsilon(z)} [R(y; D)]^{\frac{\gamma^2}{2}dy} dy$.

In such a way (mean value theorem) $\lim_{\epsilon \to 0} R_\epsilon(z; D) = R(z; D)$. In other words we proved the following proposition:

Proposition 3.1.1 (Expected quantum area given circle averages differences)**.** Let $z \in D$, $0 < \epsilon < \epsilon_0$ s.t. $B_{\epsilon_0}(z) \subseteq D$. Then

$$\mathbb{E}[\mu(B_\epsilon(z))|h_\epsilon(z) - h_{\epsilon_0}(z)] = \mu^1(B_\epsilon(z))$$

where $\mu^1(B_\epsilon(z)) = \pi\epsilon^{\gamma Q}(\frac{R_\epsilon(z;D)}{\epsilon_0})^{\frac{\gamma^2}{2}} exp(\gamma(h_\epsilon(z) - h_{\epsilon_0}(z))$ and, as always, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

We continue with a very similar reasoning as before, but on the quantity $\mathbb{E}[\mu_h(B_\epsilon(z))|h_\epsilon(z)]$ instead. This variable has actually a more important role: in principle we could have started directly from this case, but the example before strongly helps in understanding the underlying technique.

This time we select $\tilde{g}_1(y) \doteq \frac{\xi^z_\epsilon(y)}{\|\xi^z_\epsilon\|_\nabla}$ and since $\|\xi^z_\epsilon\|^2_\nabla = (\xi^z_\epsilon, \xi^z_\epsilon)_\nabla = \xi^z_\epsilon(z)$ we obtain $\tilde{h}^1(y) = h_\epsilon(z)\frac{\xi^z_\epsilon(y)}{\xi^z_\epsilon(z)}$. By using the already studied equality $Var h_\epsilon(z) = \xi^z_\epsilon(z) = -\log(\epsilon/R(z; D))$, we deduce $Var\tilde{h}^1(y) = Var h_\epsilon(z)(\frac{\xi^z_\epsilon(y)}{\xi^z_\epsilon(z)})^2 = \frac{(\xi^z_\epsilon(y))^2}{\xi^z_\epsilon(z)}$.

The next step is to compute the quantity $\tilde{\mu}^1(dy) = exp(\gamma\tilde{h}^1(y) - \frac{\gamma^2}{2}Var\tilde{h}^1(y) + \frac{\gamma^2}{2}log R(y; D))dy$. Differently from before, a good approximation is much longer (but not deep) to achieve, and for sake of simplicity we prefer to leave the computation directly to the reference [2] in favor of going directly to the point. This is: $\tilde{\mu}^1(B_\epsilon(z)) \simeq \mu_\odot(B_\epsilon(z))$ where $\simeq$ means that the limit $\epsilon \to 0$ of their ratio tends to 1. The object $\mu_\odot$ (which is not required to be a measure) is defined as $\mu_\odot(B_\epsilon(z)) \doteq \pi\epsilon^{\gamma Q}e^{\gamma h_\epsilon(z)}$. In other words, the following claim holds:

Proposition 3.1.2 (Expected quantum area given circle averages)**.** Let $z \in D$, $0 < \epsilon$ s.t. $B_\epsilon(z) \subseteq D$. Then

$$\mathbb{E}[\mu(B_\epsilon(z))|h_\epsilon(z)] \simeq \pi\epsilon^{\gamma Q}e^{\gamma h_\epsilon(z)}$$

where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

Note that the quantity $\mu_\odot$ can also be used for reformulating the approximation appearing in the first case: $\mu_1(B_\epsilon(z)) = \pi\epsilon_0^2 R_\epsilon(z; D)^{\frac{\gamma^2}{2}} \frac{\mu_\odot(B_\epsilon(z))}{\mu_\odot(B_{\epsilon_0}(z))}$.

Moreover if we define (for $\epsilon \leq \epsilon_0$):

$$t \doteq -\log(\epsilon/\epsilon_0)$$

and

$$V_t \doteq h_\epsilon(z) - h_{\epsilon_0}(z)$$

we can rewrite both the area expectations as

$$\mathbb{E}[\mu(B_\epsilon(z))|h_\epsilon(z) - h_{\epsilon_0}(z)] = \mu^1(B_\epsilon(z)) = \pi\epsilon_0^2 R_\epsilon(z; D)e^{\gamma V_t - \gamma Q t}$$

and

$$\mathbb{E}[\mu(B_\epsilon(z))|h_\epsilon(z)] \simeq \mu_\odot(B_\epsilon(z)) = \mu_\odot(B_{\epsilon_0}(z))e^{\gamma V_t - \gamma Q t}$$

In other words, see how both the expressions are product of two parts. The first has not any deep consequence, but the key lies on the fact that the second factors are exactly the same, and they are exponential of drifted Brownian motions *independent of the point* z. This independence will play an important role soon (in the next section, it is beyond the shift from $\Theta(dh|z)$ to Θ). In few words, the moral of this section is: *expected quantum areas w.r.t. circle averages are governed by Brownian motions.*

3.2 A weak KPZ formula

This section starts with a new definition motivated by the construction done before. As sometimes implicitly done before, if there is a quantity depending on many random variables $A, B, ...$, we put subscripts on $\mathbb{E}$, like $\mathbb{E}_A$ for stressing w.r.t. which variable the expectation is taken. As usual, the symbol μ^0 refers to Lebesgue measure, i.e. Liouville Quantum Gravity when $\gamma = 0$.

