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A Fredholm Alternative for Linear Functional Differential Equations of Mixed Type

Bachelor's Thesis by Sa Wu

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In [2] Mallet-Paret introduces a Fredholm Alternative theorem for asymptotically hyperbolic linear functional differential equations of mixed type and first order. We provide a generalisation to equations of arbitrary order by retracing Mallet-Paret's approach and extending definitions and concepts in a natural way. Moreover we reproduce some results from [3] to highlight one application of presented theory on nonlinear equations.

Contents

Nomenclature

$\langle \cdot, \cdot \rangle$	dot product in $\mathbb{C}^d$	14
$\Delta_{c,L_0}(\xi)$	$-c\Delta_{-c^{-1}L_0}$	47
Δ_L	see (1.9)	12
$\mathcal{I}$	Identity operator	9
$\mathfrak{i}$	imaginary unit	9
$\operatorname{ind} A$	index of a Fredholm operator A	10
$\ker A$	$A^{-1}(\{0\})$, kernel of a linear operator A	8
λ^s	unique negative eigenvalue at ∞ for a linearization about 1	65
λ^u	unique positive eigenvalue at $-\infty$ for a linearization about -1	65
λ^+	unique real positive eigenvalue	50
λ^-	unique real negative eigenvalue	50
$\Lambda_{c,L}$	$-c\Lambda_{-c^{-1}L}$	47
Λ_L	Solution Operator	12
$\mathfrak{K}_{c,L}$	$\ker \Lambda_{c,L}$	47
$\mathfrak{K}_L$	kernel of Λ_L in all $W^{k,p}$	34
$\mathfrak{K}_L^p$	kernel of Λ_L in $W^{k,p}$	34
$\mathfrak{R}_{c,L}$	$\Lambda_{c,L}(W^{1,\infty})$	47
$\mathfrak{S}$	Schwartz space, rapidly decreasing functions	13
$\mathfrak{S}'$	dual of $\mathfrak{S}$, space of tempered distributions	13
$\lVert \cdot \rVert_X, \lVert \cdot \rVert$	norm in normed space $(X, \lVert \cdot \rVert_X)$, canonical norms in $\mathbb{C}^d$ or $\mathbb{C}^{d\times d} \cong B(\mathbb{C}^d)$	9
$\lVert f \rVert_{C_B^k(J)}$	$\sup_{1\leq i\leq k} \sup_{\xi\in J} \lVert f^{(i)}(\xi) \rVert$	10
$\overline{A}$	norm topology closure of a set A	40

$\overline{A}^{w*}$	weak-$*$ closure of a set A	40
$\overline{x}$	complex conjugate of $x \in \mathbb{C}^d$ for some $d \in \mathbb{N}$	14
τ_ξ	translation operator	11
φ	$\in \mathfrak{S}$, a rapidly decreasing function	13
$\widehat{f}, \widehat{T}$	Fourier transform of a function f, of a tempered distribution T	14
$\widetilde{f}, \widetilde{T}$	inverse Fourier transform of a function f, of a tempered distribution T	14
A^T	dual of a linear operator A	9
A_i	$A_{i,0}$	47
A_Σ	$\sum_{i=1}^N A_i$	48
$A_{i,j}, B_{i,j}$	$\in (\mathbb{C}^{d\times d})^{\mathbb{R}}$, measurable and uniformly bounded	8
$B(X)$	$B(X,X)$, bounded linear operators $X \to X$	8
$B(X,Y)$	bounded linear operators $X \to Y$	8
$C^k(J)$	$C^k(J, \mathbb{C}^d)$	10
$C^k_B(J)$	$\big(\{f \in C^k(J) \| \|f\|_{C^k_B(J)} < \infty\}, \|f\|_{C^k_B(J)}\big)$	10
d, k, N	$\in \mathbb{N}$, some integers	8
D^k	differential operator	9
$E^\perp$	$\{f \in X^* \| \forall e \in E : f(e) = 0\}$, annihilator of $E \subseteq X$	9
$f * g$	convolution of f and g	13
$f^{(k)}$	k-th weak derivative of f	9
$F_{\perp X}$	$\{x \in X \| \forall f \in F : f(x) = 0\}$, annihilator of $F \subseteq X^*$	9
$F_\perp$	$\{x \in X \| \forall f \in F : f(x) = 0\}$, annihilator of $F \subseteq X^*$	9
i, j, l	$\in \mathbb{N}_0$, some other integers	8
L'	$\in B(W^{k-1,p}, L^p)$	11
$L(\xi)$	$\in R^{k-1}$, $\xi \in \mathbb{R}$	11
L^p	$L^p\big(\mathbb{R}, \mathbb{C}^d\big)$, Lebesgue space	10
L_+	constant coefficient limiting operator at ∞	48

