Orthogonal Wavelets on the Interval

Frank Koster *, Michael Griebel †

Abstract

In this paper we generalize the constructions [5], [8] and [21] of wavelets on the interval. These schemes give boundary modifications of compactly supported orthogonal wavelets $\psi \in L_2(\mathbf{R})$ with $supp\ \psi = [-N+1, N]$, where N denotes the number of vanishing moments of ψ. Our new scheme overcomes this restrictive condition. Furthermore, the constructions [5], [8], [21] involve Gram matrices for explicit orthogonalization steps. These Gram matrices tend to be very ill conditioned for increasing N. It is shown that for the present scheme the condition numbers of the resulting matrices are smaller by orders of magnitude. Therefore our scheme is numerically more stable. We also point out how wavelets can be obtained satisfying homogeneous Dirichlet or Neumann conditions.

In addition, we deal with the requirement of the discrete wavelet transform on the interval to find, e.g. from nodal values of $u \in L_2([0,1])$, an approximation $\tilde{u}$ by a linear combination of dilated scaling functions. We present and compare two methods. One method has already been used in [5] and [21]. Our experiments show that this scheme is not suited for data compression on the interval. The other method however is designed for data compression applications and leads to cheap and very well conditioned approximation mappings.

Key words. multiresolution analysis, wavelet, boundary conditions, Coiflets

AMS subject classification. 35C10, 42C15, 46E35

1 Introduction

Wavelets have become an approved tool in numerical applications ranging from signal processing to the solution of PDEs. The crucial feature of wavelets for these purposes is the approximation property of the corresponding scaling functions. Together with Bernstein type inequalities this leads to norm equivalences between scales of Sobolev norms and scales of norms of weighted coefficient sequences [9]. Usually wavelets and scaling functions, respectively, are defined in the shift invariant setting which allows for the use of the Fourier

*IAM Universität Bonn; email:koster@iam.uni-bonn.de

†IAM Universität Bonn; email:griebel@iam.uni-bonn.de

machinery. In this case the approximation property only holds for spaces of periodic functions or functions defined on **R**. But in applications, quite often one has to deal with functions defined on bounded domains. Thus, techniques had been developed to adapt the wavelets from the shift invariant case to the interval.

Boundary modifications of semi-orthogonal Chui-Wang wavelets [6] were constructed in [7]. Biorthogonal wavelets with spline scaling functions have been treated by [1] and [10]. The later work contains extensive tables concerning the constants for the equivalence of the L_2-norm and the l_2-sequence norm. The numbers reported there exponentially grow with increasing number N of vanishing moments of the (biorthogonal) wavelets. Of course, one is interested in possibly tight bounds in the norm equivalence. For orthogonal wavelets these constants are always 1. This explains the interest in boundary modifications of compactly supported *orthogonal* wavelets.

A first technique for Daubechies wavelets (in the following called Daublets) [12] was proposed by Meyer [19]. This scheme suffers from two disadvantages. The cardinality of the subspaces $V_j^{[0,1]} \subset L_2[0,1]$ is not a power of 2. However, this is required for a number of applications, e.g. wavelet packets. Furthermore the Gram matrix involved in the orthogonalization step is very ill conditioned. This leads to a significant loss of accuracy and therefore limits the practical use to wavelets with small N. To overcome these drawbacks, in [5], [8], another technique was developed. There, suitable linear combinations of scaling functions from the shift invariant setting are used to define auxiliary boundary adapted scaling functions which preserve the approximation property. The Gram matrix required to orthonormalize the auxiliary scaling functions is determined by a linear system which is well conditioned. But unfortunately, the Gram matrix itself is not. A more stable technique was proposed by [21]. However, a serious drawback of [5], [8] and [21] is that the underlying wavelet ψ must have support $supp\ \psi = [-N+1, N]$. This excludes e.g. Coiflets [13], [22] which play an important role in numerical applications [3], [4].

The main concern of this paper is to give a general scheme for the boundary modification of orthogonal wavelets. This scheme can be applied to all compactly supported orthogonal wavelets which are at least piecewise continuous. As in previous constructions the approximation property of the original scaling functions is preserved. We will also show how wavelets can be obtained satisfying homogeneous Dirichlet or Neumann conditions. Furthermore we will show that our scheme yields significant improvements in the condition numbers of the Gram matrices involved in the orthogonalization step.

Besides the construction of wavelets adapted to the interval there is another issue of practical interest. The discrete wavelet transform is just a basis transform from a basis of scaling functions to a basis of wavelets. Thus, in many applications, it is necessary first to calculate from some given nodal values of a function u an approximation $\tilde{u}$ which is a linear combination of (dilated) scaling functions. Here a *careful* approximation step is important. A common technique is to use the approximation which is exact for polynomials of a certain degree. This technique has some drawbacks which make it not appropriate for data compression. Therefore, we present another technique which is well suited for such applications.

This paper is organized as follows. In the second section some topics from multiresolution analysis and compactly supported orthogonal wavelets are reviewed and basic notations are introduced. The third section deals with our generalized construction of boundary adapted orthogonal wavelets. First the scaling functions adapted to the half open interval $[0, \infty[$ are defined and an outline is given how to obtain scaling functions with homogeneous boundary conditions. Then, we carry the technique for the half open interval $[0, \infty[$ over to the bounded interval $[0, 1]$. Thereafter, we deal with the wavelets adapted to the interval and compare the numerical stability of the present general scheme and the techniques proposed in [8] and [21]. In the fourth section we consider the case of discretely sampled data on the interval. We present two techniques to obtain from nodal values of u an approximation $\tilde{u}$ in a suitable, finite dimensional subspace $V_j^{[0,1]}$. The first technique addresses the needs of data (image) compression. Since this approximation introduces a non-unitary mapping from the discrete nodal values to the discrete coefficients with respect to the basis of scaling functions, a significant deterioration of the compression properties may occur. However, we present some optimization technique which almost perfectly remedies this problem. Here, once again, it is important that we are not restricted to orthogonal wavelets with almost symmetric support $supp\ \psi = [-N+1, N]$. The second approach which we consider is quite familiar in the context of wavelets adapted to the interval [5], [21]. We show that this technique is not appropriate for data compression.

2 Preliminaries

A multiresolution analysis (MRA) of $L_2(\mathbf{R})$ is a sequence of nested, closed subspaces $\{V_j\}_{j \in \mathbf{N}} \subset L_2(\mathbf{R})$, which is dense in $L_2(\mathbf{R})$, i.e.

$$clos_{L_2}(\bigcup_{j \geq 0} V_j) = L_2(\mathbf{R}) \quad ; \quad V_j \subset V_{j+1} \quad \text{for } j \geq 0$$

The idea of a MRA is to expand functions $u \in L_2(\mathbf{R})$ in terms of the complementary spaces $W_j := V_{j+1} \ominus V_j$:

$$u = u^0 + \sum_{j \geq 0}^{\infty} u_j \quad ; \quad u^0 \in V_0 \quad , \quad u_j \in W_j \quad . \tag{1}$$

Depending on the approximation properties of the V_j and the smoothness of u this expansion converges very fast in j and allows for a compact representation of a reasonable approximation of u by a finite sum in (1).

The starting point for the efforts of the present paper are spaces V_j and W_j generated by dilates and translates of mother functions ϕ and ψ respectively, i.e.

$$\begin{aligned} V_j &= span\{\phi_k^j\}_{k \in \mathbf{Z}} \quad ; \quad \phi_k^j(x) := 2^{j/2}\phi(2^j x - k) \quad , \\ W_j &= span\{\psi_k^j\}_{k \in \mathbf{Z}} \quad ; \quad \psi_k^j(x) := 2^{j/2}\psi(2^j x - k) \quad . \end{aligned}$$

The functions ϕ_k^j are so-called scaling functions and the ψ_k^j are the so-called wavelets. This name comes from the fact that in many cases ψ has vanishing mean $\int \psi dx = 0$, which induces a wave like graph. The ϕ_k^j are called 'scaling' functions, because $V_j \subset V_{j+1}$ implies the existence of real or complex coefficients $\{h_l\}_{l\in\mathbf{Z}}$, which relate the basis functions of two subsequent V_j by a scaling equation

$$\phi_l^j(x) = \sum_{r\in\mathbf{Z}} h_{r-2l}\phi_r^{j+1}(x) \quad . \tag{2}$$

In addition to these simple construction rules of dilation and translation, the scaling functions and the wavelets considered in this paper exhibit some orthogonality relations

$$\int_{\mathbf{R}} \phi_k^j \phi_l^j dx = \int_{\mathbf{R}} \psi_k^j \psi_l^j dx = \delta_{kl} \quad \text{and} \quad \int_{\mathbf{R}} \psi_k^j \phi_l^j dx = 0 \quad k,l \in \mathbf{Z} \quad \text{and} \quad j \geq 0 \quad , \tag{3}$$

which, by $W_j \subset V_{j+1} \perp W_{j+1}$, imply $\int_{\mathbf{R}} \psi_k^i \psi_l^j dx = 0$ for $i \neq j$ and hence give an orthogonal decomposition in (1). Whenever a scaling function ϕ with $\int_{\mathbf{R}} \phi_k^j \phi_l^j dx = \delta_{kl}$ is known, an appropriate mother wavelet ψ can be defined by $\psi(x) = \sum_{k\in\mathbf{Z}} g_k \phi_k^1(x)$ or more generally

$$\psi_l^j(x) = \sum_{r\in\mathbf{Z}} g_{r-2l}\phi_r^{j+1}(x) \quad ; \quad g_k = (-1)^k h_{1-k} \quad . \tag{4}$$

The wavelets ψ_l^j must be such linear combinations of $\{\phi_r^{j+1}\}$, because of $W_j \subset V_{j+1}$. The particular form of the $\{g_k\}$ ensures the orthogonality relations. This setting is not unique, since all masks $\{g_k := (-1)^k h_{1+2q-k}\}$ ($q \in \mathbf{Z}$ arbitrary) lead to wavelets with the desired properties.

At this stage we make a further restriction of the general setting and consider only scaling functions and wavelets of compact support. This can be shown to be equivalent to the compact support of the mask $\{h_k\}_{k\in\mathbf{Z}}$, see [14]. One of the breakthroughs in wavelet theory was the construction of such compactly supported scaling functions and wavelets by Daubechies. Well known families of compactly supported orthonormal wavelets are Daublets(N), Symmlets(N) or Coiflets(N), see [12], [13]. The different ϕ and ψ of these families depend on an order parameter N, which stands for the fact that polynomials of degree less than N are infinite linear combinations of the translates $\{\phi_k^0\}$, i.e. in the pointwise sense there holds

$$p(x) = \sum_{k\in\mathbf{Z}} p_k \phi_k^0(x) \quad . \tag{5}$$

Although p is not in $L_2(\mathbf{R})$ and therefore not in V_0 it is in $L_2^{loc}(\mathbf{R})$. Due to the compact support of ψ one has

$$\int_{\mathbf{R}} p(x)\psi(x)dx = \int_{\mathbf{R}} \sum_k p_k \phi_k^0(x)\psi(x)dx = \sum_k p_k \int_{\mathbf{R}} \phi_k^0(x)\psi(x)dx = 0 \quad .$$

Since the polynomial p was arbitrary ($deg\ p < N$), ψ and therefore all ψ_l^j have vanishing moments

$$\int_{\mathbf{R}} x^r \psi_l^j dx = 0 \quad \text{for } 0 \le r < N \quad . \tag{6}$$

Now, consider the expansion of $u \in L_2(\mathbf{R})$ in terms of the orthogonal wavelet basis

$$u = \sum_l c_l^0 \phi_l^0 + \sum_{j=0}^{\infty} \sum_l d_l^j \psi_l^j \quad ; \quad c_l^0 = \int_{\mathbf{R}} u \phi_l^0 dx \quad ; \quad d_l^j = \int_{\mathbf{R}} u \psi_l^j dx \quad . \tag{7}$$

Using the vanishing moments property (6) for smooth functions $u \in C^M$ ($M \le N$) the coefficients d_l^j can be estimated as follows:

$$\begin{aligned}
|d_l^j| &= 2^{j/2} \left| \int_{\mathbf{R}} u(x) \psi(2^j x - l) dx \right| = 2^{-j/2} \left| \int_{\mathbf{R}} u(2^{-j} y + 2^{-j} l) \psi(y) dy \right| \\
&= 2^{-j/2} \left| \int_{\mathbf{R}} \left[\sum_{r=0}^{M-1} \frac{u^{(r)}(2^{-j} l)}{r!} (2^{-j} y)^r + \frac{u^{(M)}(\eta(j,l,y))}{M!} (2^{-j} y)^M \right] \psi(y) dy \right| \\
&= \frac{2^{-j(M+1/2)}}{M!} \left| \int_{\mathbf{R}} u^{(M)}(\eta(j,l,y)) y^M \psi(y) dy \right| \\
&\le C 2^{-j(M+1/2)} \max_{\eta \in supp\ \psi_l^j} |u^{(M)}(\eta)|
\end{aligned} \tag{8}$$

with some constant C depending only on M and ψ. Hence, the decay of the coefficients d_l^j with increasing scale j expresses the *global* smoothness properties of u by the factor 2^{-jM} and also the *local* smoothness around $2^{-j} l$ by the factor $\max |u^{(M)}(\eta)|$. This makes a wavelet expansion (7) very desirable, since it immediately allows for an adaptive approximation of u by only the large coefficients d_l^j.