Definition 3.2.1 (Tilde-balls)**.** We indicate with $\tilde{B}^\delta(z)$ the random Euclidean ball centered in z with radius e^{-T_A}, where $A \doteq -\frac{log\delta}{\gamma}$ and $T_A \doteq inf\{t : -V_t + Qt = A\}$. In other words, this is the largest z-centered euclidean ball contained in D for which $e^{\gamma V_t - \gamma Q t} = \delta$.

Since the following will frequently appear it is convenient to define:

Definition 3.2.2 (KPZ relation for reals)**.** Let α, β non-negative real numbers. We say that they satisfy the KPZ relation iff $\alpha = \frac{\gamma^2}{4}\beta^2 + (1 - \frac{\gamma^2}{4})\beta$. In such a case, we just write $KPZ(\alpha, \beta)$. The quantity γ is intended to be the coefficient in $[0, 2)$ appearing in the LQG definition.

The just introduced Tilde balls are interesting since they allow to derive a "weak" version of the KPZ formula:

Theorem 3.2.1 (weak KPZ formula). Let X be a (possibly random) measurable subset of D. Fix $\gamma \in [0, 2)$, and let μ be a LQG on D. Suppose X and μ to be independent. Then if we have:

$$\lim_{\epsilon \to 0} \frac{\log \mathbb{E}_X[\mu^0\{z \in D : B_\epsilon(z) \cap X \neq \emptyset\}]}{\log \epsilon^2} = x$$

then it follows that:

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}_{X,h}[\mu_h\{z \in D : \tilde{B}^\delta(z) \cap X \neq \emptyset\}]}{\log \delta} = \Delta$$

where $KPZ(x, \Delta)$ is satisfied.

Proof. The core quantity in the theorem is surely $\mathbb{E}_h[\mu_h\{z \in D : \tilde{B}^\delta(z) \cap X \neq \emptyset\}]$. Proposition 2.3.1 suggests the proportionality $\mathbb{E}_h[\mu_h\{z \in D : \tilde{B}^\delta(z) \cap X \neq \emptyset\}] = Z\Theta\{(z,h) : \tilde{B}^\delta(z) \cap X \neq \emptyset\}$, which means that by changing our working measure to Θ, we can replace an expectation computation with a probability one.

Consider for the moment the conditioned measure $\Theta(dh|z)$. We already know that under such a measure, original circle averages have now the law of $h_\epsilon(z) - \gamma(\log \epsilon - R(z; D))$ consequently the process $V_t = h_\epsilon(z) - h_{\epsilon_0}(z)$ is now in law equivalent to $h_\epsilon(z) - h_{\epsilon_0}(z) - \gamma(\log(\frac{\epsilon}{\epsilon_0})) = V_t + \gamma t$ since t was originally defined as $t = -\log(\frac{\epsilon}{\epsilon_0})$. All these computations mean that under the probability measure $\Theta(dh|z)$ the law T_A becomes the same as $T_A = inf\{t : B_t + at = A\}$ where B_t is a standard Brownian motion in $\Theta(dh|z)$ and $a = Q - \gamma > 0$. But since it happens independently on $z \in D$, the law of T_A is preserved if we work under the full unconditioned measure $\Theta(dhdz)$, and so we do.

Going further, by replacing $\epsilon = e^{T_A}$ in the assumed euclidean limit, we obtain

$$\lim_{T_A \to \infty} \frac{\log \mathbb{E}_X[\mu^0\{z \in D : \tilde{B}^\delta(z) \cap X \neq \emptyset\}]}{-2T_A} = x$$

so we deduce $\mathbb{E}_X[\mu^0\{z \in D : \tilde{B}^\delta(z) \cap X \neq \emptyset\}] \simeq exp(-2xT_A)$, in the sense that the ratio of the log of these quantities tends to 1 as $T_A \to \infty$.

Define now the quantity $q_A \doteq \Theta\{(h,z) : B_{e^{-T_A}}(z) = \tilde{B}^\delta(z) \cap X \neq \emptyset\}$, representing the probability that tilde-balls hit the set X. But since these sets are governed by a Brownian motion, whose related hitting time is in the definition of T_A, it is possible to approximate $\mathbb{E}_X[q_A] \simeq \mathbb{E}[exp(-2xT_A)]$, and actually this is the only step in which not all details are specified; the author apologizes for this, and in case of big issues suggests to relay on the main source [2].