L_-	constant coefficient limiting operator at $-\infty$	48
M'	$\in B(W^{k-1,p}, L^p)$	13
$M(\xi)$	$\in R^{k-1}$, $\xi \in \mathbb{R}$	13
R^k	$B(C_B^k([r_-, r_+]), \mathbb{C}^d)$	11
r_-, r_+	$\in \mathbb{R}$, minimal and maximal shift	8
r_i	$\in \mathbb{R}$, where $1 \leq i \leq N$, some shifts	8
T, T_f	$\in \mathfrak{S}'$, tempered distributions, the latter induced by a function f	13
$T^{(k)}$	k-th derivative of the tempered distribution T	14
$W_0^{1,\infty}$	$\{x \in W^{1,\infty} \mid x(0) = 0\}$	67
$W^{k,p}$	$\{f \in L^p \mid \forall 0 \leq j \leq k : f^{(k)} \in L^p\}$, Sobolev space	10
W^k	$W^{k,2}$	10
X^*	dual space of a normed linear space X	9
$Y^{\mathbb{N}}$	set of all Y valued sequences	8
Y^X	$\{f : X \to Y\}$ set of of all Y valued functions over X	8

Numbers

1 Introduction

In this thesis we shall discuss certain aspects of the linear functional differential equations of mixed type

$$x^{(k)}(\xi) = \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}(\xi)x^{(j)}(\xi + r_i) + h(\xi) \tag{1.1}$$

where $x : \mathbb{R} \to \mathbb{C}^d$.

In [2] a Fredholm Alternative theorem for an asymptotically hyperbolic class of these equations for $k = 1$ is presented. One way of directly extending this work to higher order equations is the standard method of rewriting them to systems of first order equations. This approach has the drawback of having to infer properties from a much more complex system of equations with characteristics not directly apparent from study of (1.1).

Therefore we shall recapitulate some of the results presented in [2] and attempt to extend them to arbitrary $k \in \mathbb{N}$ using the approach of directly retracing the steps made in the original paper and properly extending presented results and definitions to fit the the case $k \in \mathbb{N}$.

The first parts of this thesis are dedicated to unifying notation and recalling some well known results for later use.

Then we construct a Green's function for hyperbolic constant coefficient operators without and with a sufficiently small perturbation by using characteristics of tempered distributions and the Fourier transform.

The last steps towards our Fredholm Alternative theorem are directed towards showing that, assuming asymptotic hyperbolicity, we are always dealing with Fredholm operators and therefore a Fredholm Alternative holds. Moreover we show that we need not examine the dual operators but can instead use a kind of generalized dual operator defined on nicer spaces.

The very last part is dedicated to a faithful reproduction of some results from [3] to highlight one possible application of presented theory of linear equations to nontrivial nonlinear equations.

Most of this thesis is but a technical extension of [2] and a reproduction of certain parts from [3] using most of the proofs and methods presented therein. This thesis basically contains all steps necessary for the author to arrive at the given results. Hence some parts may vary greatly from [2] or [3] while others show much greater resemblance.