With respect to numerical applications of (7) it is sufficient to consider the case that u is in some V_J, i.e

$$u = \sum_l c_l^J \phi_l^J \in V_J \quad ; \quad c_l^J = \int_{\mathbf{R}} u \phi_l^J dx \tag{9}$$

with known coefficients c_l^J. The coefficients of the wavelet expansion, which is now finite in scale j,

$$u = \sum_l c_l^0 \phi_l^0 + \sum_{j=0}^{J-1} \sum_l d_l^j \psi_l^j$$

can be determined from the coefficients c_l^J recursively from $j = J-1$ down to $j = 0$ as follows:

$$\begin{aligned}
d_l^j &= \int_{\mathbf{R}} u \psi_l^j dx = \int_{\mathbf{R}} u \sum_r g_{r-2l} \phi_r^{j+1} dx = \sum_r g_{r-2l} c_r^{j+1} \\
c_l^j &= \int_{\mathbf{R}} u \phi_l^j dx = \sum_r h_{r-2l} c_r^{j+1} \quad .
\end{aligned}$$

These steps are the pyramidal scheme of the Fast Wavelet Transform, i.e. we have

$$\begin{array}{ccccccc} \{c_l^J\} & \xrightarrow{h} & \{c_l^{J-1}\} & \xrightarrow{h} & \cdots & \xrightarrow{h} & \{c_l^0\} \\ & \overset{g}{\searrow} & \{d_l^{J-1}\} & \overset{g}{\searrow} & \cdots & \overset{g}{\searrow} & \{d_l^0\} \end{array} .$$

If the coefficients $h_{-L}, ..., h_R$ are non vanishing and if u has compact support such that only the coefficients c_l^J with $|l| < C(u,\phi)2^J$ are distinct from zero, then the whole algorithm is finite and requires order $\mathcal{O}(2^J(L+R))$ operations. For most families of wavelets and scaling functions $L+R$ is of order N, i.e. a stronger decay of d_l^j for smooth functions u is paid with more operations for the transform.

The prerequisite of the fast wavelet transform was the explicit knowledge of expansion (9) in some subspace $V_J \subset L_2$, i.e. if $u \notin V_J$ or if the exact coefficients c_l^J are not known, first one has to find an approximation

$$Q_J u = \tilde{u} = \sum_l \tilde{c}_l^J \phi_l^J \in V_J$$

in some V_J. This approximation should be such that the $\tilde{d}_l^j$, which stem from the transform applied to $\tilde{c}_l^J$, essentially behave in the same manner as the d_l^j. Clearly the best choice for Q_J would be the orthogonal projection onto V_J, i.e. $\tilde{c}_l^J = c_l^J = \int_{\mathbf{R}} u\phi_l^J dx$. But, in numerical applications there is often only finite information about $u \in L_2$, e.g. collocation values $u(x_l^J)$ on some grid, which only in special cases is sufficient to allow for the best approximation. However, one can approximate the integrals c_l^J in some sense. Two particular techniques for this are described in the fourth section of this paper.

A major drawback of the construction of the MRA by dilation and translation as described above is that it does not apply to spaces of functions over bounded domains, e.g. $L_2([0,1])$. The main concern of this paper is to build up from very general compactly supported ϕ and ψ a MRA $\{V_j^{[0,1]}\}$ and $\{W_j^{[0,1]}\}$ of $L_2([0,1])$, while preserving as many of the nice features of the previous construction as possible such as simple scaling equations and orthogonality relations, which are required for the fast wavelet transform.

3 Boundary Fitted Wavelets

In this section we introduce step by step the general construction of a boundary adaptation for very general underlying compactly supported wavelets. Like former schemes, first we will only treat the construction of scaling functions on the half interval $[0,\infty[$. Then, we will show how to extend this result to obtain basis functions of the particular nested subspaces $V_j^{[0,1]}$ of $L_2([0,1])$. Explicit knowledge of the dimension of these subspaces will turn out to be useful in subsection 3.4, where boundary adapted wavelets are derived. The last subsection is concerned with the comparison of the present scheme and earlier constructions. In particular we will compare the schemes with respect to their numerical stability. As a main result it will turn out that our method yields significant improvements in the numerical stability.

3.1 MRA of $L_2([0,\infty[)$

In the following, we introduce a MRA of $L_2([0,\infty[)$. The main idea for the construction of boundary adapted scaling functions is to employ linear combinations of translates of ϕ and to leave the other translates unchanged. Let $\mathbf{h}$ be the compactly supported mask, i.e.

$$supp\ \mathbf{h} = \{-L,..,R\} \quad ; \quad h_{-L} \neq 0 \quad \text{and} \quad h_R \neq 0 \quad ; \quad (L, R \in \mathbf{N}\backslash 0) \ . \tag{10}$$

Then, up to a constant factor, there exists only one function ϕ, which fulfills the scaling equation (2) and which has compact support. More precisely, in this case the support of ϕ is $[-L, R]$. In the construction we will additionally require

$$\int_{-L}^{-L+1} |\phi|^2 dx \neq 0 \ , \ \int_{R-1}^{R} |\phi|^2 dx \neq 0 \quad . \tag{11}$$

E.g. if ϕ is piecewise continuous, these conditions are necessarily satisfied, as shown by the following argument. By iterated use of (2) one can determine the value $\phi(x)$ from the values $\phi(\mathbf{Z})$ if x has finite binary expansion (see [14], chapter 7.2). In particular one has $\phi(-L+1/2) = \sqrt{2}h_{-L}\phi(-L+1)$. Using this and similar relations one successively arrives at

$$\phi|_{[-L,-L+1]} = 0 \quad \Rightarrow \quad \phi(\mathbf{Z}) = 0 \quad .$$

But then ϕ vanishes for all x with finite binary expansion and thus, everywhere.

Our aim is to find a set of scaling functions $\{\varphi_i\}_{i\in\mathbf{N}}$ with the following properties:

(i) $\{\varphi_i\}_{i\in\mathbf{N}}$ is an orthonormal family of functions,

(ii) $supp\ \varphi_i \subset [0,\infty[$,

(iii) the functions φ_i fulfill a scaling equation, i.e. they induce a MRA,

(iv) the φ_i recover all polynomials of degree less than N on $[0,\infty[$.

A moments thought yields that, due to $supp\ \phi \subset [-L,\infty[$, the set $\{\phi_k^0\}_{k\geq L}$ satisfies the first three conditions but not the fourth. As mentioned before, we will augment a similar set $\{\phi_k^0\}_{k\geq K}$ of translates by some linear combinations of restrictions of $\{\phi_k^0\}_{k<K}$ on $[0,\infty[$ to obtain all properties **(i)-(iv)**. Here, K is such that

$$\max\{L, N\} \leq K \leq \max\{L+1, N\} \quad . \tag{12}$$

Since conditions **(i)-(iii)** even hold for $K = L$ one might wonder why not to take simply $K = L$. As it will be shown in subsection 3.2, polynomial recovery requires $K \geq N$. Furthermore, in this subsection, we will also show how to incorporate homogeneous boundary conditions for the boundary scaling functions. In subsection 3.5 we will see that it is required to be free to set $K = L$ or $K = L+1$ when $L > N$ to fulfill some technical conditions.

Since in the following lines we are mainly working on the 0th scale $j = 0$, it will be convenient to use the abbreviations

$$\phi_k = \phi_k^0 \quad \text{and} \quad \psi_k = \psi_k^0 \quad .$$

Taking into account equation (10) it turns out that only the translates $\phi_{-R+1}, \phi_{-R+2}, \ldots$ do not vanish on $[0, \infty[$. Thus it is sufficient to define the edge scaling functions by linear combinations of the components of the following vector of functions

$$\Phi^c := (\phi_{-R+1}, \ldots, \phi_{K-1})^T \tag{13}$$

restricted to $[0, \infty[$. This immediately rises the question how many edge scaling functions are necessary. For example for Coiflets, the shifts ϕ_l are nicely localized in $x = l$. Thus, it will be convenient to define K edge scaling functions which exactly replace the shifts $\phi_0, .., \phi_{K-1}$. This is by no means the only way, but leads to reasonable finite dimensions of $V_j^{[0,1]}$ like 2^j.

At this stage some remarks on the notation are in order. In this paper we will drop the usual convention that indices for components of vectors or matrices begin with 1. Instead we will denote by $H := (H_{l,r})_{l\in\Lambda, r\in\Sigma}$ matrices with lth row $H_{l,.}$ $(l \in \Lambda)$ and rth column $H_{.,r}$ $(r \in \Sigma)$, where Λ and Σ are suitable index sets. To multiply a matrix from the right by a vector, the row index set of the vector must coincide with the column index set of the matrix. E.g. Φ^c has row index set $\{-R+1, ..., K-1\}$ and thus can only be multiplied with matrices having column index set $\{-R+1, ..., K-1\}$. We will use the notation $H \in M(\Lambda, \Sigma)$ to explicitly specify the row and column index sets, or the notation $H \in M(\#\Lambda, \#\Sigma)$ when only the dimension of H is of particular interest.

Now, let us introduce auxiliary edge scaling functions

$$\tilde{\Phi}(x) := \begin{pmatrix} \tilde{\phi}_0 \\ \vdots \\ \tilde{\phi}_{K-1} \end{pmatrix}(x) \quad := \quad A\Phi^c(x) \cdot \chi_{[0,\infty[}(x) \quad , \tag{14}$$

where $A \in M(0...K-1, -R+1...K-1)$ and $\chi_{[0,\infty[}$ denotes the characteristic function of $[0, \infty[$.

Lemma 3.1. *The component functions of* $\tilde{\Phi}$ *are orthogonal to the inner translates* $\{\phi_k\}_{k\geq K}$. *They are linear independent, iff* $rgA = K$. *Furthermore they fulfill the equation*

$$\tilde{\Phi}(x) = \sqrt{2}(A\mathcal{H}^c\Phi^c(2x) + A\mathcal{H}^p\Phi^p(2x)) \cdot \chi_{[0,\infty[}(2x) \quad , \tag{15}$$

where the matrices $\mathcal{H}^c$, $\mathcal{H}^p \in M(K+R-1, K+R-1)$ *and the vector* Φ^p *are defined by*

$$\begin{aligned} \mathcal{H}^c_{l,r} &:= h_{r-2l} \quad \textit{for} \quad -R+1 \leq l, r \leq K-1 \\ \mathcal{H}^p_{l,r} &:= h_{r-2l} \quad \textit{for} \quad -R+1 \leq l \leq K-1 \ ; \ K \leq r \leq 2K+R-2 \\ \Phi^p &:= (\phi_K, \ldots, \phi_{2K+R-2})^T \end{aligned} \tag{16}$$

Proof. Orthogonality to the inner translates follows from the orthogonality of $\{\phi_i\}_{i\in\mathbf{Z}}$ and $supp\ \phi_k \subset [0,\infty[$ for $k \geq K$. The staggered support of the components of Φ^c and condition (11) show, that even $\Phi^c\chi_{[0,\infty[}$ are linear independent functions. This proves that linear independence of $\tilde{\Phi}$ is equivalent to $rg\ A = K$.

Similar to $\mathcal{H}^p$ and Φ^p, respectively, define $\mathcal{H}^m$ and Φ^m by

$$\mathcal{H}^m_{l,r} := h_{r-2l} \quad \text{and} \quad \Phi^m_r = \phi_r \quad \text{for} \quad -R+1 \leq l \leq K-1 \ ; \ -2R-L+2 \leq r \leq -R \quad .$$

Then, from (2) follows

$$\Phi^c(x) = \sqrt{2}\left(\mathcal{H}^m\Phi^m + \mathcal{H}^c\Phi^c + \mathcal{H}^p\Phi^p\right)(2x) \quad . \tag{17}$$

Now, $supp\ \Phi^p \subset [0,\infty[$ and $supp\ \Phi^m \subset]\infty, 0]$ yield equation (15). □

Unfortunately equation (15) is not a scaling equation, since $\tilde{\Phi}$ does not occur on the right hand side with dilated argument $2x$. To replace the first term of the right hand side of (15) by a term involving $\tilde{\Phi}$ requires additional conditions on A, which are derived in the next lemma.

Lemma 3.2. *With the prerequisites of Lemma 3.1, $\tilde{\Phi}$ exhibits a scaling equation*

$$\tilde{\Phi}(x) = \sqrt{2}(D\tilde{\Phi}(2x) + B\Phi^p(2x)) \quad ; \quad B := A\mathcal{H}^p \quad , \tag{18}$$

for some $D \in M(0...K-1, 0...K-1)$ if and only if

$$A\mathcal{H}^c = DA \quad . \tag{19}$$

Proof. Subtraction of (18) and (17) yields

$$(A\mathcal{H}^c - DA)\Phi^c\chi_{[0,\infty[}(2x) = 0 \quad ,$$

which together with the linear independence of $\Phi^c\chi_{[0,\infty[}(2x)$ proves the assertion. □

Now we show how to find such pairs A and D satisfying all the previous conditions.

Lemma 3.3. *Consider the Jordan or Schur decomposition of $\mathcal{H}^{cT}$,*

$$\mathcal{H}^{cT} = TJT^{-1} \quad , \tag{20}$$

where J is some upper triangular matrix. Let S be the $K \times (K+R-1)$ matrix $S = (I_K|\mathbf{0})$ with I_K being the $K \times K$ identity matrix. Then

$$A := ST^T \quad \text{and} \quad D = SJ^TS^T \tag{21}$$

satisfy (19) and $rg\ A = K$.