The next move is to alternatively compute $\mathbb{E}[exp(-2xT_A)]$ by using a classic stopping time argument. Choose β as positive solution to $2x = \beta a + \frac{\beta^2}{2}$. For each $t < T_A$ we have $B_t + at \leq A$ and so the exponential martingale $exp(\beta B_t - \beta^2 \frac{t}{2})$ is bounded by a constant since the same happens for its exponent $\beta B_t - \beta^2 \frac{t}{2} \leq \beta A - (\beta a + \frac{\beta^2}{2})t \leq \beta A$. Since δ can be chosen small enough to have $A > 0$, and since is the drift $a > 0$, the hitting time is almost surely finite; so we use optimal sampling obtaining $1 = \mathbb{E}[exp(\beta B_{T_A} - \beta^2 \frac{T_A}{2})] = \mathbb{E}[exp(-T_A(a\beta + \frac{\beta^2}{2}) + A\beta)]$ since $B_{T_A} + T_A a = A$, but the latest equality implies, by recalling the relation between x and β, $\mathbb{E}[exp(-2T_A x)] = exp(-A\beta)$. On the other hand $A = -\frac{\log \delta}{\gamma}$ and so $exp(-A\beta) = \delta^{\frac{\beta}{\gamma}}$. Setting now $\Delta = \frac{\beta}{\gamma}$ we deduce: $\mathbb{E}[exp(-2xT_A)] = \delta^\Delta$.

The conclusion comes now naturally by direct computation:

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}_{X,h}[\mu_h\{z \in D : \tilde{B}^\delta(z) \cap X \neq \emptyset\}]}{\log \delta} = \lim_{\delta \to 0} \frac{\log Z\mathbb{E}_X \Theta\{(h,z) : \tilde{B}^\delta(z) \cap X \neq \emptyset\}}{\log \delta} =$$

$$= 0 + \lim_{\delta \to 0} \frac{\log \mathbb{E}_X \Theta\{(h,z) : \tilde{B}^\delta(z) \cap X \neq \emptyset\}}{\log \delta} = \lim_{\delta \to 0} \frac{\log \mathbb{E}_X[q_A]}{\log \delta} =$$

$$= \lim_{\delta \to 0} \frac{\log \mathbb{E}[exp(-2xT_A)]}{\log \delta} = \lim_{\delta \to 0} \frac{\log \delta^{\Delta}}{\log \delta} = \Delta$$

.

Finally, the property $KPZ(x, \Delta)$ follows immediately by construction. □

Before concluding this section, we would like to underline a key intuition learned during the proof: very roughly speaking, for sets governed by Brownian motions (Tilde-balls in this case), a KPZ relation seems reasonable to be expected.

3.3 KPZ relation: intuition

In this chapter we comment a final important result known as "KPZ relation", discovered by Knizhnik, Polyakov and Zamolodchikov in the late '80 in the Physics context of conformal field theory. It is very close to the theorem explored in the section before, except that instead of Tilde balls $\tilde{B}^{\delta}(z)$ we use:

Definition 3.3.1 (Quantum balls)**.** Let $z \in D$. We define $B^{\delta}(z)$ the quantum ball of area δ, centered in z, as the euclidean ball $B_{\epsilon}(z)$ with random radius $\epsilon = sup\{k : \mu(B_k(z)) \leq \delta\}$. Observe that for the deterministic (Lebesgue) case $\gamma = 0$, we have $B^{\delta}(z) = B_{\epsilon}(z)$ where $\delta = \pi\epsilon^2$.

Before introducing the main theorem it is nice to underline some geometrical intuition. As always μ^0 indicates the Lebesgue measure. For the whole section, we assume the set $X \subseteq D$ to be possibly random (it is not relevant according to which distribution), always measurable and *independent* of the quantum gravity μ. We say that:

Definition 3.3.2 (Euclidean Scaling Exponent)**.** The set X has *euclidean scaling exponent* x, abbreviated with $ESE(X) = x$ iff :

$$\lim_{\epsilon \to 0} \frac{\log \mathbb{E}_X[\mu^0\{z \in D : B_{\epsilon}(z) \cap X \neq \emptyset\}]}{\log \epsilon^2} = x$$

There is of course the quantum counterpart.

Definition 3.3.3 (Quantum Scaling Exponent)**.** The set $X \subset D$ has *quantum scaling exponent* Δ, abbreviated with $QSE(X) = \Delta$ iff it realizes the same limit in the setting of Liouville:

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}_{X,\mu}[\mu_h\{z \in D : B^{\delta}(z) \cap X \neq \emptyset\}]}{\log \delta} = \Delta$$

Note how in the first limit the denominator refers to the area of an euclidean ball B_{ϵ}, evolving like ϵ^2, while in the second it refers to δ, i.e. by construction the area of quantum balls. Said that, the full KPZ formula can be simply interpreted as a statement like "$ESE(X) = x \implies QSE(X) = \Delta$", where x and Δ are forced to respect the KPZ formula introduced in the section before. It is therefore a relationship between a **geometric** property of in the euclidean setting and a similar one under the influence of gravity.