1.1 (Preliminaries and) Notation

This section covers some of the notational aspects of this thesis and some standing assumptions and definitions. Basically we follow notation set forth in [4], [5], [2] and [2] with some alterations due to notational collisions and personal taste of the author.

Definition 1.1.1
Let

$$d, k, N \in \mathbb{N} \qquad i, j, l \in \mathbb{N}_0 \qquad p, q \in [0, \infty] \qquad \xi, \nu, \mu \in \mathbb{C}$$

. p, q are usually conjugated, that is $p^{-1} + q^{-1} = 1$ and sometimes $\xi, \nu, \mu \in \mathbb{R}$. $\forall 1 \leq i \leq N, \forall 0 \leq j \leq k-1$ let the matrix coefficient functions

$$A_{i,j} \in (\mathbb{C}^{d\times d})^{\mathbb{R}} \qquad B_{i,j} \in (\mathbb{C}^{d\times d})^{\mathbb{R}}$$

be measurable and uniformly bounded, that is

$$\exists C > 0 : \forall 1 \leq i \leq N : \forall 0 \leq j \leq k-1 : \forall \xi \in \mathbb{R} : \quad \|A_{i,j}(\xi)\| \leq C \wedge \|B_{i,j}(\xi)\| \leq C$$

. For $1 \leq i \leq N$ we call the quantities $r_i \in \mathbb{R}$ *shifts*. We introduce the notational restriction that the r_i are distinct and $r_1 = 0$. We write our maximal and minimal shifts as

$$r_- := \min_{1\leq i\leq N} r_i \qquad r_+ := \max_{1\leq i\leq N} r_i$$

.

For any two sets X, Y we write Y^X for the set of all functions $f : X \to Y$. In particular $X^{\mathbb{N}}$ is the set of all X valued sequences and $\mathbb{C}^d$ the "complex euclidean space" of dimension d.

We assume all used linear spaces to be over the complex field $\mathbb{C}$. For any two normed linear spaces X, Y by $B(X,Y) \subseteq Y^X$ we denote the space of bounded linear operators from $X \to Y$ and by $B(X) = B(X,X)$ those mapping back to X. The kernel of a linear operator $A : X \to Y$ is designated by

$$\ker A := A^{-1}(\{0\}) = \{x \in X | Ax = 0\}$$

.

We write X^* for the dual of X, that is $B(X, \mathbb{C})$. For $E \subseteq X$, $F \subseteq X^*$ we define their (topological) annihilators to be

$$E^\perp := \{f \in X^* | \forall e \in E : f(e) = 0\}$$
$$F_\perp := \{x \in X | \forall f \in F : f(x) = 0\}$$

. If it is not clear in which spaces we are operating, e.g. if $F \subseteq X_1^*, F \subseteq X_2^*$ we shall add a subscript

$$F_{\perp X} := \{x \in X | \forall f \in F : f(x) = 0\} \tag{1.2}$$

. If $A \in B(X,Y)$ then $A^T \in B(Y^*, X^*)$ defined by

$$(A^T f)(x) = f(Ax)$$

is called its *dual/conjugate* operator. By the very definition of the dual operator A^T we have a series of nice properties related to A, e.g.

$$\ker A^T = A(X)^\perp \qquad\qquad \ker A = A^T(Y^*)_\perp$$

For a matrix $A \in \mathbb{C}^{d\times d}$, A^T is the transposed operator/matrix which coincides with the dual operator.

We stick to the standard convention of writing normed spaces $(X, \|\cdot\|_X)$ as only X unless this notation is ambiguous. For the canonical euclidean norm in $\mathbb{C}^d$ and the associated operator norm in $\mathbb{C}^{d\times d}$ we write $\|\cdot\|$.

For suitable f by $f^{(k)}$ we denote its k-th (weak) derivative. In some cases it might be more practical to write differentiation as an operator

$$D^k : W^{k,p} \to L^p \qquad\qquad f \mapsto f^{(k)}$$

. sometimes using other spaces as domain and range.