Proof. From (20) follows $ST^T\mathcal{H}^c = SJ^TT^T$. Since J^T is a lower triangular matrix we have $SJ^T = SJ^TS^TS$. Hence,

$$\begin{aligned} A\mathcal{H}^c = ST^T\mathcal{H}^c &= SJ^TT^T \\ &= (SJ^TS^T)ST^T = DA \quad . \end{aligned}$$ □

At the present stage we are able to build functions $\tilde{\Phi}$, which fulfill a scaling equation. Since the inner scaling functions ϕ_k $(k \geq K)$ are linear combinations of only $\{\phi_l^1\}_{l\geq K}$, there holds

$$\{\tilde{\phi}_0(x),..,\tilde{\phi}_{K-1}(x),\phi_K(x),...\} \quad \in \quad span\{\tilde{\phi}_0(2x),..,\tilde{\phi}_{K-1}(2x),\phi_K(2x),...\} \quad .$$

Thus, conditions **(ii)-(iii)** are established. The functions $\tilde{\Phi}$ are orthogonal to the inner translates ϕ_k and they are linear independent. Under some weak assumptions on ϕ (e.g. piecewise continuity) this is sufficient for the Gram matrix

$$G^S := \int_0^\infty \tilde{\Phi}(x)\tilde{\Phi}^H(x)dx$$

to be non-singular. Hence, we can define orthonormalized functions

$$\Phi^{left} := (G^S)^{-1/2}\tilde{\Phi} \quad ,$$

which together with $\{\phi_k\}_{k\geq K}$ form an orthonormal system of functions on $L_2([0,\infty[$. More generally we could use any decomposition $G^S = MM^H$ and then set $\Phi^{left} := M^{-1}\tilde{\Phi}$. The functions Φ^{left} then fulfill the scaling equation

$$\Phi^{left}(x) = \sqrt{2}(H\Phi^{left}(2x) + E\Phi^p(2x)) \quad , \tag{22}$$

where $H := (G^S)^{-1/2}D(G^S)^{+1/2}$ and $E := (G^S)^{-1/2}B$. This yields condition **(i)**. It remains to treat the polynomial recovery and to determine the Gram matrix G^S for the orthogonalization step. This is the concern of

Lemma 3.4. *With the previous definitions, the Gram matrix G^S is the unique solution of the linear system of equations*

$$G^S = DG^SD^H + BB^H \quad . \tag{23}$$

Proof. Multiplication of equation (18) by its Hermitian (conjugate transpose) and integration yields (23). Rewriting (23) in terms of its components and exploiting the lower triangular shape of D (see (21)) we obtain

$$\begin{aligned} (BB^H)_{l,k} &= (G^S)_{l,k} - (DG^SD^H)_{l,k} \\ &= G^S_{l,k}(1 - D_{l,l}\bar{D}_{k,k}) - \sum_{j=0}^{k-1} D_{l,l}G^S_{l,j}\bar{D}_{k,j} - \sum_{i=0}^{l-1}\sum_{j=0}^{k} D_{l,i}G^S_{i,j}\bar{D}_{k,j} \quad . \end{aligned} \tag{24}$$

This shows that the entry $(G^S)_{l,k}$ is unique, provided $(1 - D_{l,l}\bar{D}_{k,k})$ does not vanish. In addition (24) gives a nice algorithm for the recursive solution of (23). To prove $(1 - D_{l,l}\bar{D}_{k,k}) \neq 0$ for $0 \leq l, k < K$, we show that all eigenvalues of $\mathcal{H}^c$ are smaller than one by modulus. Since the diagonal entries $D_{l,l}$ are a subset of these eigenvalues this would complete the proof.

In a first step we consider matrices $\mathcal{H}^c_{(K)}$ defined by (16) for varying parameter K. Then, for the spectrum σ of $\mathcal{H}^c_{(K)}$, there holds

$$\sigma(\mathcal{H}^c_{(L+1)}) = \sigma(\mathcal{H}^c_{(L)}) \bigcup \{h_{-L}\} \quad \text{and} \quad \sigma(\mathcal{H}^c_{(K+1)}) = \sigma(\mathcal{H}^c_{(K)}) \bigcup \{0\} \quad \text{for } K \geq L+1 \quad .$$

To see this, observe that $\mathcal{H}^c_{(K)}$ is the upper left submatrix of $\mathcal{H}^c_{(K+1)}$ and that the last (new) row of $\mathcal{H}^c_{(K+1)}$ reads $(\mathcal{H}^c_{(K+1)})_{K,.} = (h_{r-2K})_{-R+1 \leq r \leq K}$. Thus

$$(\mathcal{H}^c_{(L+1)})_{L,.} = (0, ..., 0, h_{-L}) \quad \text{and} \quad (\mathcal{H}^c_{(K+1)})_{K,.} = (0, ..., 0, 0) \quad \text{for } K > L \quad .$$

Hence, we can restrict our considerations to the case $K = L + 1$, i.e. $\mathcal{H}^c = \mathcal{H}^c_{(L+1)}$. Multiplication of (17) by its Hermitian and integration over the real line yields

$$I = \mathcal{H}^m \mathcal{H}^{mH} + \mathcal{H}^c \mathcal{H}^{cH} + \mathcal{H}^p \mathcal{H}^{pH} \quad .$$

Obviously non of the eigenvalues of $\mathcal{H}^c \mathcal{H}^{cH}$ can be equal or larger than one if $\mathcal{H}^m \mathcal{H}^{mH} + \mathcal{H}^p \mathcal{H}^{pH}$ is non singular. Then, the modulus of all eigenvalues of $\mathcal{H}^c$ is smaller than one.

Since in terms of the mask $\mathbf{h}$ the orthogonality relation $\int_{\mathbf{R}} \phi \phi_k dx = \delta_{0,k}$ reads $\sum_l h_l h_{l-2k} = \delta_{0,k}$, $R + L + 1$ must be even. Thus, for the particular case $K = L + 1$, the matrix $\mathcal{H}^p$ is of the following shape:

$$\mathcal{H}^p = \begin{pmatrix} \multicolumn{6}{c}{\mathbf{0}} \\ \hline h_R & & & & & \\ \dots & h_{R-1} & h_R & & & \\ & \ddots & \ddots & & & \\ & & & \dots & h_{R-1} & h_R \end{pmatrix} \begin{array}{l} \} \ \frac{L+R-1}{2} \ \text{rows} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \ \frac{L+R+1}{2} \ \text{rows} \quad . \end{array} \tag{25}$$

Hence, $rg\mathcal{H}^p = \frac{L+R+1}{2}$. Furthermore $\mathcal{H}^p \mathcal{H}^{pH}$ has the following block form: $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Box \end{array}\right)$. Likewise there holds

$$\mathcal{H}^m = \begin{pmatrix} h_{-L} & h_{-L+1} & \dots & & & & \\ & & h_{-L} & h_{-L+1} & \dots & & \\ & & & & \ddots & \ddots & \\ & & & & & h_{-L} & h_{-L+1} \\ \hline \multicolumn{7}{c}{\mathbf{0}} \end{pmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \ \frac{L+R-1}{2} \ \text{rows} \\ \} \ \frac{L+R+1}{2} \ \text{rows} \quad . \end{array} \tag{26}$$

Therefore $\mathcal{H}^m \mathcal{H}^{mH} + \mathcal{H}^p \mathcal{H}^{pH}$ is of full rank due to its block form $\left(\begin{array}{c|c} \Box & 0 \\ \hline 0 & \Box \end{array}\right)$ and the full rank of the diagonal blocks. $\Box$

3.2 Polynomial Recovery and Boundary Conditions

In this subsection we show that the setting of the previous subsection still allows for the construction of boundary fitted scaling functions which recover polynomials of degree less than N. Due to the choice of the matrices A and D this leads to certain conditions on the eigenvectors of $(\mathcal{H}^c)^T$. In the second part we will show how to incorporate homogeneous boundary conditions in the construction of Φ^{left}.

Consider any basis $\{p_0, ..., p_{N-1}\}$ of the polynomials of degree less than N. Because of equation (5) there exist coefficients $\{p_{r,k}\}_{k\in\mathbf{Z}}$ $(0 \leq r < N)$ such that pointwise

$$p_r(x) = \sum_k p_{r,k}\phi_k(x)$$

holds. The coefficients $p_{r,k}$ are the values of some other polynomials P_r at the arguments k, see (27). A straightforward way to show this is to pick the particular basis $p_r(x) = \frac{x^r}{r!}$. Using the orthogonality of the translates ϕ_k, substituting $y = x - k$ and using binomial expansion of $(y + k)^r$ one arrives at

$$p_{r,k} = \int_{\mathbf{R}} \frac{x^r}{r!}\phi(x-k)dx = \sum_{s=0}^{r} \frac{C_{r-s}k^s}{s!} =: P_r(k) \quad \text{where} \quad C_s := \int_{\mathbf{R}} \phi(y)\frac{y^s}{s!}dy \quad . \tag{27}$$

By the scaling equation (2) one can recursively determine the moments C_s for $s \in \mathbf{N}$:

$$C_0 = 1 \quad ; \quad C_s = \frac{1}{2^s - 1}\sum_{n=1}^{s}\left(\frac{1}{\sqrt{2}}\sum_{l\in\mathbf{Z}} h_l \frac{l^n}{n!}\right) C_{s-n} \quad (s > 0) \quad . \tag{28}$$

Explicit knowledge of the P_r is useful in connection with the following

Lemma 3.5. *For $K \geq L$ the matrix $(\mathcal{H}^c)^T$ has eigenvalues $\lambda_r = 2^{-r-1/2}$ $(0 \leq r < N)$ and corresponding eigenvectors $\Pi_r = (P_r(k))_{-R+1\leq k<K}$. If we choose the first N rows of A as $A_{r,.} := \Pi_r^T$, then all polynomials of degree less than N are recovered by the edge scaling functions and the inner translates ϕ_k, i.e. we have pointwise for $x \geq 0$*

$$\frac{x^r}{r!} = \tilde{\phi}_r(x) + \sum_{k\geq K} P_r(k)\phi_k(x) \quad ; \quad 0 \leq r < N \quad . \tag{29}$$

Proof. The first statement concerning the eigenvalues follows from [21] Lemma 3, which states

$$\frac{1}{2^r}P_r(k) = \sqrt{2}\sum_{l\in\mathbf{Z}} P_r(l)h_{k-2l} \quad ; \quad k \in \mathbf{Z} \quad . \tag{30}$$

The range of l in the sum in (30) is restricted to

$$-R + 1 - R \leq k - R \ \leq \ 2l \ \leq \ k + L < K + L \quad ,$$

because $h_{k-2l} \neq 0$ only for $-L \leq k - 2l \leq R$. If we restrict the range of k to $\{-R + 1, ..., K - 1\}$, then the range of l is restricted to $-R + 1 \leq l < K$. Hence, the Π_r are eigenvectors of $\mathcal{H}^{cT}$. The second claim (29) is an immediate consequence. □

At the present stage we are able to find boundary modifications of scaling functions, which recover polynomials of the same degree as the inner scaling functions. Hence, the approximation properties are preserved. Furthermore one is interested in functions which fulfill specific boundary conditions. The most common conditions are that the functions or their derivatives vanish at the boundary.

In the subsequent lines we will show how to obtain scaling functions with such homogeneous boundary conditions. We assume $\phi \in C^s$ and look for edge scaling functions satisfying

$$\phi_l^{left(\beta)}(0) := \frac{\partial^\beta \phi_l^{left}}{\partial x^\beta}(0) = 0 \quad ; \quad 0 \leq l < K \quad ,$$

where $0 \leq \beta \leq \lfloor s \rfloor$. It is sufficient to find suitable edge functions $\tilde{\Phi}$ with

$$\tilde{\Phi}^{(\beta)}(0) := (\phi_0^{left(\beta)}, ..., \phi_{K-1}^{left\ (\beta)})^T(0) = 0 \quad , \tag{31}$$

since the orthogonalization step $\tilde{\Phi} \to \Phi^{left}$ preserves such homogeneous conditions.

In order to find $\tilde{\Phi}$ we start with other auxiliary edge functions $\hat{\Phi}$ which do not satisfy any boundary condition. If there is a matrix U of appropriate size with $U\hat{\Phi}^{(\beta)}(0) = 0$, then $\tilde{\Phi} = U\hat{\Phi}$ fulfills (31). However, this setting rises the following questions:

- How can $\hat{\Phi}$, U be defined such that a scaling equation similar to (22) holds for $U\hat{\Phi}$?
- How can $\hat{\Phi}$ be defined such that there are K linear independent functions $U\hat{\Phi}$?
- Which polynomials are recovered by $U\hat{\Phi}$?

We will give the answer to the first two questions and postpone the third point to the end of this subsection. There, it will be shown that we do not have to recover *all* polynomials of degree less than N to establish estimates similar to (8). Thus, we can choose K from

$$K \in \{\max\{L, N-1\},\ \max\{L+1, N-1\}\} \quad . \tag{32}$$

Now analogously to the previous construction for $A, D, \tilde{\Phi}$, define the matrices $\hat{A} \in M(K+1, K+R-1)$, $\hat{D} \in M(K+1, K+1)$ by suitable eigenvectors and eigenvalues of $\mathcal{H}^c$ and set

$$\hat{\Phi} := \hat{A}\Phi^c(x) \cdot \chi_{[0,\infty[} \quad .$$

Again the first N rows of $\hat{A}$ are the Π_r from Lemma 3.5. We define $K+1$ functions $\hat{\phi}_r$ instead of K functions as in equation (14). The additional function is used to compensate for the loss of one boundary scaling function induced by the multiplication with U, which will be a $K \times (K+1)$ matrix. With this setting there holds $rg\hat{A} = K+1$. $\hat{A}$ together with $\hat{D}$ satisfies (19).

Lemma 3.6. *Let $U \in M(K, K+1)$ with $rg\ U = K$ and $U\hat{\Phi}^{(\beta)}(0) = 0$. Then the inverse*

$$\hat{V} = \begin{pmatrix} U \\ \hat{\Phi}^{(\beta)T}(0) \end{pmatrix}^{-1}$$

exists. Let $\hat{V} = (V, b)$; $V \in M(K+1, K)$ be a partition of $\hat{V}$. Then

$$A := U\hat{A} \quad \text{and} \quad D := U\hat{D}V$$

satisfy condition (19). In addition for $\tilde{\Phi} = U\hat{\Phi}$ there holds $\tilde{\Phi}^{(\beta)}(0) = 0$.