Theorem 3.3.1 (**Final theorem: KPZ relation for Liouville Quantum Gravity**)**.** Let $D \subseteq \mathbb{C}$ a bounded, open subset with smooth boundary. Let $\gamma \in [0, 2)$, and μ be the Liouville Quantum Gravity associated to γ. If $\tilde{D}$ be a compact subset of D and X is a (possibly random, but independent of μ) subset of D, the following equation holds:

$$ESE(X \cap \tilde{D}) = x \implies QSE(X \cap \tilde{D}) = \Delta, KPZ(x, \Delta)$$

The proof of this theorem is reasonably long, consequently for the moment we briefly give an intuitive idea of the used strategy, and try to fill in more details and explanations in the next section.

Intuition. It is reasonable to follow the same pattern of what described in the section before ("weak KPZ formula"), except for one important step. Indeed, if it is possible to repeat such a proof in almost every detail, one must be careful because the quantum balls $B^\delta(z)$ behave differently from the Brownian ones $\tilde{B}^\delta(z)$. But on the other hand, $B^\delta(z)$ can be approximated by the balls $\tilde{B}^\delta(z)$, making so possible to reduce to the previous case and concluding the proof (the compact set $\tilde{D}$ is required at this approximation stage). Let's try to understand *why* we could expect such an approximation. The key point is in the fact that $\tilde{B}^\delta(z)$ are quantities governed by a Brownian motion. Consequently, if we managed to find a similar property for the quantum balls $B^\delta(z)$ too, it would be at least reasonable to expect that. By definition, $B^\delta(z)$ strongly depends on $\mu(B_\epsilon(z))$; on the other hand (we'll see how) such a quantity is very close to $\mathbb{E}[\mu(B_\epsilon(z))|h_\epsilon(z)]$. But the last entity is approximated by $\mu_\odot(B_\epsilon(z))$ (as studied in section 3.1). Since, as previously pointed out, these are random variables strongly governed by a Brownian motion, the reasoning actually finds a nice logical conclusion. □

3.4 KPZ relation: more proofs

The aim of this section is to give a formal proof of theorem 3.3.1. The author must apologize because not *every* detail is carefully explained (so almost all the proofs have to be considered "partial"), *but* the reached compromise is surely helpful for filling in anyway a lot of skipped steps in the given references and for strengthening many intuitive ideas. We start with some preliminary lemmas:

Lemma 3.4.1 (logLQG admits exponential tail)**.** Let $D = \mathbb{D} = B_1(0)$ be the unit disc and fix $\gamma \in [0, 2)$. Let $\mu = e^{\gamma h(z)} dz$ be the Liouville Quantum Gravity as before. Then the real-valued random variable $A = \log \mu(B_{\frac{1}{2}}(0))$ admits a superexponential decay. More precisely, $P_A(\eta) \doteq \mathbb{P}[A < \eta] < e^{-C\eta^2}$ for some fixed constant C and sufficiently negative values of η.

Proof (partial). The proof is based on a domain subdivision that will allow, at the end, to use a recursive strategy. Let h' be the projection of h onto the space $Harm[B_{\frac{1}{4}}(\frac{1}{4}) \cap B_{\frac{1}{4}}(-\frac{1}{4})]$ of distributions on $\mathbb{D}$ which are harmonic when restricted on the union above. By using the Spatial Markov Property (section 1.6) it is known its complementary to be $H(B_{\frac{1}{4}}(\frac{1}{4}) \cap B_{\frac{1}{4}}(-\frac{1}{4}))$, i.e. the usual Sobolev space but with the two discs rather than "D". Consequently $h - h'$ has the law of a Gaussian Free Field. More precisely, $h - h' = h_+ + h_-$ where the former is a GFF on $B_{1/4}(1/4)$ and the latter on $B_{1/4}(-1/4)$. They are independent by using the Markov property again.

Let's define $B_- \doteq B_{\frac{1}{8}}(-1/4)$, $B_+ \doteq B_{\frac{1}{8}}(1/4)$ and the real-valued random variable $\bar{h} : \omega \mapsto \inf_{\rho \in C_c^\infty(B_-) \cup C_c^\infty(B_+)}(h'(\omega), \rho)$. In other words, $\bar{h}$ is the inf of h' over the union of B_+ and B_- (being h' a random distribution on the full set $\mathbb{D}$, the evaluation $(h'(\omega), \rho)$ is completely meaningful).

Note the existence of conformal functions $\Gamma_+ : B_{\frac{1}{4}}(1/4) \to \mathbb{D}$, $z \mapsto 4z - 1$ and $\Gamma_- : B_{\frac{1}{4}}(-1/4) \to \mathbb{D}$, $z \mapsto 4z + 1$ mapping exactly B_- and B_+ to $B_{1/2}(0)$, respectively (where the last is the quantity in the lemma's description).

Define so $A_- \doteq \log \mu_{h-h'}(B_-)$ and $A_+ \doteq \log \mu_{h-h'}(B_+)$. We would like to establish a relation between the original case "A" and the described quantities. The solution is to use LQG conformal covariance (see section 2.5) with maps Γ_+ and Γ_-, obtaining directly $\mu(B_{1/2}(0)) = \mu_{h-h'}(B_-)e^{\gamma Q \log 4}$ which implies $A = A_- + \gamma Q \log 4$ and $A = A_+ + \gamma Q \log 4$ by taking the logarithm on both sides. The described equations represents the first key point of the proof.