All "=" involving L^p functions of some sorts are generally to be read in L^p sense, i.e., equality only holds almost everywhere. Additionally instead of normal (non equivalence class) functions we shall always consider the associated Lebesgue class. Conversely we always take preferably smooth representatives for L^p functions.

By "$\mathcal{I}$" we shall denote the identity operator in the space it is used and by "$\mathfrak{i}$" the imaginary unit.

We recall Landau Notation and write

$$O(f, Y) \text{ for } \xi \to \pm\infty := \{h \in Y^{\mathbb{R}} : \exists \tau \in \mathbb{R} : \exists C > 0 : \forall \pm\xi \geq \pm\tau : \|h(\xi)\|_Y \leq C\|f(\xi)\|_X\} \tag{1.3}$$

where $f \in X^{\mathbb{R}}$ and X,Y are some normed spaces. Additionally we shall use the shorthands $O(f)$ whenever Y should be clear from context and $f = g+O(h)$ for some functions $f, g \in Y^{\mathbb{R}}$, $h \in X^{\mathbb{R}}$ to represent

$$\exists \tilde{h} \in O(h) = 0(h, Y) : f = g + \tilde{h} \tag{1.4}$$

.

We may sometimes use an expression as an abbreviated notation for the function it represents, e.g.: we may use $\mathfrak{i}\xi$ to denote the function $\xi \mapsto \mathfrak{i}\xi$, $O(f(\xi))$ as a shorthand for $O(\xi \mapsto f(\xi)) = O(f)$ or (x_n) for the sequence $n \mapsto x_n$.

Definition 1.1.2

Let $J \subseteq \mathbb{R}$ be an open interval. The Lebesgue and Sobolev spaces will be written as

$$\begin{aligned}
L^p &:= L^p\big(\mathbb{R}, \mathbb{C}^d\big)\\
W^{k,p} &:= \{f \in L^p | \forall 0 \leq j \leq k : f^{(k)} \in L^p\}\\
W^k &:= W^{k,2}\\
C^k(J) &:= C^k(J, \mathbb{C}^d)\\
C^k_B(J) &:= \Big(\{f \in C^k(J) | \|f\|_{C^k_B(J)} < \infty\}, \|\cdot\|_{C^k_B(J)}\Big)
\end{aligned}$$

where

$$\|f\|_{C^k_B(J)} = \sup_{0\leq j\leq k} \sup_{\xi\in J} \|f^{(i)}(\xi)\|$$

.

Additionally, if $\overline{J}$ is used for the closure of a compact interval J, we write

$$C^k(\overline{J}) \coloneqq \{f \in (\mathbb{C}^d)^{\overline{J}} | f|_J \in C^k(J) \wedge \forall 0 \leq i \leq k : f^{(i)} \text{ continuously extendable to } \overline{J}\}$$
$$C^k_B(\overline{J}) \coloneqq \{f \in C^k(\overline{J}) | \|f\|_{C^k_B(\overline{J})} < \infty\}$$

with respectively extended norms and evaluations at the border by continuous extension. Some more care is actually needed here if said spaces are used for closures of more general sets but as we are only using those spaces for the compact interval $[r_-, r_+]$ we will not and need not dwelve any further here.

We recall that the notion of weak differentiability is stronger for smaller dimension. In particular, as we are considering functions on $\mathbb{R}^1$, every $f \in W^{1,p}$ is absolutely continuous (i.e. in the same equivalence class as an absolutely continuous function) with an (in a classical sense) almost everywhere defined derivative in L^p and furthermore the following continuous embeddings hold.

$$\forall 0 \leq j < k : \quad W^{k,p} \to C^j_B$$
$$\forall 0 \leq j < k : \forall p \leq q \leq \infty : \quad W^{k,p} \to W^{j,q}$$

Most of those results can be found in fairly any book on partial differential equations. One nice account of Sobolev space theory can be found in [1].