Proof. By Lemma 3.5 there holds

$$\hat{\phi}_l^{(\beta)}(0) = \delta_{l,\beta} \quad \text{for} \quad 0 \le l < N \quad . \tag{33}$$

Thus, $\hat{\Phi}^{(\beta)}(0) \neq 0$ and $(U^T, \hat{\Phi}^{(\beta)}(0))$ is non-singular. By condition (19) we have

$$\hat{A}\mathcal{H}^c = \hat{D}\hat{A} \quad \Rightarrow \quad U\hat{A}\mathcal{H}^c = U\hat{D}\hat{A} \quad .$$

Assume there holds the relation $U\hat{D} = U\hat{D}VU$. Then clearly

$$A\mathcal{H}^c = U\hat{A}\mathcal{H}^c = U\hat{D}\hat{A} = U\hat{D}VU\hat{A} = DA \quad .$$

This would complete the proof. To show $U\hat{D} = U\hat{D}VU$ we consider

$$\begin{aligned} U\hat{D} = U\hat{D}VU \Leftrightarrow U\hat{D}(V,b) &= U\hat{D}VU\hat{V} \\ &= U\hat{D}V(I_K, 0) = (U\hat{D}V, 0) \quad . \end{aligned}$$

Thus, we have to show $U\hat{D}b = 0$. By definition of $\hat{V}$ there holds $Ub = 0$. Hence, b is a multiple of $\hat{\Phi}^{(\beta)}(0)$. Furthermore evaluation of the derivative of $\hat{\Phi}(x) = \sqrt{2}(\hat{D}\hat{\Phi}(2x) + \hat{A}\mathcal{H}^p\Phi^p(2x))$ at zero yields

$$\hat{\Phi}^{(\beta)}(0) = 2^{\beta+1/2}\hat{D}\hat{\Phi}^{(\beta)}(0) \quad .$$

Hence, $\hat{D}b$ is a multiple of $\hat{\Phi}^{(\beta)}(0)$ and $U\hat{D}b = 0$. □

Now we can consider the approximation properties of $U\hat{\Phi}$. Assume the eigenvalue $2^{-(\beta+1/2)}$ of $\hat{D}$ has multiplicity one. Then

$$\hat{\phi}_l^{(\beta)} = \delta_{l,\beta} \quad ; \quad 0 \le l < K+1 \quad .$$

Multiplication by U simply discards $\hat{\phi}_\beta$ from $\hat{\Phi}$. Hence, the particular polynomial $\frac{x^\beta}{\beta!}$ is not recovered by $U\hat{\Phi}$. Let ψ_l^{left} denote one of the edge wavelets corresponding to the boundary adapted scaling functions. Their proper definition will be given in subsection 3.4. These wavelets will fulfill orthogonality relations

$$\psi_l^{left} \perp \phi_0^{left}, .., \phi_{K-1}^{left}, \phi_K, ... \quad .$$

Hence, we can only expect

$$\int_0^\infty \psi_l^{left} \frac{x^r}{r!} dx = 0 \quad \text{for} \quad 0 \le r < N \quad ; \quad r \ne \beta \quad .$$

The point is that for $u \in C^M$ with $u^{(\beta)}(0) = 0$ ($\beta \le M$) we can again establish the estimate $|d_l^j| \le C 2^{-j(M+1/2)}$. The argument uses Taylor expansion, but now in 0 for all edge wavelets $2^{j/2}\psi_l^{left}(2^j x)$:

$$\begin{aligned} |d_l^j| &= \left| \int_0^\infty u(x) 2^{j/2} \psi_l^{left}(2^j x) dx \right| = 2^{-j/2} \left| \int_0^\infty u(2^{-j}y) \psi_l^{left}(y) dy \right| \\ &= 2^{-j/2} \left| \int_0^\infty \left[\sum_{r=0}^{M-1} \frac{u^{(r)}(0)}{r!} (2^{-j}y)^r + \frac{u^{(M)}(\eta(j,y))}{M!} (2^{-j}y)^M \right] \psi_l^{left}(y) dy \right| \\ &= 2^{-j/2} \left| \int_0^\infty \left[\frac{u^{(\beta)}(0)}{\beta!} (2^{-j}y)^\beta + \frac{u^{(M)}(\eta(j,y))}{M!} (2^{-j}y)^M \right] \psi_l^{left}(y) dy \right| \qquad (34) \\ &= \frac{2^{-j(M+1/2)}}{M!} \left| \int_{\mathbf{R}} u^{(M)}(\eta(j,y)) y^M \psi_l^{left}(y) dy \right| \quad . \end{aligned}$$

For the appropriate subspace of L_2, the asymptotic behaviour of the wavelet coefficients is not affected by the restricted degree of polynomial recovery.

3.3 MRA on the Interval

In this section we carry the ideas of the previous subsections over to the unit interval $[0,1]$. In subsections 3.1 and 3.2 we have defined scaling functions Φ^{left} adapted to $[0,\infty[$. In a similar fashion we can define scaling functions Φ^{right} adapted to $]-\infty,1]$. Then, for sufficiently large dilation $2^j x$, both $\Phi^{left}(2^j x)$ and $\Phi^{right}(2^j x)$ have support in $[0,1]$. Thus, we can use certain scaling functions of $V_j^{[0,\infty[}$ and $V_j^{]-\infty,1]}$, respectively, to span $V_j^{[0,1]}$. This will be called the merging step. Figure 1 depicts the basic idea. For reasons of simplicity the scaling functions are sketched as linear functions. Note, that the restriction to sufficiently large dilations avoids the technical complications of e.g. adapting Φ^{left} to the right edge.

Instead of rewriting the boundary construction for the right edge, we apply all the steps (definition of $\hat{\Phi}$, enforcement of boundary conditions and orthogonalization) to mirrored scaling functions [8], [21]. To this end, consider the mirror operator

$$\mathcal{T}_\alpha : L_2(\mathbf{R}) \to L_2(\mathbf{R}) \quad ; \quad \mathcal{T}_\alpha f(x) = f(\alpha - x) \quad .$$

Here, $\alpha \in \{0,1\}$ is a parameter which is associated to the finite dimension of the subspaces $V_j^{[0,1]}$ (the role of α will be explained in more details at the end of this subsection). In this setting the function $\phi^{\mathcal{T}} := \mathcal{T}_\alpha \phi$ satisfies a scaling equation (2) with mask $\{h_l^{\mathcal{T}} := h_{\alpha - l}\}$. Furthermore this function and the mask $\mathbf{h}^{\mathcal{T}}$, respectively, have support

$$supp\ \phi^{\mathcal{T}} = [-L^{\mathcal{T}}, R^{\mathcal{T}}] \quad ; \quad L^{\mathcal{T}} := R - \alpha \quad , \quad R^{\mathcal{T}} := L + \alpha \quad .$$

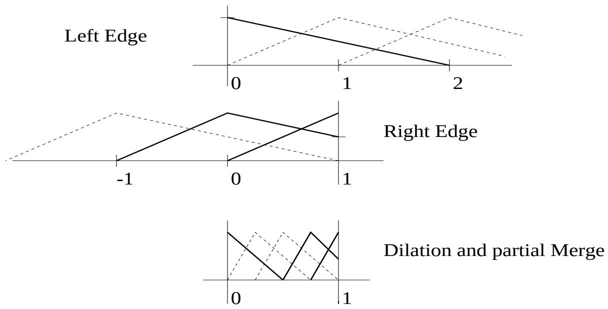

Figure 1: Merge of left and right MRA

With this mask we arrive at edge scaling functions $\Phi^{\mathcal{T},left} \in L_2([0,\infty[)$, which involve scaling matrices $H^{\mathcal{T}}$ and $E^{\mathcal{T}}$ similar to H and E in (22). For convenience we will denote the number of these edge scaling functions by K^{right}.

Application of $\mathcal{T}_1$ to $\Phi^{\mathcal{T},left}, \phi^{\mathcal{T}}_{K^{right}}, \phi^{\mathcal{T}}_{K^{right}+1}, \ldots$ yields the desired MRA of $L_2(]-\infty,1])$ with boundary adapted scaling functions

$$\Phi^{right} := \mathcal{T}_1 \Phi^{\mathcal{T},left} \quad .$$

Furthermore the inner scaling functions, e.g. $\mathcal{T}_1\phi^{\mathcal{T}}_{K^{right}}$, $\mathcal{T}_1\phi^{\mathcal{T}}_{K^{right}+1}, \ldots$, are again translates of ϕ. For j sufficiently large, i.e. more precisely

$$2K + R - 2 \;\leq\; 2^j - \alpha - K^{right} \qquad \text{and} \qquad K \;\leq\; 2^j - \alpha - (2K^{right} + R^{\mathcal{T}} - 2) \quad ,$$

we can apply the merging step and define vectors of dilated functions

$$\Phi^j(x) := 2^{j/2}(\phi^{left}_0, .., \phi^{left}_{K-1}, \phi_K, ..., \phi_{2^j-\alpha-K^{right}}, \phi^{right}_{K^{right}-1}, .., \phi^{right}_0)^T(2^j x) \quad . \tag{35}$$

These vectors form a basis of $V_j^{[0,1]}$. In addition, $\{V_j^{[0,1]}\}_{j\geq j_0}$ is a sequence of nested subspaces, since there holds the scaling equation

$$\Phi^j(x) = \sqrt{2}\mathcal{M}_j \Phi^{j+1}(x) \quad , \tag{36}$$

where $\mathcal{M}_j$ is the $(2^j + 1 - \alpha) \times (2^{j+1} + 1 - \alpha)$ matrix

$$\mathcal{M}_j = \left(\begin{array}{c|c|c} H & E & \\ \hline & \mathcal{H}_j & \\ \hline & E^{right} & H^{right} \end{array}\right) \quad ; \quad \mathcal{H}_j = (h_{r-2l})_{\substack{K \leq l \leq 2^j - \alpha - K^{right} \\ K \leq r \leq 2^{j+1} - \alpha - K^{right}}} \quad .$$

The matrices H^{right} and E^{right} are simply the above mentioned matrices $H^{\mathcal{T}}$ and $E^{\mathcal{T}}$ with rows and columns in reversed order.

By counting the number of basis functions it turns out that the subspaces $V_j^{[0,1]}$ have dimension $2^j + 1 - \alpha$. For $\alpha = 1$ we obtain 2^j scaling functions. Such powers of 2 are required for many applications of wavelets (wavelet packets, image compression). On the other hand the construction of subspaces by dilation with a factor 2 induces dyadic grids with mesh size 2^{-j}. These grids have 2^j+1 grid points. Thus an identification of a scaling function ϕ_l^j with one of the dyadic points $l2^{-j}$ is not possible. The result of the merging step is sketched in Figure 2 for this situation. Identification of the scaling functions with

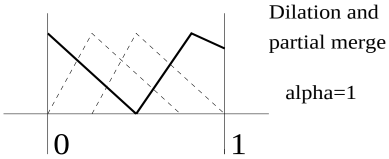

Figure 2: Merge of left and right MRA for $\alpha = 1$

dyadic points is desired, e.g. for Coiflets, where the inner products $< u, \phi_l^j >$ can be approximated by $u(l2^{-j})$ for $K \le l \le 2^j - \alpha - K^{right}$.

3.4 Wavelets on the Interval

Our next concern are wavelets which span the complementary space

$$W_j^{[0,1]} = (I - P_{V_j^{[0,1]}})V_{j+1}^{[0,1]} \quad , \tag{37}$$

where P_X denotes the orthogonal projection onto X. Similar to the scaling functions, we want to preserve as much as possible from the original MRA. From relation

$$\psi_k^j(x) = \sqrt{2} \sum_{2k-R+1 \le r \le 2k+L+1} g_{r-2k}\phi_r^{j+1}(x) \tag{38}$$

we infer that for $\lfloor \frac{K+R}{2} \rfloor \le k \le 2^j - 1 - \lfloor \frac{K^{right}+R^{\mathcal{T}}}{2} \rfloor$ the wavelets ψ_k^j are linear combinations of the inner scaling functions ϕ_r^{j+1} of the next finer level. Since $dim W_j^{[0,1]} = 2^j$, it is sufficient to determine at the left edge $\hat{K} := \lfloor \frac{K+R}{2} \rfloor$ and at the right edge $\hat{K}^{right} := \lfloor \frac{K^{right}+R^{\mathcal{T}}}{2} \rfloor$ boundary adapted wavelets. We present the construction for the left edge only, since the mirror trick of the previous subsection applies here as well. It will be convenient to introduce the infinite matrix $\mathcal{G}^p$

$$(\mathcal{G}^p)_{l,r} \quad = \quad \begin{cases} g_{r-2l} & \min\{K, L+1\} \le r \\ 0 & 0 \le r < \min\{K, L+1\} \end{cases} \qquad 0 \le l \le \hat{K} \quad .$$

Like the matrix $\mathcal{H}^p$ from (25), $\mathcal{G}^p$ is a staggered banded matrix, but here no row vanishes. Easy to derive properties of $\mathcal{G}^p$ are

$$rg\ \mathcal{G}^p = \hat{K} \quad \text{and} \quad \mathcal{G}^p_{\cdot,r} = 0 \quad \text{for} \quad r \ge K + R - 1 \quad .$$

Hence, all columns $\mathcal{G}^p_{.,r}$ vanish, except those with $\min\{K, L+1\} \le r \le K+R-2$. With help of $\mathcal{G}^p$ we can define auxiliary edge wavelets $\tilde{\Psi}$, which share most of the properties of the final edge wavelets Ψ^{left}.