Another important step is given by the upcoming inequality. Since $B_{1/2}(0) \supseteq B_{1/8}(1/4) = B_+$ we have:

$$\mu_h(B_{1/2}(0)) \geq \mu_h(B_+) = \int_{B_+} e^{\gamma h(z)} dz \overset{?}{\geq} e^{\gamma \bar{h}} \int_{B_+} e^{\gamma (h-h')(z)} dz = e^{\gamma \bar{h}} \mu_{h-h'}(B_+)$$

where the step (?) needs a lot of attention. It would be spontaneous just to write $e^{\gamma h(z)} = e^{\gamma (h-h')(z)} e^{h'(z)}$ and then use the fact that $\bar{h} \leq h'$, but the object "$h'(z)$" doesn't actually exist! Recall that h' is only a random distribution (a generic one, not necessarily a GFF) and so there is no way to evaluate it punctually. This problem doesn't hold for $(h - h')(z)$, since (as seen) it has the law of a Gaussian Free Field.

To imitate the circle average technique helps in solving this issue. Let ρ_ϵ^z be the normalized mass on $\partial B_\epsilon(z)$. As well as Gaussians converges in distribution to Dirac deltas, we can choose a family $\{\psi_n\}_{n \in \mathbb{N}}$ of test functions converging to ρ_ϵ^z (think about "bumps around the circle"). It's consequently completely legitimate to write $(h, \psi_n) = (h - h', \psi_n) + (h', \psi_n) \geq (h - h', \psi_n) + \bar{h}$ since, by construction, $\bar{h} \leq (h', \psi_n)$ for each $n \in \mathbb{N}$. By taking the limit $n \to \infty$ we obtain $(h, \rho_\epsilon^z) \geq (h - h', \rho_\epsilon^z) + \bar{h}$, where everything is justified by the same reasoning that is beyond proposition 1.8.2, where the main idea (omitted in this work, but available from the reference [3]), is that when a measure is Greenian (like $\rho_\epsilon^z \in \mathcal{M}$), approximations with test functions are carried on with Gaussian Free Field. At this point, circle averages allow to define respectively approximations μ_ϵ leading to the final desired inequality. Since by definition $A_+ = \log \mu_{h-h'}(B_+)$ (and the same with B_-), we obtain $A \geq A_+ + \gamma \bar{h}$ and $A \geq A_- + \gamma \bar{h}$.

Summing up, in this first half of the proof we deduced:

- Property $(*)$: $A_- = A_+ = \gamma Q \log 4 - A$

- Property $(**)$: $A \geq \max\{A_+, A_-\} + \gamma\bar{h}$

where the equalities must be intended to be in law. For reason of time we do not manage to cover in details the second part of the proof, but we think it is very worthy to intuitively understand why these formulas are useful and how the final claim can follow.

The strategy can be described as, first of all, use harmonicity of $\bar{h}$ for showing that it admits a superexponential decay. Then it can be shown how for a specific value, say η_0, the exponential inequality holds for $P_A(\eta_0)$, too. But then it holds for A_+ or A_- by using equation $(*)$ and so $(**)$ allows to "convert" it for the full set A, obtaining a new suitable value $\eta_1 < \eta_0$. The procedure repeats recursively giving rise to a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ for which the superexponential decay of P_A holds. Finally, by using monotonicity of P_A is clear that this decay must generally happen for every sufficiently negative value, concluding the proof. □

This lemma is used for proving the following result, which is the formal equivalent of the approximation $\mu(B_\epsilon(z)) \approx \mathbb{E}[\mu(B_\epsilon(z))|h_\epsilon(z)]$ stated in the "intuitive proof" of the KPZ relation described at the end of the section before. This is obtained by recalling how $\mu_\odot(B_\epsilon(z)) \simeq \mathbb{E}[\mu(B_\epsilon(z))|h_\epsilon(z)]$ and using the fact that:

Lemma 3.4.2 (Expected area is not far from actual area)**.** Fix z and ϵ so that $B_\epsilon(z) \subseteq D$. Conditioned on $h_{\epsilon'}(z)$, for all $\epsilon' \geq \epsilon$, we have that:

$$\mathbb{P}[\frac{\mu(B_\epsilon(z))}{\mu_\odot(B_\epsilon(z))} < e^\eta] \leq C_1 e^{-C_2\eta^2}$$

for some positive constants C_1 and C_2 independent of $\eta \leq 0$, z, D and the values of $h_{\epsilon'}(z)$ for $\epsilon' \geq \epsilon$.

Sketch of the proof. Recall that $\mu_\odot(B_\epsilon(z)) = \pi\epsilon^{\gamma Q} e^{\gamma h_\epsilon(z)}$, and so for a fixed ϵ we want to show that the probability

$$A \doteq \log \frac{\mu(B_\epsilon(z))}{\pi\epsilon^{\gamma Q} e^{\gamma h_\epsilon(z)}} \leq \eta$$

for $\eta \leq 0$ decays quadratically exponentially in η. This result can be obtained by adapting the same reasoning in the proof before. □

We continue by introducing a compacted-version of the Rooted Probability measure, needed during the final approximation argument.