Definition 1.1.3

We shall recall the notion of Fredholm operators and Fredholm Alternatives.

Let X, Y be two Banach spaces. $A \in B(X, Y)$ is called a *Fredholm Operator* or simply *Fredholm* iff

$$\dim\ker A < \infty \qquad\qquad \operatorname{codim} A(X) < \infty$$

. Then the *index* or *Fredholm index* of A is defined as the integer

$$\operatorname{ind} A \coloneqq \dim\ker A - \operatorname{codim} A(X)$$

For such a Fredholm operator the following properties are met.

1. A is surjective iff $\operatorname{ind} A = \dim\ker A$
2. A is injective iff $\dim\ker A = 0$
3. A is bijective iff $\operatorname{ind} A = \dim\ker A = 0$
4. $A(X)$ is closed
5. $A(X) = (\ker A^T)_\perp$ and $A^T(Y^*) = (\ker A)^\perp$

6. $\operatorname{codim} A^T(Y^*) = \dim\ker A$

7. $\operatorname{codim} A(X) = \dim\ker A^T = \dim\ker A - \operatorname{ind} A$

8. $A^T : Y^* \to X^*$ is Fredholm and $\quad \operatorname{ind} A^T = -\operatorname{ind} A$

5, 6, 7 represent a *Fredholm Alternative.*
The interested reader is referred to [5].

Definition 1.1.4
Let $f : \mathbb{R} \to \mathbb{C}^d, \xi \mapsto f(\xi)$. For $\xi \in \mathbb{R}$ we introduce a translation operator

$$\begin{aligned} \tau_\xi : (\mathbb{C}^d)^{\mathbb{R}} &\to (\mathbb{C}^d)^{\mathbb{R}} \\ \big(\tau_\xi f\big)(\nu) &:= f(\nu - \xi) \end{aligned}$$

Related to (1.1) are the family of bounded operators

$$\begin{aligned} L(\xi) : C_B^{k-1}([r_-, r_+]) &\to \mathbb{C}^d \\ \varphi &\mapsto \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}(\xi)(\tau_{-\xi}\varphi^{(j)})(r_i) \end{aligned}$$

,where $\xi \in \mathbb{R}$, and the operator

$$\begin{aligned} L' : W^{k-1,p} &\to L^p \\ (L'f)(\xi) &= \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}(\xi) f(\xi + r_i) = \text{“}L(\xi)(\tau_{-\xi} f)\text{”} \end{aligned} \tag{1.5}$$

.

Since the matrix coefficients are assumed to be uniformly bounded those are all bounded operators. To facilitate notation we define

$$R^k := B(C_B^k([r_-, r_+]), \mathbb{C}^d)$$

. .
With these operators (1.1) can be written as

$$\begin{aligned} x^{(k)}(\xi) &= L(\xi)\big(\tau_{-\xi} x\big) \\ x^{(k)} &= L' x \end{aligned} \tag{1.6}$$

. In written words this means that the k-th derivative of the funcion x at any point depends on a shifted version of the whole function.

We define a *solution* to (1.1) to be a function $x \in W^{k,p}$ satisfying (1.1) in L^p sense. In this context we might add that if $x \in W^{k,p}$ then $x \in C_B^{k-1}(\mathbb{R})$ so $\forall \xi \in J : \tau_{-\xi} x \in C_B^{k-1}([r_-, r_+])$. Hence the first expression in (1.6) is indeed applicable.

Associated with equations (1.1), (1.6) is another linear bounded operator defined by

$$\begin{aligned} \Lambda_L : W^{k,p} &\to L^p \\ x &\mapsto x^{(k)} - L'x \\ (\Lambda_L x)(\xi) &= x^{(k)}(\xi) - L(\xi)\tau_{-\xi}x \end{aligned} \tag{1.7}$$

. Using this notation (1.1) for $h \in L^p$ can be rewritten to

$$\Lambda_L x = h \tag{1.8}$$

.