Lemma 3.7. *Let $Q \in M(n, \hat{K})$ be the (possibly empty) matrix whose columns span the null space of $(H,E)(\mathcal{G}^p_{.,0..K+R-2})^H \in M(K, \hat{K})$, i.e.*

$$(H,E)(\mathcal{G}^p_{.,0..K+R-2})^H Q = 0 \quad \text{and} \quad n = dim\ Ker(H,E)(\mathcal{G}^p_{.,0..K+R-2})^H \quad . \tag{39}$$

Then for

$$\tilde{\Psi} := W \begin{pmatrix} \Phi^{left} \\ \Phi^p \end{pmatrix} := \left(\begin{array}{c|c} I_K - H^H H & -H^H E \\ \hline \multicolumn{2}{c}{Q^H \mathcal{G}^p_{.,0..K+R-2}} \end{array} \right) \begin{pmatrix} \Phi^{left} \\ \Phi^p \end{pmatrix} \tag{40}$$

there holds

(i) $\tilde{\Psi} \perp V_0^{[0,\infty[}$,

(ii) $\tilde{\Psi} \perp \psi_k \ \ (k \ge \hat{K})$,

(iii) $rg\ W \ge \hat{K}$.

Before we prove this lemma we show how to define edge wavelets Ψ^{left} with this prerequisites. Consider the Schur decomposition of WW^H

$$T\ diag(\mu_1, ..., \mu_{K+n})\ T^H = WW^H \quad .$$

Without loss of generality we can assume $\mu_1 \ge ... \ge \mu_{\hat{K}} > 0$. Then Ψ^{left} is defined by

$$\Psi^{left}(x) = \sqrt{2}\ diag(\mu_1, ..., \mu_{\hat{K}})^{-1/2} S T^H W \begin{pmatrix} \Phi^{left} \\ \Phi^p \end{pmatrix} (2x) \quad , \tag{41}$$

where $S = (I_{\hat{K}} \mid 0) \in M(\hat{K}, K)$. This is the orthogonalization and extraction of $\hat{K}$ linear independent wavelets from $\tilde{\Psi}$. Note, that the role of the Gram matrix is played here by the diagonal matrix $diag(\mu_1, ..., \mu_{\hat{K}})$.

Proof. Within the setting of a MRA of $L_2([0,\infty[)$ it is convenient to define the infinite vector of functions $\Phi^\infty := (\phi_0^{left}, .., \phi_{K-1}^{left}, \phi_K, ...)^T$, for which the scaling equation derived in the previous subsections reads

$$\Phi^\infty(x) = \sqrt{2}\mathcal{M}_\infty \Phi^\infty(2x) \quad ; \quad \mathcal{M}_\infty = \left(\begin{array}{c|c} H & (E,\ 0..) \\ \hline 0 & \mathcal{H}_\infty \end{array} \right) \quad .$$

Here $\mathcal{H}_\infty = (h_{r-2l})_{K \le l,r}$ denotes the infinite version of $\mathcal{H}_j$ from equation (36). According to (37) the complementary space $W_0^{[0,\infty[}$ has a generating system

$$\begin{aligned} (I - P_{V_0^{[0,\infty[}})\Phi^\infty(2x) &= \Phi^\infty(2x) - \int \Phi^\infty(2y)(\Phi^\infty)^H(y)dy\Phi^\infty(x) \\ &= (I - \mathcal{M}_\infty^H \mathcal{M}_\infty)\Phi^\infty(2x) \\ &= \left(\begin{array}{c|c} I_K - H^H H & -H^H(E, 0\ ..) \\ \hline -(E,0\ ..)^H H & I - (E,0\ ..)^H(E,0\ ..) - \mathcal{H}_\infty^H \mathcal{H}_\infty \end{array} \right) \Phi^\infty(2x) \quad . \end{aligned}$$

In the following steps of the proof matrix products will occur which involve parts of the bi-infinite matrices $(h_{r-2l})_{l,r\in\mathbf{Z}}$ and $(g_{r-2l})_{l,r\in\mathbf{Z}}$. Examples for such parts are $\mathcal{H}_\infty$ and $\mathcal{G}^p$. The products can be evaluated using the matrix products

$$\sum_{i\in\mathbf{Z}} h_{i-2l}h_{r-2i} = \sum_{i\in\mathbf{Z}} g_{i-2l}g_{r-2i} = \delta_{l,r} \quad \text{and} \sum_{i\in\mathbf{Z}} h_{i-2l}g_{r-2i} = 0 \quad ; \quad l,r\in\mathbf{Z} \quad . \tag{42}$$

These relations are the orthogonality relations (3) in terms of the coefficients $\{h_k\}$ and $\{g_k\}$. From (42) follows $(\mathcal{G}^p_{.,K..\infty})\mathcal{H}^H_\infty = 0$ and therefore

$$W = \left(\frac{I_K \mid 0...}{Q^H\mathcal{G}^p}\right)(I - \mathcal{M}^H_\infty\mathcal{M}_\infty) \quad .$$

Hence, the wavelets $\tilde{\Psi}$ are linear combinations of $(I - P_{V_0^{[0,\infty[}})\Phi^\infty(2x) \in W_0^{[0,\infty[}$. This proves the first assertion **(i)**.

In a similar fashion we prove the second assertion **(ii)**. The inner wavelets are related to $\Phi^\infty(2x)$ with the infinite matrix $\mathcal{G}_\infty = (g_{r-2l})_{\hat{K}\le l\ ,\ 0\le r}$ by

$$\begin{pmatrix} \psi_{\hat{K}} \\ \psi_{\hat{K}+1} \\ \vdots \end{pmatrix}(x) = \sqrt{2}\mathcal{G}_\infty\Phi^\infty(2x) \quad .$$

The first K columns of $\mathcal{G}_\infty$ are zeros, since the inner wavelets are linear combinations of the inner scaling functions $\phi_k(2x)$ $(k \ge K)$ only. From (42) one obtains

$$\mathcal{H}_\infty\mathcal{G}^H_\infty = 0 \quad , \quad \mathcal{G}^p\mathcal{G}^H_\infty = 0 \quad \text{and} \quad (H, E, 0...)(\mathcal{G}_\infty)^H = 0 \quad .$$

Thus, $W\mathcal{G}^H_\infty = 0$ and $\tilde{\Psi}\ \perp\ \psi_k$ $(k \ge \hat{K})$.

It remains to show that $rg\ W \ge \hat{K}$. First we will estimate the rank of the upper left block $I_K - H^H H$ of W. Let $Z \in M(m, \hat{K})$ be the (possibly empty) matrix, whose columns span the left kernel of $I_K - H^H H$, i.e

$$Z(I_K - H^H H) = 0 \quad .$$

Obviously there holds $rg\ Z = rg\ ZH^H H = rg\ ZH^H$ and $(Z)H^H = (ZH^H H)H^H$. From the later relation we obtain

$$\begin{aligned} ZH^H = ZH^H(HH^H) \stackrel{(22),(23)}{=} ZH^H(I_K - EE^H) &\Leftrightarrow ZH^H(EE^H) = 0 \\ &\Leftrightarrow (ZH^H)E = 0 \quad . \end{aligned}$$

Hence, $rg\ E \le K - rg\ Z$ and therefore $rg\ (I_K - H^H H) \ge rg\ E$. Now we have to show that the lower rows of W increase the rank of W in the desired manner. Here, we consider two different cases:

(1) $K \in \{L, L+1\}$: In this case the first K columns of $\mathcal{G}^p$ vanish. Thus, there holds $(H, E)\mathcal{G}^{pH} = (0, E)\mathcal{G}^{pH}$. From this follows $rg\ (H, E)\mathcal{G}^{pH} \le rg\ E$ and therefore

$$\hat{K} - rg\ E \le rg\ Q = rg\ (\mathcal{G}^p_{\cdot,0..K+R-2})^H Q \quad .$$

The lower left block $Q^H \mathcal{G}^p_{\cdot,0..K-1}$ of W vanishes. The upper left block $I_K - H^H H$ has rank $\ge rg\ E$ and the lower right block $Q^H \mathcal{G}^p_{\cdot,K..K+R-2}$ has rank $\ge \hat{K} - rg\ E$. Thus, we have $rg\ W \ge \hat{K}$.

(2) $L + 2 \le K \in \{N-1,\ N\}$: In this case there holds $rg\ E = \min\{K,\ \hat{K}\}$. This follows from the particular form of $\mathcal{H}^p$ (see (25)) and from the linear independence of the last K columns of A. The later property holds, because the entries of the last K columns of A are the values of K linear independent polynomials at K different arguments.

For $K \ge \hat{K}$ this yields $rg\ W \ge rg\ (I_K - H^H H) \ge rg\ E = \hat{K}$.

The case $K < \hat{K}$ is a little bit more complicated. We use $rg\ W = rg\ WW^H$. Again by equations (22) and (23) one finds

$$WW^H = \left(\begin{array}{c|c} I_K - H^H H & M^H \\ \hline M & Q^H \mathcal{G}^p \mathcal{G}^{pH} Q \end{array}\right) \qquad \text{where} \qquad M = Q^H \mathcal{G}^p_{\cdot,0..K-1} \quad .$$

Since $rg\ (I_K - H^H H) = rg\ E = K$ one can perform a block elimination step, which does not change the rank, i.e. we obtain

$$\left(\begin{array}{c|c} I_K - H^H H & M^H \\ \hline 0 & Q^H \mathcal{G}^p \mathcal{G}^{pH} Q - M(I_K - H^H H)^{-1} M^H \end{array}\right) \quad .$$

The lower right block can be written as

$$Q^H \mathcal{G}^p \left(I - \left(\begin{array}{c|c} (I_K - H^H H)^{-1} & 0 \\ \hline 0 & 0 \end{array}\right)\right) \mathcal{G}^{pH} Q \quad . \tag{43}$$

Zero is not an eigenvalue of $H^H H$ in the present case $K \in \{N-1, N\}$, since zero is not an eigenvalue of $H = (G^S)^{-1/2} D (G^S)^{+1/2}$. This follows from the particular setting of D ($\hat{D}$) in Lemma 3.5. Therefore, the inner expression $(I - (\boxplus))$ of (43) is non-singular. Hence, the rank of (43) is equal to $rg\ Q^H \mathcal{G}^p = \hat{K} - K$. This completes the proof.

□

3.5 Modifications and Numerical Stability of the Scheme

If one considers the previous results, it appears that there are still free parameters in the construction. E.g., for $K > N$ the choice of the $K - N$ eigenvalues and eigenvectors different from those in Lemma 3.5 is free. Another possibility is dealt with in the following

Lemma 3.8. *Let $U \in M(K,K)$ be any non-singular matrix. Then $A^U := UA$ and $D^U := UDU^{-1}$ satisfy the prerequisites of Lemma 3.2. One can define modified auxiliary edge scaling functions $\tilde{\Phi}^U := U\tilde{\Phi}$. The Gram matrix G^{SU} of $\tilde{\Phi}^U$ is $G^{SU} = UG^SU^H$.*

This lemma may be used to improve the numerical stability of the boundary construction. But, first we will deal with some technical problem:

If one builds the boundary modifications of real scaling functions with real mask $\mathbf{h}$, one wants to obtain real edge scaling functions. However, if $K > N$ it is possible that there are complex entries in D and A. With help of Lemma 3.8 one can define real auxiliary edge scaling functions $\tilde{\Phi}^U$ with real masks. For reasons of simplicity assume that the Jordan matrix J in (20) is a diagonal matrix. It is necessary to choose not only the complex eigenvalue λ_1 but also its conjugate $\lambda_2 = \bar{\lambda}_1$ to build D. This is always possible if K is properly chosen within the bounds of conditions (12) or (32). Without loss of generality let $D = diag(\lambda_1, \lambda_2, *)$. The following matrix U removes the imaginary parts introduced by $\lambda_1,\ \lambda_2$ in D:

$$U = \left(\begin{array}{cc|c} -\frac{1}{\lambda_2} & -\frac{1}{\lambda_1} & \\ -1 & 1 & \\ \hline & & I_{K-2} \end{array}\right) \quad ; \quad D^U = \left(\begin{array}{cc|c} 0 & 1 & \\ -\lambda_1\lambda_2 & -(\lambda_1+\lambda_2) & \\ \hline & & * \end{array}\right) \quad .$$

Whenever $\mathcal{H}^c$ has only real entries, the eigenvectors v_i $(i = 1, 2)$ associated to λ_i are also conjugate. Hence, the first two rows of A^U are also real. D^U is not a lower triangular matrix and algorithm (24) does not apply to obtain G^{SU}. But we can first compute G^S by (24) and then G^{SU} by UG^SU^H.

An important criterion for the quality of the construction of wavelets on the interval is the numerical stability. The critical point is that ill conditioned Gram matrices G^S and $G^W = diag(\mu_1, .., \mu_{\hat{K}}$ (see (41)) lead to a loss of accuracy of the coefficients $H,\ E,\ ...$ and therefore, to deviations from exact L_2-orthogonality. Successive application of the wavelet transform (WT) and its inverse (IWT) to some data $\{c_k^J\}$ introduces round-off errors

$$\epsilon = \frac{\sum_l |(IWT \circ WT\{c_k^J\})_l - c_l^J|^2}{\sum_l |c_l^J|^2} \quad , \tag{44}$$

which can be larger by orders of magnitude than the machine accuracy.

One may wonder whether we can find U such that G^{SU} and G^W are well conditioned. Clearly there is U such that $G^{SU} = I$ (by Cholesky decomposition of G^S), but then the problems are shifted to the computation of U and the gain is rather doubtful. Another problem arises in the construction of wavelets, i.e. in (41). Here

$$\begin{aligned} I - H^H H &= I - (G^{SU})^{1/2} D^{UH} (G^{SU})^{-1} D^U (G^{SU})^{1/2} \\ &\sim I - D^{UH}(G^{SU})^{-1} D^U G^{SU} \\ &\sim I - D^H (G^S)^{-1} D G^S \quad , \end{aligned} \tag{45}$$

where $\sim$ denotes spectral equivalence. Hence, e.g. for $K = N$ the condition number of $G^W = I - H^H H$ does not depend on U. In such cases no improvement of the calculation of the wavelet filter coefficients can be achieved.

In our numerical experiments we tested different preconditioning matrices U.