Definition 3.4.1 (The measure $\Theta^{\tilde{D}}$)**.** Let $\tilde{D}$ be a fixed compact subset of D. We define $\Theta^{\tilde{D}}$ to be the rooted probability measure Θ conditioned on $\tilde{D}$, i.e. $\Theta^{\tilde{D}}(dh, dz) = \frac{\Theta(dh, dz\cap D)}{\Theta^z(\tilde{D})}$.

Lemma 3.4.3 (To have tilda balls contained in quantum one is always not hopeless)**.** Let $\epsilon_0 \doteq \sup\{\epsilon' : B_{\epsilon'}(\tilde{D}) \subset D\}$, fix $\delta > 0$ and consider the two balls $B^\delta(z)$ and $\tilde{B}^\delta(z)$. Then we have:

$$\mathbb{P}_R[\tilde{B}^\delta(z) \subseteq B^\delta(z)] > c$$

where $c > 0$ is a strictly positive constant independent of D, $\tilde{D}$ and δ. The measure $\mathbb{P}_R(\cdot) = \mathbb{P}(\cdot|e^{T_A})$ is the conditional probability given the radius of the tilde-ball $\tilde{B}^\delta(z)$ (recall it to be e^{T_A} by definition and/or construction).

At this point we are ready for giving a proof for the original KPZ relation.

Proof of theorem 3.3.1. There are many aspects we'd like to comment, but the easier strategy is just to start with the limit computation. During the whole proof we mean by $\mathbb{P}$ the original probability measure, by Θ the rooted one and $\Theta^{\tilde{D}}$ is the measure defined some lines above. Important point: from now on all the expectations are intended to be w.r.t. $\Theta^{\tilde{D}}$, if not differently specified (e.g. by the symbols $\mathbb{E}^{\mathbb{P}}$, ...). Start directly with:

$$\lim_{\delta\to 0} \frac{\log \mathbb{E}^{\mathbb{P}}_{X,h}[\mu_h\{z \in D : B^\delta(z) \cap X \cap \tilde{D} \neq \emptyset\}]}{\log \delta} = \lim_{\delta\to 0} \frac{\log \Theta((h,z) : B^\delta(z) \cap X \cap \tilde{D} \neq \emptyset)}{\log \delta} =$$

$$\lim_{\delta\to 0} \frac{\log \Theta^{\tilde{D}}((h,z) : B^\delta(z) \cap X \neq \emptyset)}{\log \delta}$$

by using the proportionality "$\mathbb{E}_h[\mu_h] = Z\theta$" (proposition 2.3.1); Z is a constant vanishing after the splitting in the limit, and the same happens for $\Theta^z(\tilde{D})$ appearing from the definition of $\Theta^{\tilde{D}}$. Let's introduce $\bar{\epsilon} \doteq$ radius of $B^\delta(z)$, i.e. $\bar{\epsilon}$ is the random variable satisfying $B_{\bar{\epsilon}}(z) = B^\delta(z)$. If $\bar{T}_A \doteq -\log(\bar{\epsilon}/\epsilon_0)$, then one can use again a similar argument as before for checking the approximation $\Theta^{\tilde{D}}((h,z) : B_{\bar{\epsilon}}(z) \cap X \neq \emptyset) \approx \mathbb{E}[exp(-2x\bar{T}_A)]$ (as described in the weak KPZ formula; this is the step where having the Euclidean Scaling Exponent equal to x comes into play). Recall that $A = -\log/\gamma$ and so $\delta \to$ iff $A \to \infty$. Assume for a moment to have:

$$\lim_{A\to\infty} \frac{\log \mathbb{E}[exp(-2x\bar{T}_A)]}{\log \mathbb{E}[exp(-2xT_A)]} = 1$$

Call it equation $(*)$, and see how we can continue the limit computation as:

$$\lim_{\delta\to 0} \frac{\log \Theta^{\tilde{D}}((h,z) : B^\delta(z) \cap X \neq \emptyset)}{\log \delta} = \lim_{\delta\to 0} \frac{\log \mathbb{E}[exp(-2x\bar{T}_A)]}{\log \delta} =$$

$$\lim_{\delta\to 0} \frac{\log \mathbb{E}[exp(-2xT_A)]}{\log \delta} \frac{\log \mathbb{E}[exp(-2x\bar{T}_A)]}{\log \mathbb{E}[exp(-2xT_A)]} = \Delta$$

by using $(*)$ and the stopping time computation $\mathbb{E}[exp(-2xT_A)] = \delta^\Delta$ already done for the weak formula case. As already pointed out, the expectations here are computed w.r.t. $\Theta^{\tilde{D}}$ rather than Θ, but this is absolutely not a problem since the law of T_A (stopping time of a drifted Brownian Motion) didn't depend on z, and so it remains the same under the new setting allowing to justify the stopping time result. In other words, in this step we learned an important key idea: for proving the full KPZ relation is enough to check equation $(*)$.