Definition 1.1.5
Assume L in Equation (1.6) is constant (as a family in ξ), i.e. the system is a constant coefficient system and $\xi \mapsto L(\xi)$ is a constant function. Then $A_{i,j}(\xi) = A_{i,j}$ are independent from ξ as well.

Associated with such a constant coefficient system is the *characteristic equation*

$$0 = \det \Delta_L(\xi) := \det\Big(\xi^k I - \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\xi^j e^{\xi r_i}\Big) \tag{1.9}$$

.

A constant coefficient system or more simply L is called *hyperbolic* iff

$$\forall \nu \in \mathbb{R} : \det \Delta_L(\mathrm{i}\nu) \neq 0 \tag{1.10}$$

.

Assume a function of the form $\nu \mapsto e^{\xi\nu} f(0)$ solves the equation with $f \neq 0$. Then $\forall \xi \in \mathbb{R} : \Delta_L(\xi) = 0$. Hence there are no such solutions for imaginary ξ.

Definition 1.1.6
Often we shall write the operators $L(\xi)$ and L' in Definition 1.1.4 as sums of a constant coefficient operator and a linear perturbation term and study our system for suitably small perturbations. We write $\forall \xi \in \mathbb{R}$

$$L(\xi) = L_0 + M(\xi) \tag{1.11}$$

$$\begin{aligned} M(\xi) : C_B^{k-1}([r_-, r_+]) &\to \mathbb{C}^d \\ \varphi &\mapsto \sum_{i=1}^{N}\sum_{j=0}^{k} B_{i,j}(\xi)\varphi^{(j)}(r_i) \end{aligned} \tag{1.12}$$

$$\begin{aligned} M' : W^{k-1,p} &\to L^p(J) \\ \big(M'x\big)(\xi) &= \sum_{i=1}^{N}\sum_{j=0}^{k} B_{i,j}(\xi)\varphi^{(j)}(r_i) = \text{“}M(\xi)\big(\tau_{-\xi}x\big)\text{”} \end{aligned} \tag{1.13}$$

$$\Lambda_L = \Lambda_{L_0} - M' \tag{1.14}$$

.

If there exists an L_0 as in (1.11) such that

$$\lim_{\xi\to\infty} \|M(\xi)\|_{R^{k-1}} = 0 \tag{1.15}$$

, then the system (1.6) or simply L is called *asymptotically autonomous at* $+\infty$. Then of course, if $A_{i,j}$ denote the matrix coefficients of L_0

$$\forall 1 \le i \le N : \forall 0 \le j \le k-1 : \lim_{\xi\to\infty} A_{i,j}(\xi) = A_{i,j} \tag{1.16}$$

.

If additionally L_0 is hyperbolic then L is called *asymptotically hyperbolic at* $+\infty$. Analogous definitions apply to $-\infty$.

L will be called *asymptotically autonomous* if it is asymptotically autonomous at both $+\infty$ and $-\infty$ and analogously we call L *asymptotically hyperbolic* if it is asymptotically hyperbolic at both $+\infty$ and $-\infty$.

We point out that the limiting equations for $+\infty$ and $-\infty$ need not be the same.

Definition 1.1.7

If $f \in L^p$ and $g \in L^q$ such that $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ then

$$(f * g)(\xi) := \int_{\mathbb{R}} f(\xi - \nu) g(\nu) \mathrm{d}\nu$$

defines an L^r function. In particular the convolution of two integrable functions is again integrable.