- In [5],[8] the entries of A are the values of the polynomials $\tilde{P}_r(x) := \binom{x+N-1}{r}$ $(0 \leq r < N)$ instead of the values of P_r. This replacement corresponds to $U = U^{\tilde{P}} := blockdiag(\tilde{P}P^{-1}, I_{K-N})$. P and $\tilde{P}$ denote the matrices which transform the monomials to $\{P_r\}_{0\leq r<N}$ and $\{\tilde{P}_r\}_{0\leq r<N}$ respectively.
- In [21] the polynomials $\{P_r\}$ are used for the auxiliary scaling functions. Thus, we tested $U = I$. However, note that in [21] the edge wavelets are defined differently from (40), (41). More precisely, in (40) the auxiliary edge wavelets are defined from the orthogonal scaling functions, while in [21] they are defined from the auxiliary edge scaling functions $\tilde{\Phi}$. The comparison between the latter setting and (40), (41) will be given at the end of this subsection.
- We also tested $U = U^B := diag^{-1}(\sum_i |B_{l,i}|)$. This is based on the observation that for the smallest eigenvalue of G^{SU} there holds $\lambda_{min}(G^{SU}) \geq \lambda_{min}(UBB^HU^H)$.

Note that, [5],[8] and [21] treat only Daublets and Symmlets with least asymmetric support $(K = N = R = L+1)$. We emphasize this restriction, since our present general framework allows to shift the support of the scaling functions. This plays an important role for the quality the quadrature methods as described in the next section.

The numerical computation of the filter coefficients H, E, .. was carried out on a SGI workstation with `MATLAB`'s double precision matrix routines. Condition numbers were computed using the `cond` routine which employs a SVD. For Symmlets and Daublets we used the masks $\mathbf{h}$ computed with the `MATLAB` tool-box *Uvi_wave* [25]. For Coiflets($N = 2, 4$) we used the coefficients reported in [13], while for Coiflets($N = 6$) we used the more accurate coefficients from [3]. For $N = 8$ we could not obtain coefficients good enough to recover the exact eigenvalues of $\mathcal{H}^c$ with reasonable accuracy (see Lemma 3.5). Thus, no results for $N = 8$ are presented. For the Coiflet family there are different scaling functions with the same degree of polynomial recovery. Our experience is that those with least asymmetric support yield the most stable constructions. We present only these results. The auxiliary scaling functions and therefore the Gram matrices G^{SU} depend on the particular choice of the matrix D (see Lemma 3.3). To achieve a high accuracy, our implementation tests the different possibilities with respect to the condition numbers of the Gram matrices. This is done by a simple loop over all possible D's. The particular D, which leads to a minimum of $\kappa(G^{SU}) + \kappa(G^W)$ is selected.

The numerical experiments concerning the round-off errors (44) were carried out with double precision. These experiments were repeated with smooth and oscillating data $\{c_k^J\}$, but the order of magnitude of the round-off errors was almost independent of the data.

In Table 1 condition numbers and round-off errors are shown for different wavelets on the interval (without specific boundary conditions) and for different U. The boundary scaling functions (wavelets) differ for the left and right boundary due to the asymmetry of ϕ and ψ. Thus we present the condition numbers for both boundaries. In the following we discuss the major results of Table 1:

- In all experiments the condition numbers of G^{SU} for $U = U^{\tilde{P}}$ are dramatically higher than for $U = I$ or $U = U^B$ respectively. The loss of accuracy of the filter coefficients leads to large round-off errors. A sufficiently accurate calculation of the boundary filter coefficients is possible only for a degree of polynomial recovery lower than that for $U = I$ or $U = U^B$. Therefore, we can not recommend the choice $U = U^{\tilde{P}}$.

  Note, that all results for Symmlets(8) with $U = U^{\tilde{P}}$ are questionable, since the ill-conditioning of G^{SU} affects the computation of the edge wavelets and thus G^W. However, even for $N \leq 6$, where double precision is sufficient for a reasonable accuracy, $U = U^{\tilde{P}}$ yields worse results than $U = I$ or $U = U^B$.

- For Symmlets the condition numbers and round-off errors are similar for $U = I$ and $U = U^B$ ($U = I$ is slightly better). For Coiflets the third approach $U = U^B$ yields the better results. But, the differences are rather small.

- For Symmlets the condition numbers of G^W do not change for different U ($K = N$). For Coiflets things are a little bit more complicated. There it holds that $\hat{K} \leq K$ and $\hat{K}^{right} \leq K^{right}$. Hence, the eigenvalues μ_i of the diagonal blocks $I - H^H H$ and $Q^T \mathcal{G}^p \mathcal{G}^{pH} Q$ of WW^H are not affected by U. But, they are affected by the particular choice of D (see the remarks above). In our case the condition numbers $\kappa(G^{Wleft})$ and $U = I$, $U^{\tilde{P}}$ agree (same D), but differ from those for $U = U^B$ (different D).

- The condition numbers of G^{SU} are better or of the same order than those of G^W. Hence, even significant improvements in the conditioning of G^{SU} can not significantly improve the complete construction.

We finish this subsection with the comparison of the condition numbers of G^W with respect to the construction of subsection 3.4 (present scheme) and the one of [21]. There, the auxiliary edge wavelets are defined from the auxiliary scaling functions $\tilde{\Phi}$. In the present scheme this is done using the orthogonal scaling functions Φ^{left} and, in addition, a number of inner scaling functions to maintain sufficiently many linear independent auxiliary edge wavelets.

The condition numbers for both approaches are given in Table 2. One observes that the present scheme is much more stable than that of [21]. This allows for the boundary adaptation of scaling functions with higher degree of polynomial recovery than [21].

To summarize all results of this section one can say that we have generalized the previous schemes for wavelets on the interval proposed in [5], [8] and [21]. Our generalized scheme can be applied to a very broad class of compactly supported orthogonal wavelets, since no direct relation between the degree of polynomial recovery and the support of the scaling function, i.e. $N = R = L + 1$ is required. In particular this allows for the boundary

	N	U	scaling functions G^{SUleft}	scaling functions $G^{SUright}$	wavelets G^{Wleft}	wavelets G^{Wright}	round-off e. ϵ
Symmlets	2	$U^{\tilde{P}}$	9.738e+00	2.638e+02	2.094e+01	8.281e+01	1.025e-27
		I	1.888e+01	2.053e+01	2.094e+01	8.281e+01	2.285e-29
		U^B	1.583e+01	1.652e+01	2.094e+01	8.2812e+01	5.8098e-29
	3	$U^{\tilde{P}}$	1.485e+06	1.988e+02	1.252e+03	5.847e+02	1.143e-20
		I	1.078e+03	5.859e+02	1.252e+03	5.847e+02	1.759e-26
		U^B	4.596e+02	3.721e+02	1.252e+03	5.847e+02	2.609e-26
	4	$U^{\tilde{P}}$	1.220e+07	5.501e+05	1.279e+04	1.033e+05	6.574e-19
		I	1.184e+04	9.598e+03	1.279e+04	1.033e+05	1.560e-23
		U^B	1.772e+04	1.040e+04	1.279e+04	1.033e+05	3.277e-23
	5	$U^{\tilde{P}}$	4.0105e+09	1.891e+08	4.994e+05	3.001e+05	3.4445e-13
		I	2.1515e+05	4.227e+05	4.994e+05	3.001e+05	1.5859e-20
		U^B	3.2026e+05	4.206e+05	4.994e+05	3.001e+05	5.2201e-20
	6	$U^{\tilde{P}}$	1.085e+11	1.522e+12	1.565e+07	3.629e+07	1.585e-07
		I	1.021e+07	1.277e+07	1.565e+07	3.629e+07	6.586e-16
		U^B	1.039e+07	1.039e+07	1.565e+07	3.629e+07	1.941e-15
	7	$U^{\tilde{P}}$	3.8215e+15	1.5733e+13	5.3581e+07	6.0980e+08	2.3113e-01
		I	8.2813e+08	3.3709e+08	8.8201e+07	6.0980e+08	2.9712e-12
		U^B	1.1687e+09	2.7213e+08	8.8201e+07	6.0980e+08	5.8081e-12
	8	$U^{\tilde{P}}$	2.710e+16	3.493e+17	1.928e+08	1.114e+07	5.409e-01
		I	1.358e+10	1.688e+10	2.734e+10	8.419e+09	3.655e-10
		U^B	1.218e+10	1.972e+10	2.734e+10	8.419e+09	1.161e-08
Coiflets	2	$U^{\tilde{P}}$	1.114e+02	1.680e+01	1.204e+03	2.278e+01	2.003e-24
		I	1.664e+01	1.680e+01	1.204e+03	2.278e+01	1.970e-24
		U^B	1.724e+01	1.601e+01	1.204e+03	2.278e+01	1.971e-24
	4	$U^{\tilde{P}}$	1.286e+07	6.473e+05	3.542e+04	1.149e+05	1.263e-19
		I	2.720e+05	1.095e+04	3.706e+04	1.149e+05	1.100e-21
		U^B	3.914e+04	1.201e+04	3.542e+04	1.149e+05	4.681e-22
	6	$U^{\tilde{P}}$	1.283e+13	9.119e+10	1.066e+10	2.642e+07	3.149e-07
		I	3.720e+11	1.170e+07	2.921e+10	2.642e+07	4.107e-08
		U^B	7.049e+07	1.141e+07	1.066e+10	2.642e+07	3.592e-11

Table 1: spectral condition numbers of Gram matrices and round-off errors (44)

		scheme [21]		present scheme	
	N	G^{Wleft}	G^{Wright}	G^{Wleft}	G^{Wright}
Symmlets	2	6.912e+01	6.181e+01	2.094e+01	8.281e+01
	3	2.133e+02	3.003e+04	1.252e+03	5.847e+02
	4	1.536e+05	1.505e+07	1.279e+04	1.033e+05
	5	4.703e+07	3.598e+08	4.994e+05	3.001e+05
	6	5.477e+10	9.652e+10	1.565e+07	3.629e+07
	7	8.936e+11	2.618e+13	5.358e+07	6.098e+08
	8	9.012e+15	1.028e+14	2.734e+10	8.419e+09

Table 2: condition numbers for different definitions of the wavelets

modification of Coiflets, which play an important role in numerical applications. Furthermore, we have shown that our scheme yields significant improvements in the numerical stability of the construction of the boundary adapted wavelets. Hence, one can achieve more accurate filter coefficients and deal with scaling functions which recover polynomials of a higher degree than before.

4 Quadrature Formulae

As mentioned already in the second section the knowledge of (boundary adapted) scaling functions and wavelets is not sufficient in numerical applications. To apply the fast wavelet transform to $u \in L_2(\mathbf{R})$ or $u \in L_2([0,1])$ the inner products $c_l^J =< u, \phi_l^J >$ for $l \in \Sigma$ (Σ some index set) are required. But in many practical applications only discrete nodal values $\{u(x_l^J)\}_{l\in\Sigma}$ are given. Hence, one has to find an approximation $\tilde{u}$ for which the exact evaluation of $\tilde{c}_l^J :=< \tilde{u}, \phi_l^J >$ is carried out. Then, the resulting values $\tilde{c}_l^J$ are used as approximations to c_l^J. To keep things simple the approximation step $\{u(x_l^J)\} \to \tilde{u}$ should be linear and $\tilde{u}$ should be in V_J, i.e we are looking for a matrix $Q \in M(\Sigma, \Sigma)$ (Σ some suitable index set) such that

$$\tilde{u}(x) = (u(x_l^J))_{l\in\Sigma}^T Q \Phi^J(x) \quad . \tag{46}$$

In case of a MRA of $L_2(\mathbf{R})$ and dyadic grid points ($x_l^J = l2^{-J}$) ($l \in \mathbf{Z}$) it is convenient to use for this task a bi-infinite Toeplitz matrix, i.e

$$Q_{l,k} = w_{l-k} \quad ; \quad l, k \in \mathbf{Z} \quad , \tag{47}$$

where the w_{l-k} are some weights. A common choice is to use weights which lead to the exact quadrature $c_l^J = \tilde{c}_l^J$ if u is a polynomial of a certain degree. In subsection 4.2 we will describe the modification of this approach to boundary scaling functions in more detail. In the following we directly analyze the approximation error $u - \tilde{u}$ for more general matrices Q.

Lemma 4.1 (Theorem 2.4 [24]). *If $\varphi_l(x) := (Q\Phi^J)_l(x)$ fulfills the relation*

$$\forall y \in supp\ \Phi^J: \quad \sum_{l\in\mathbf{Z}} (y-l)^r \varphi_l(y) = \sum_{l\in\mathbf{Z}} (-l)^r \varphi_l(0) = const(r) \quad 0 \le r < N \quad , \tag{48}$$

and $u \in C^M$ $(M \le N)$ then there holds for $x \in supp\ \Phi^J$

$$\tilde{u}(x) = 2^{J/2} \sum_l \varphi_l(0) u(x + 2^J l) + \mathcal{O}(2^{-J(M+1/2)}) \quad . \tag{49}$$

Note, that if we are working on an interval, we identify u in (49) with it's smooth extension onto $\mathbf{R}$.

Equation (49) implies that $\tilde{u}$ is as smooth as u, except for the higher order term. Whenever we are interested in the compression properties of the wavelet transform only, the size of the quadrature error $|c_l^J - \tilde{c}_l^J|$ is of minor interest. It is more important to preserve the compression properties of the wavelet coefficients d_l^j gained from $\{c_l^J\}$. From (49) follows that for the wavelet coefficients $\tilde{d}_l^j$ with respect to $\{\tilde{c}_l^J\}$ there holds

$$\exists C: \ |d_l^j| \le C 2^{-j(M+1/2)} \quad \Rightarrow \quad \exists \tilde{C}: \ |\tilde{d}_l^j| \le \tilde{C} 2^{-j(M+1/2)} \quad \text{for } 0 \le j < J \quad .$$

This expresses the preservation of the asymptotic compression properties. In particular we have

$$|d_l^j - \tilde{d}_l^j| \le (C + \tilde{C}) 2^{-j(M+1/2)} \quad .$$

For the compactly supported scaling functions ϕ considered in this paper there holds (48), i.e

$$\forall y \in \mathbf{R}: \quad \sum_{l\in\mathbf{Z}} (y-l)^r \phi_l(y) = \sum_{l\in\mathbf{Z}} (-l)^r \phi_l(0) \quad . \tag{50}$$

Thus, for the circulant matrix Q of (47), equation (48) holds also for $\varphi_l = \sum_k w_{l-k}\phi_k$.