At this point we continue by checking $(*)$ but with an inequality instead, i.e.

$$\lim_{A\to\infty} \frac{\log \mathbb{E}[exp(-2x\bar{T}_A]}{\log \mathbb{E}[exp(-2xT_A)]} \leq 1$$

We start with "approximated" stopping times T_{aA} for $0 < a < 1$ (where aA is literally the product between the two numbers). It is useful because by observing the behavior of quantum and tilde balls on the event $\bar{T_A} < T_{aA}$, it's possible to use technique like lemma 3.4.1 for deducing that the probability of $\bar{T_A} < T_{aA}$ decays superexponentially in A. In other words we have $\Theta^{\tilde{D}}(\bar{T_A} \geq T_{aA}) \geq 1 - e^{-A^2 K}$, consequently $\Theta^{\tilde{D}}(e^{-2xT_{aA}} \geq e^{-2x\bar{T_A}}) \geq 1 - e^{-A^2 K}$ and finally $\mathbb{E}[e^{-2xT_{aA}}] \geq \mathbb{E}[e^{-2x\bar{T_A}}](1 - e^{-A^2 K})$ by using Markov inequality. By taking the ratio of their logarithms we see:

$$\lim_{A\to\infty} \frac{\log \mathbb{E}[exp(-2x\bar{T}_A]}{\log \mathbb{E}[exp(-2xT_{aA})]} \leq 1$$

and finally the required inequality with T_A instead of T_{aA} by taking the limit $a \to 1$ and using continuity of coefficients.

We verify now that the inequality cannot be strict, leading to the desired equation $(*)$. From lemma 3.4.3 we have that conditioned on e^{-T_A} (the radius of the tilde-ball $\tilde{B}^{\delta}(z)$), the $\Theta^{\tilde{D}}$ probability that $\bar{T_A} < T_A$ is at least $c > 0$. Of course we obtain the same result if we condition w.r.t T_A and so: $\Theta^{\tilde{D}}(\bar{T_A} < T_A | T_A) > c$, implying $\Theta^{\tilde{D}}(e^{-2x\bar{T_A}} > e^{-2xT_A} | T_A) > c$. By using the Markov inequality again: $\mathbb{E}[e^{-2x\bar{T_A}} | T_A] \geq e^{-2xT_A} c$ and finally $\mathbb{E}[e^{-2x\bar{T_A}}] \geq c\mathbb{E}[e^{-2xT_A}]$ by taking expectation again in a way to eliminate conditioning. At this point, two are the possible cases:

- we have $c = 1$, concluding the proof immediately;
- we have $0 < c < 1$. If we assume the starting inequality to be strict, we obtain:

  $$1 > \lim_{A\to\infty} \frac{\log \mathbb{E}[exp(-2x\bar{T}_A]}{\log \mathbb{E}[exp(-2xT_A)]} \geq \frac{\log c}{\log \mathbb{E}[exp(-2xT_A)]} + 1$$

  implying:

  $$\frac{\log c}{\log \mathbb{E}[exp(-2xT_A)]} < 0$$

  and so $\mathbb{E}[e^{-2xT_A}]$ *strictly* less than one for every $x > 0$, which cannot be true because x can be chosen arbitrarily small.

□

We conclude this section by remarking how the KPZ relation can be slightly generalized. Instead of choosing a random subset $X \subseteq D$ and then consider balls intersecting with it, one could choose a collection $\mathcal{X}$ of balls with centers in $\tilde{D}$, and then count the points that are centers of balls contained in this set. In other words, the following theorem holds:

Theorem 3.4.1 (Extended KPZ relation)**.** Let $\mathcal{X}$ be any random measurable set of the set of balls of the form $B_\epsilon(z)$ for $\epsilon > 0$ and z in a fixed compact subset $\tilde{D}$ of D. Fix $\gamma \in [0, 2)$. Then if:

$$\lim_{\epsilon\to 0} \frac{\mathbb{E}_{\mathcal{X}}[\mu^0\{z \in D : B_\epsilon(z) \in \mathcal{X}\}]}{\log \epsilon^2} = x$$

then it follows that, when $\mathcal{X}$ and μ are chosen independently, we have:

$$\lim_{\delta\to 0} \frac{\mathbb{E}_{\mathcal{X},h}[\mu_h\{z \in D : B^\delta(z) \in \mathcal{X}\}]}{\log \delta} = \Delta$$

with $KPZ(x, \Delta)$.

The proof of this result follows literally *exactly* the same steps of what already done previously (use the mean proportionality for switching to the probability measure $\Theta^{\tilde{D}}$, approximate the probability with an expectation of a stopping time, and finally compute it via martingale sampling theorem). We chosen to point out the first formulation rather than this one, reachable by choosing the family $\mathcal{X} = \{B_\epsilon(z) : B_\epsilon(z) \cap X\}$, because the former has a more clear intuition beyond and, since it deals with "areas scaling", also a more understandable geometrical meaning.