But, clearly from the point of compression, there is no need to use other weights than $w_q = \delta_{0,q}$, which correspond to a one point quadrature formula. Unfortunately the edge scaling functions Φ^{left} and Φ^{right} do not fulfill (48). Therefore, e.g. the one point quadrature employed in the boundary adapted MRA may lead to strong wavelet coefficients $\tilde{d}_l^j$ next to the boundaries on all scales. An example for this effect is given in Figures 3 and 4. The right figure shows the magnitude of the $\tilde{d}_l^j$ for different scales j. The coefficients according to the inner wavelets strongly decay with increasing level but the coefficients near the boundaries do not. In this experiment we used Coiflets ($N = 2$). Similar results are obtained with other wavelets (Daublets, Symmlets) and other degrees of polynomial recovery N.

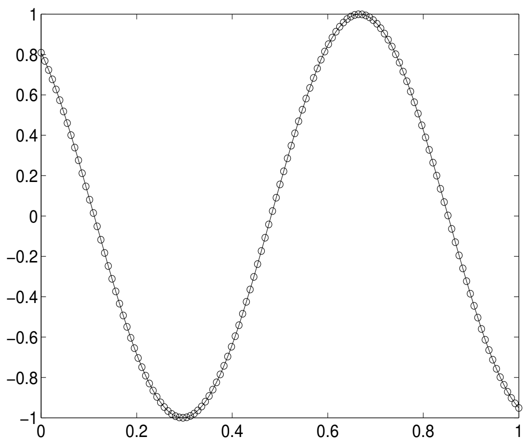

Figure 3: smooth function u

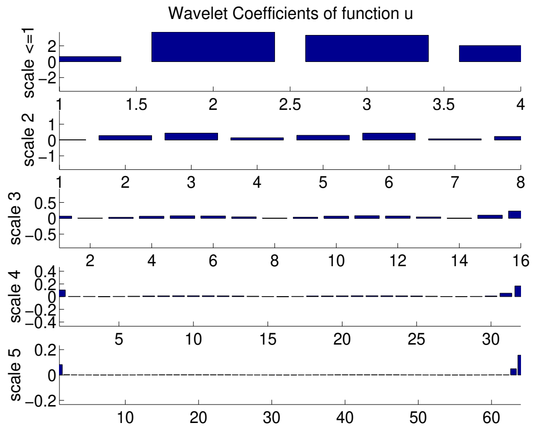

Figure 4: wavelet coefficients

The usage of wavelets which are adapted to the interval like the ones described in the previous section does not remove such boundary artifacts. Therefore, we have to use more sophisticated quadrature formulae - at least near the boundaries. This is subject of the following subsections.

4.1 Method 1

The method introduced in this subsection is especially designed for applications of data or image compression. There, as a rule, wavelets without specific prescribed boundary conditions (Dirichlet, Neumann) are used. Another demand is that the quadrature mapping Q should be non singular, to allow for, in principle, loss-less data compression. We describe the approach for the left boundary only, since the right boundary is treated in a similar fashion.

To obtain an inexpensive technique we use the one point quadrature for the inner scaling functions and use only the first K nodal values for the coefficients $\tilde{c}_0^J, .., \tilde{c}_{K-1}^J$. Thus, we have to care about the upper left block $Q^{left} \in M(K,K)$ of Q. The key idea to define Q^{left} is that $\varphi_l := (Q^{left}\Phi^{left})_l$ should satisfy condition (48) for $y \geq 0$. Then, the prerequisites of Lemma 4.1 are fulfilled again. Let us define the abbreviation

$$\mathcal{R}(z) := \left(z^0, .., z^{N-1}\right)^T$$

for the vector whose components are the first N powers of z. Using this notation, condition (48) reads

$$\forall y \geq 0 : \quad \big(\mathcal{R}(y), .., \mathcal{R}(y-K+1)\big) Q^{left}\Phi^{left}(y) + \sum_{K \leq l} \mathcal{R}(y-l)\phi_l(y) \quad = \quad const \quad .$$

Because of (50) Q^{left} fulfills this condition if

$$\big(\mathcal{R}(y), .., \mathcal{R}(y-K+1)\big) Q^{left}\Phi^{left}(y) = \big(\mathcal{R}(y+R-1), ..., \mathcal{R}(y-K+1)\big)\Phi^c(y) \quad , \qquad (51)$$

where $\Phi^c = (\phi_{-R+1}, ..., \phi_{K-1})^T$ from (13). Expansion of $\mathcal{R}(y-l)$ for $y \neq 0$ yields a non-singular, lower triangular matrix C such that

$$\mathcal{R}(y-l) = YCY^{-1}\mathcal{R}(l), \text{ where } Y = diag(y^0, ..., y^{N-1}) \quad .$$

(Since C is lower triangular there is no singularity at $y = 0$.) Let P denote the transformation matrix, which maps the monomials y^r to the polynomials $\{P_l\}$ from (27), i.e. $(P_0, .., P_{N-1})^T(x) = P\mathcal{R}(x)$. Then, by construction of Φ^{left} and the linear independence of $\Phi^c \chi_{[0,\infty[}$ condition (51) is equivalent to

$$YCY^{-1}\big(\mathcal{R}(0), .., \mathcal{R}(K-1)\big)Q(G^{SU})^{-1/2}U \begin{pmatrix} P & 0 \\ 0 & I_{K-N} \end{pmatrix} \begin{pmatrix} \Theta \\ A_{N+1..K,.} \end{pmatrix} = YCY^{-1}\Theta \quad , \tag{52}$$

where $\Theta = \big(\mathcal{R}(-R+1), ..., \mathcal{R}(K-1)\big)$. Clearly any matrix

$$Q^{left} = W^{-1} \begin{pmatrix} P^{-1} & 0 \\ 0 & I_{K-N} \end{pmatrix} U^{-1}(G^{SU})^{+1/2} \tag{53}$$

satisfies (52). Here,

$$W := \left(\frac{\mathcal{R}(0) \quad .. \quad \mathcal{R}(K-1)}{*} \right) \in M(K,K) \tag{54}$$

denotes a non-singular matrix with prescribed first N rows. Quadrature matrices defined in this way fulfill all our demands. They are non-singular and lead to a smooth approximation $\tilde{u}$. Applied to the example from Figure 4 we now obtain wavelet coefficients without significant boundary artifacts, compare Figure 5.

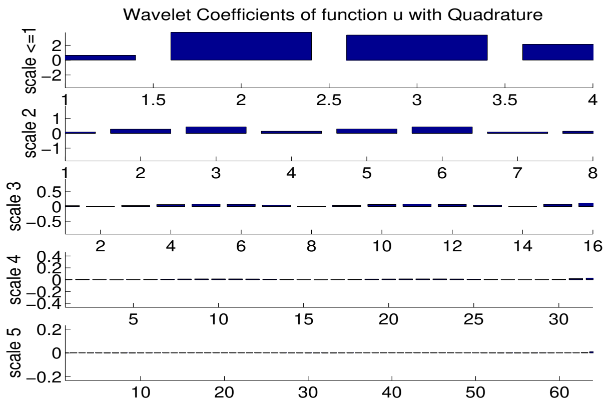

Figure 5: wavelet coefficients with quadrature

Here, some remarks are in order. An advantage of L_2-orthogonal wavelets in connection with compression by non-linear thresholding is that they allow for tight estimates for the error introduced by discarding small coefficients in the wavelet representation:

$$\|u - (u - d_l^j \psi_l^j)\|_0^2 = \|d_l^j \psi_l^j\|_0^2 = |d_l^j|^2 \quad .$$

This property is desirable also for the l_2-norm of the discrete data $\{u(x_l^J)\}$ from which we start. In fact for the one point quadrature one has $\sum_l |u(x_l^J)|^2 = \sum_l |\tilde{c}_l^J|^2 = \sum_{j,l} |\tilde{d}_l^j|^2$.

Now, the matrix Q^{left} introduces a non-unitary step in this scheme. By (46) we have

$$\sum_{l=0}^{K-1} |\tilde{c}_l^J|^2 = (u_0, .., u_{K-1}) Q^{left} {Q^{left}}^H (u_0, .., u_{K-1})^H \quad .$$

Thus, the largest and the smallest eigenvalue λ_{max} and λ_{min}, respectively, of $Q^{left}{Q^{left}}^H$ yield upper and lower bounds for the deviation from exact l_2-orthogonality. Clearly, one is interested in quadrature matrices with small condition number

$$\kappa^{left} = \sqrt{\frac{\lambda_{max}(Q^{left}{Q^{left}}^H)}{\lambda_{min}(Q^{left}{Q^{left}}^H)}} \quad .$$

This gives rise to the question how one can influence κ^{left} by the construction of Q^{left}. To start with, we write $Q^{left}{Q^{left}}^H$ as

$$Q^{left}{Q^{left}}^H = W^{-1} \begin{pmatrix} \Theta \\ A_{N+1..K,.} \end{pmatrix} \int_0^\infty \Phi^c \Phi^{cH} dx \begin{pmatrix} \Theta \\ A_{N+1..K,.} \end{pmatrix}^H W^{-H} \quad .$$

This implies that the matrix U from subsection 3.5 does not change the condition number of Q. Besides a shift of the support of ϕ, i.e. to localize ϕ around zero by changing L and R, suitable ways to influence κ^{left} are

- if $K > N$: suitable choice of $A_{N+1..K,.}$ and $D_{N+1..K,.}$ (see Lemmas 3.3 and 3.5),
- if $K > N$: suitable choice of W (see equations (53) and (54)).

The first point exploits the fact that the Jordan decomposition of $\mathcal{H}^c$ (see Lemma 3.3) is not unique. We assume that the first N Jordan blocks are associated to the eigenvalues $\{2^{-1/2}, .., 2^{-N+1/2}\}$ to establish the polynomial recovery. But, the other Jordan blocks may be permuted arbitrarily leading to different candidates $A_{N+1..K,.}$ and $D_{N+1..K,.}$. Hence, the first point can be implemented by probing all possibilities, which are of moderate number. The second point requires non-linear optimization. Instead of direct optimization of κ^{left}, which is rather involved, we can optimize the upper bound

$$f(W) = trace(QQ^T) trace((QQ^T)^{-1}) \quad .$$

For this bound we can easily derive the directional gradients

$$\begin{aligned} \frac{1}{2}\nabla_R f(W) &= trace(W^{-1}TW^{-H}) trace(R^H T^{-1} W) - \\ & \quad trace(W^{-1}RW^{-1}TW^{-H}) trace(W^H T^{-1} W) \\ \text{where } T &= \begin{pmatrix} \Theta \\ A_{N+1..K,.} \end{pmatrix} \int_0^\infty \Phi^c \Phi^{cH} dx \begin{pmatrix} \Theta \\ A_{N+1..K,.} \end{pmatrix}^H \quad , \end{aligned}$$

given ϕ, $\mathbf{h}$ with particular shifted support
set $\kappa^{opt,left} = \infty$
/* *outer optimization loop* */ for each possible (A, D) from Lemmas 3.3 and 3.5 do
compute boundary scaling functions Φ^{left} from particular A and D
/* *'local' non-linear optimization for given A, D* */   find $\kappa^{*,left} = min\{\kappa^{left} \mid W$ as in (54) $\}$ by gradient type scheme
if $(\kappa^{*,left} < \kappa^{opt,left})$ then $\kappa^{opt,left} = \kappa^{*,left}$, $\Phi^{opt,left} = \Phi^{left}$, $Q^{opt,left} = Q^{*,left}$
next A and D

Table 3: Optimization algorithm

needed for fast gradient type optimization schemes. In our experiments we used a scheme by Armijo [2]. The complete optimization algorithm for the left boundary reads:

A similar algorithm has been used for the right boundary. To figure out the effect of optimization on non-linear thresholding, consider the experiment described in Table 4. In

given nodal values $u_{\mathbf{l}} := u(2^{-7}\mathbf{l})$; $0 \leq l_1, l_2 < 2^7$ of the sample function u depicted in Figure 6.
apply quadrature, i.e. multiply the rows of $\{u_{\mathbf{l}}\}$ by Q and then multiply the columns of this product by Q
apply two-dimensional discrete wavelet transform
nonlinear thresholding of wavelet coefficients for different rates of compression (CR), i.e. cut off of small wavelet coefficients for different cut-off values
inverse wavelet transform
'inverse quadrature' by suitable multiplications by Q^{-1} to obtain $\{u_{\mathbf{l}}^{CR}\}$

Table 4: Experiment data compression

this experiment we use the more asymmetric Coiflets of order $N = 4$ with $L = 4, R = 7$. First we dropped the nonlinear optimization step in the algorithm of Table 3 and used Q^{left} (Q^{right}) with respect to some initial choice of W. The quadrature matrices $Q^{opt,left}$ and $Q^{opt,right}$ (optimized only with respect to A and D) have condition numbers $\kappa^{opt,left} = 2.55$ and $\kappa^{opt,right} = 625.42$, respectively. We denote this case as 'non-optimized'. Applying the non-linear optimization in Table 3 yields quadrature matrices with condition numbers $\kappa^{opt,left} = 2.55$ and $\kappa^{opt,right} = 2.30$. This case is denoted as 'optimized'. Figure 7 depicts the relative error ϵ^{CR} for different rates of compression (CR)

$$\epsilon^{CR} = \frac{\sum_{\mathbf{l}} |u_{\mathbf{l}} - u_{\mathbf{l}}^{CR}|^2}{\sum_{\mathbf{l}} |u_{\mathbf{l}}|^2} \qquad ; \qquad CR = \frac{\text{number of retained coefficients}}{\text{number of all coefficients}} \quad .$$

Especially for the interesting range of $\epsilon^{CR} = 10^{-5}..10^{-3}$ the resulting compression rates dif-

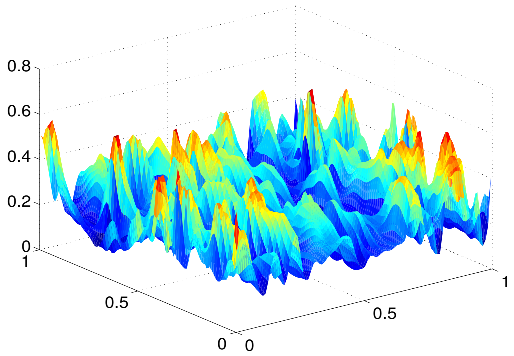

Figure 6: 2D function u

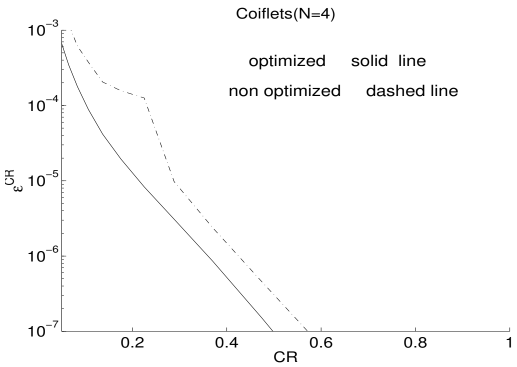

Figure 7: well/ill cond. quadrature

fer by factors of up to 2.5. This emphasizes the importance of well conditioned quadrature matrices.

We finish this subsection with the comparison of (completely optimized) condition numbers $\kappa^{opt,left}$ and $\kappa^{opt,right}$ for often used families of scaling functions including modifications by shifting the support. For Coiflets of order $N = 4$ we present the results for the least asymmetric subfamily and the most asymmetric subfamily. These subfamilies are not simply shifted versions of each other. Instead they correspond to different solutions of a non-linear system of equations arising in the construction of the scaling function ϕ [13]. For Coiflets shifting makes no sense, since then the property of vanishing moments of the *scaling function* is destroyed. Since for $N = 2,\ 3$ there is no difference between Daublets and Symmlets, we do not present the results for Symmlets (N=2).

Table 5 shows the condition numbers. It is interesting to note that the condition numbers for e.g. Daublets strongly depend on the shift of the support of ϕ. For Daublets the best results are obtained with the most asymmetric support. The usual assumption of least asymmetric support $[-N+1, N]$ leads to very ill conditioned matrices. The point is that former boundary constructions [8], [21] fundamentally relied on $N = R = L + 1$. The generalization given in the present paper however allows to choose the support in an appropriate manner. For Symmlets the least asymmetric support is the best choice. The main result is that all families of scaling functions considered here can be equipped with well conditioned quadrature matrices.

quadrature I	N	L	R	$\kappa^{opt,left}$	$\kappa^{opt,right}$
Daublets	2	2	1	1.09	1.25
	4	3	4	149.57	47.57
shifted		1	6	2.76	2.95
	6	5	6	5326.00	1605.70
shifted		1	10	2.48	2.20
Symmlets	4	3	4	2.35	4.40
shifted		1	6	17.98	13.02
	6	5	6	9.77	1.89
shifted		2	9	839.02	2325.20
Coiflets	2	2	3	1.75	1.73
(least asymm.)	4	5	6	1.85	4.85
(most asymm.)	4	4	7	2.55	2.30

Table 5: Condition numbers of quadrature matrices obtained with the alg. from Table 3

4.2 Method II

Another solution to the quadrature problem was given in [5] and [21] based on works of Sweldens and Piessens [24]. The idea is to exactly evaluate the inner products $< u, \Phi^J >$ (Φ^J from (35)) for polynomials u of degree $q \geq K-1$. It is convenient to use the circulant weights w_k of (47) for the inner scaling functions. To determine these weights choose a set I of $q+1$ integers around zero, e.g. $I = \{-\lceil q/2 \rceil, .., \lfloor q/2 \rfloor\}$, and solve the the linear system of equations

$$\sum_{k \in I} R(k) w_k =< R(x), \phi(x) > \quad ; \qquad \text{where} \qquad R(x) = \Big(\frac{x^0}{0!}, ..., \frac{x^q}{q!}\Big)^T \quad .$$

The coefficients $< \frac{x^r}{r!}, \phi(x) >$ of the right hand side can be obtained from (28). The particular basis $x^r/r!$ of the polynomials leads to fairly well conditioned Van der Monde matrices $(R(k))_{k \in I}$. At the boundaries we have to find matrices $Q^{left} \in M(q, K)$ and $Q^{right} \in M(q, K^{right})$ which ensure the exact quadrature. For the left boundary this leads to the linear system of equations

$$\big(R(0), .., R(q)\big) Q^{left} = \int_0^\infty R(x) {\Phi^{left}}^T(x) dx = (G^S)^{-1/2} \int_0^\infty R(x) \tilde{\Phi}^T(x) dx \quad .$$

It remains to determine the moments $\tilde{M} := \int_0^\infty R(x)\tilde{\Phi}^T(x)dx$ of the right hand side. From the scaling equation (18), one easily derives

$$\begin{aligned} \tilde{M} &= \frac{1}{\sqrt{2}}D\tilde{M}\Lambda^{-1} + \frac{1}{\sqrt{2}}A\mathcal{H}^p S\Lambda^{-1} \quad \text{where } \Lambda = diag(2^0, ..., 2^q) \qquad (55)\\ \text{and} \quad S_{l,.} &= < R(x), \phi_l(x) >^T \quad ; \quad K \le l \le 2K + R - 2 \quad . \end{aligned}$$

These equations have a similar form as (23). Thus, the solution can be found in a similar fashion. Writing equation (55) in terms of the components we arrive at

$$\tilde{M}_{l,k}(1 - D_{l,l}\frac{1}{2^{k+1/2}}) = \sum_{i=0}^{l-1} D_{l,i}\tilde{M}_{i,k}\frac{1}{2^{k+1/2}} + \frac{1}{\sqrt{2}}(A\mathcal{H}^p S\Lambda^{-1})_{l,k} \quad .$$

Since $|\frac{1}{2^{k+1/2}}D_{l,l}| < 1$ one can recursively determine $M_{0,.}, \ M_{1,.}, ..$.

In numerical algorithms it is sometimes convenient to obtain nodal values $\tilde{u}(x_l^J)$ of a function $\tilde{u}$ which is given only in terms of it's scaling function coefficients $\tilde{c}_l^J$, i.e. we want to evaluate $\tilde{u}$ pointwise at the dyadic grid points x_l^J. This requires the knowledge of ϕ_k^J and Φ^{left^J}, Φ^{right^J} at such points. Since Φ^{left^J} and Φ^{right^J} are simply linear combinations of ϕ_k^J (restricted to $[0,1]$) we must know $2^{J/2}\phi(2^J 2^{-J}l - k) = 2^{J/2}\phi(l-k)$, i.e. the values $\phi(\mathbf{Z})$. The non vanishing values $\phi(-K+1), \ ..., \phi(R-1)$ are the entries of $\Phi^c(0)$. Since $\Phi^p(0) = \Phi^m(0) = 0$ one concludes from equation (17) that $\Phi^c(0)$ is the eigenvector of $\mathcal{H}^c$ for the eigenvalue $\frac{1}{\sqrt{2}}$.

The present method to find an approximation $\tilde{u}$ from $u(x_0^J), .., u(x_{2^J}^J)$ introduces an approximation error $(u - \tilde{u})(x_l^J)$. Of course, this error vanishes if u is a polynomial of degree less than N. However, a non-vanishing error makes the present technique not appropriate for data compression.

As an example we run the experiment from Table 4 using Symmlets of order $N = 4$ and the two quadrature methods described in this paper. Figure 8 depicts the completely different behaviour of the relative error ϵ^{CR}. For method II the relative error does not fall below 10%. This is not acceptable for data compression.

In this section we introduced two particular treatments of the quadrature problem. The first method is superior in data compression applications. Special optimization techniques and the freedom to arbitrarily shift the support of the scaling function ϕ allowed for the construction of very well conditioned quadrature matrices. The second method is not appropriate for data compression. But, due to it's analytic approach it might be usable for other applications like the solution of PDEs, where always a discretization error is present.

5 Conclusion

In this paper we introduced a general framework for the construction of a boundary adapted wavelet transform. We could show that existing constructions are special cases within our

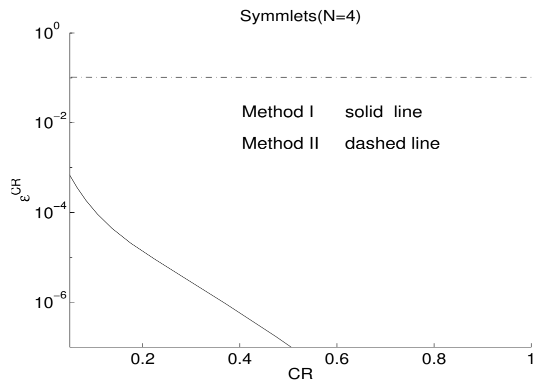

Figure 8: Comparison of different quadratures

general scheme. While the previous schemes were feasible for Symmlets and Daublets only, our general scheme allows for the boundary modification of arbitrary wavelets.

We compared the different schemes with respect to their numerical stability and derived some easy to implement stabilization techniques. Furthermore, we discussed two approaches to deal with the quadrature problem. In a numerical experiment it turned out that the method proposed in this paper (method I) is superior in data compression applications.

References

[1] L. Andersson, N. Hall, B. Jawerth, G. Peters *Wavelets on closed subsets of the real line*; Topics in Theory and Applications of Wavelets L.L. Schumaker, G. Webb (eds.) Academic Press (1994)

[2] L. Armijo *Minimization of functions having Lipschitz-continuous first partial derivatives*; Pac.J.Math. 16 (1966) Webb (eds.) Academic Press (1994)

[3] G. Beylkin, R. Coifman, V. Rokhlin *Fast Wavelet Transforms and Numerical Algorithms I*; Comm. Pure Appl. Math. Vol. XLIV (1991)

[4] G. Beylkin, J. M. Keiser *An Adaptive Pseudo-Wavelet Approach for Solving Nonlinear Partial Differential Equations*; in Multiscale Wavelet Methods for Partial Differential Equations, Vol. 6 in the Wavelet Analysis and Applications series, Academic Press, to appear

[5] G. Chiavassa, J. Liandrat *On the effective Construction of Compactly Supported Wavelets Satisfying Homogeneous BC*; App. Comp. Harm. Anal. 4 (1997)

[6] C.K. Chui, J.Z. Wang *A cardinal spline approach to wavelets*; Proc. Amer.MathSoc. 113 (1991)

[7] C.K. Chui, E. Quak *Wavelets on a Bounded Interval*; in D. Braess, L.L. Schumacher (Eds.) Num. Meth. Appr. Theory Vol. 9(1992)

[8] A. Cohen, I. Daubechies, P. Vial *Wavelets on the Interval and Fast Wavelet Transform*; App. Comp. Harm. Anal. 1 (1993)

[9] W. Dahmen *Wavelets for Operator Equations*; Preprint RWTH Aachen (1997)

[10] W. Dahmen, A. Kunoth, K. Urban *Biorthogonal Spline-Wavelets on the Interval - Stability and Moment Conditions*; IGPM Preprint RWTH Aachen (1997)

[11] W. Dahmen, C.A. Micchelli *Using the Refinement Equation for Evaluating Integrals of Wavelets*; SIAM J. Num. Anal. 30 (1993)

[12] I. Daubechies *Orthonormal Bases of compactly supported Wavelets I*; Comm. Pure Appl. Math. (1988)

[13] I. Daubechies *Orthonormal Bases of compactly supported Wavelets II*; SIAM J. Math. Anal. 24 (1993)

[14] I. Daubechies *Ten Lectures on Wavelets*; SIAM Publ. (1992)

[15] I. Daubechies, J. Lagarias *Two Scale Difference Equations I. Existence and Global Regularity of Solutions*; SIAM J. Math. Anal. 22 (1991)

[16] I. Daubechies, J. Lagarias *Two Scale Difference Equations II. Infinite Products of Matrices and Fractals*; SIAM J. Math. Anal. 23 (1992)

[17] D.L. Donoho *Unconditional bases are Optimal Bases for Data Compression and Statistical Estimation*; Preprint Stanford University (1992)

[18] G.H. Golub, C.F. Van Loan *Matrix Computations*; Johns Hopkins University Press (1989)

[19] I. Meyer *Ondelettes de l'intervalle*; Rev. Mat. Iberoamericana (1992)

[20] I. Meyer *Wavelets and Operators*; Cambrigde University Press (1992)

[21] P. Monasse, V. Perrier *Orthonormal Wavelet Bases adapted for PDEs with boundary conditions*; SIAM J. Math. Anal. Vol. 29,4 (1998)

[22] L. Monzon, G. Beylkin, W. Hereman On almost interpolating and nearly linear phase compactly supported wavelets (Coiflets) PAM report 343, Dec 1997 submitted to Applied and Computational Harmonic Analysis

[23] J. Stoer, R. Bulirsch *Numerische Mathematik*; Springer Verlag (1983)

[24] W. Sweldens, R. Piessens *Quadrature formulae and asymptotic error expansions for wavelet approximations of smooth functions*; SIAM J. Num. Anal. (1994)

[25] Departamento de Tecnoloxías das Comunicacións, Universidade de Vigo (1996), `http://www.tsc.uvigo.es/~wavelets/uvi_wave.html`