3.5 Conclusions and further developments

It is possible to develop very similar results in a range of different settings and ideas. For instance, in the notes [3], a KPZ formula concerning Minkowski dimension is proposed. We can have a quick overview of that. The hypothesis remain the same of what generally assumed in this work. Let $A \subseteq D$ be a Borelian subset of the LQG domain, and $\mathcal{S}_n$ the n-th level of the dyadic covering of D. Let S_i be the squares belonging to $\mathcal{S}_i$, for each $i \in \mathbb{N}$. As usual μ^0 denotes the Lebesgue measure. First, we define the:

- Minkowski content of A: $M_\delta(A; 2^{-n}) \doteq \sum_{i\in\mathbb{N}} \mathbb{1}_{\{S_i \cap A \neq \emptyset\}} \mu^0(S_i)^\delta$
- Euclidean Minkowski dimension: $d_M(A) \doteq \{\delta : \limsup_{n\to\infty} M_\delta(A; 2^{-n}) < \infty\}$
- Minkowski scaling exponent: $x_M(A) \doteq 1 - d_M(A)$

On the same fashion, one introduces the quantum counterparts. Here γ is the coefficient in the definition of μ:

- Quantum (random) Minkowski content: $M_\delta^\gamma(A; 2^{-n}) \doteq \sum_{i\in\mathbb{N}} \mathbb{1}_{\{S_i \cap A \neq \emptyset\}} \mu(S_i)^\delta$
- Expected quantum dimension: $q_M(A) \doteq \{\delta : \limsup_{n\to\infty} \mathbb{E}[M_\delta(A; 2^{-n})] < \infty\}$
- Minkowski scaling exponent: $\Delta_M(A) \doteq 1 - q_M(A)$

The KPZ formula is given in the following theorem:

Theorem 3.5.1 (Minkowski KPZ formula)**.** For each $A \subseteq D$ measurable, we have $x_M = (\frac{\gamma^2}{4})\Delta_M^2 + (1 - \frac{\gamma^2}{4})\Delta_M$ (i.e. $KPZ(x_M, \Delta_M)$).

As said, the proof of this result as well as further comments are available in the notes [3]. It is not discussed here for the following reasons. If one assumes the "multifractal principle of LQG" (proposition 3.4 in [3]), then it is completely straightforward. But this principle is entirely based on "Kahane's convexity inequality", whose proof is very long and in French, so the author prefers to refer to the original source.

Other interesting cases are for instance studied in [2], sections 5,6,7. We would like to point out how, if it is true that circles have always had a key role in the whole construction (e.g. we computed GFF on circle averages, considered expectation of quantum balls, etc...), on the other hand it is possible to completely replace the whole structure with the use of proper squares rather than disks. Briefly speaking, all the properties remains the same, including a final "Box-formulation" of the KPZ relation. The reason for using boxes rather than circles has very pragmatic roots, since the former are easier to implement into numerical simulations allowing a solid heuristic support to the elegant theoretical analysis.

Another aspect we would like to underline is that we have always worked with points $z \in D$, for an open subset $D \subseteq \mathbb{C}$. We managed in some way to give a meaning to the symbol "$h(z)$", and once we proved properties like the covariance structure, the spatial Markov property and the relationship with the inverse Laplacian, all the other results logically followed (of course that's not *completely* true, but good enough for underlining the key structure). It is spontaneous to ask if a similar theory can be developed for points $z \in \partial D$, this time involving e.g. line measures and analogue objects. The answer is partially positive: in principle the same strategy used for defining our random measures on D, can be adapted for random measures on ∂D, but unfortunately the new construction usually requires additional assumptions. For instance, ∂D must contain a linear piece $\_\partial D \subseteq \partial D$ (the case "=" is allowed) making possible to "define" the object $h(z)$ only for points $z \in \_\partial D$ by using averages on (complete or partial) *semicircles* (whose existence is actually guaranteed). Analytic challenges involving the Laplacian operator have now to take into account also *Neumann* boundary conditions, but all the efforts are compensated: at the end, an analogue to the KPZ formula can be deduced, too.

There are many other interesting ways for producing alternatives KPZ relations, for instance the one in the paper [8] where an heuristic heat-kernel approach is used. In the paper [9] a "Quantum Brownian Motion" is defined and then used for developing stochastic calculus under Liouville Quantum Gravity.

The fact that KPZ relations appear in so many variants and situations is well discussed in the *very* nice paper [4], where an attempt for a conceptual generalization is offered. Moreover, links to Statistical Physics and crucial connections with "true" Quantum Gravity are pointed out.

Dealing with the former, the idea is that many models are easier to study under the LQG setting (i.e. by using the metric $e^{\gamma h(z)}dz$ rather than dz), and KPZ relations might so be used for coming back to the original euclidean case after that the interesting quantities have been computed.

Finally, dealing with the latter, it seems that LQG could lead to a surface-generalization of the famous Feynman path integral. It would be *great* to explore more in this direction, but unfortunately the author has to stop here for mere reason of time, hoping that it could only be the starting point towards a fascinating journey. Thanks very much for your attention!

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