Furthermore the convolution in some sense commutes with differentiation. If $f \in W^{k,p}$, $g \in W^{l,q}$ with p, q, r conjugated as above and $k, l \in \mathbb{N}_0$ then $f * g \in W^{k+l,r}$ and

$$(f * g)^{(k+l)} = f^{(k)} * g^{(l)}$$

Definition 1.1.8

Let $\mathfrak{S} = \mathfrak{S}(\mathbb{R}, \mathbb{C}^d)$ be the (*Schwartz*) space of *rapidly decreasing* functions, that is functions $\varphi \in C^\infty(\mathbb{R}, \mathbb{C}^d)$ satisfying

$$\forall k \in \mathbb{N}_0 : \forall l \in \mathbb{N}_0 : \quad \sup_{\xi\in\mathbb{R}} \|\xi^l \varphi^{(k)}(\xi)\| < \infty$$

. Then T from its dual $\mathfrak{S}'$ is called a *tempered distribution*. Let $T \in \mathfrak{S}'$, $\varphi \in \mathfrak{S}$.

For $\varphi \in \mathfrak{S}$ the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are well defined by their integral expressions

$$\begin{aligned} \widehat{f}(\nu) = \mathcal{F}\varphi(\nu) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\mathrm{i}\nu\xi} \varphi(\xi) \mathrm{d}\xi \\ \widetilde{f}(\xi) = \mathcal{F}^{-1}\varphi(\nu) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}\nu\xi} \varphi(\nu) \mathrm{d}\nu \end{aligned} \tag{1.17}$$

in the same way these expressions define Fourier transforms for $f \in L^1$ with all of their usual properties.

We recall that

$$T^{(1)}(\varphi) \coloneqq -T(\varphi^{(1)})$$

$$\widehat{T}(\varphi) \coloneqq T(\widehat{\varphi})$$

$$\widetilde{T}(\varphi) \coloneqq T(\widetilde{\varphi})$$

$$\tau_a T(\varphi) \coloneqq T(\tau_{-a}\varphi)$$

. define tempered distributions: T's *derivative*, *Fourier transform*, *inverse Fourier transform* and *translate*.

The definition of a tempered distribution's translate may seem to have the wrong sign but if we consider a distribution defined by a function $f \in (\mathbb{C}^d)^{\mathbb{R}}$ of some special suitable sorts, that is

$$T_f(\varphi) = \int_{\mathbb{R}} \langle f(\xi), \overline{\varphi(\xi)} \rangle \mathrm{d}\xi \tag{1.18}$$

, where the brackets denote the dot product in $\mathbb{C}^d$ and $\overline{x}$ the complex conjugate of x, then $\tau_a T_f = T_{\tau_a f}$. We recall that any L^p function for arbitrary p defines a tempered distribution via the integral expression (1.18).

If f is a $C^\infty(\mathbb{R}, \mathbb{C}^{d\times d})$ function satisfying $\forall \varphi \in \mathfrak{S} : f\varphi \in \mathfrak{S}$, then $S(\varphi) = T(f\varphi)$ defines another tempered distribution. One class of such functions is the class of *slowly increasing* functions $f \in \mathfrak{D}_M$, that is functions $f \in C^\infty$ satisfying

$$\forall k \in \mathbb{N} : \exists N \in \mathbb{N} : \lim_{\xi \to \pm\infty} |\xi|^{-N} \| f^{(k)}(\xi) \| = 0$$

. For any slowly increasing function f the product of f and $T \in \mathfrak{S}'$ defined by

$$(fT)(\varphi) \coloneqq T(f\varphi)$$

is another tempered distribution. A somehow noteworthy subset of $\mathfrak{D}_M$ is the set of polynomial functions.

Using fT as notation for this product can be ambiguous, e.g. if $f \in \mathbb{C}^{d\times d}$ is a matrix then it can be applied to the tempered distribution T both as linear operator on $\mathfrak{S}'$ and as product of A as slowly increasing function with T which would be the same as applying the dual operator A^T on T. Therefore this notation is used exclusively for terms of the form $(\mathrm{i}\xi)^k$ and $e^{a\xi}$ and both together, as resulting from differentiation rules as stated below. Hence $(\mathrm{i}\xi T)(\varphi) = T(\mathrm{i}\xi\varphi)$ while $(AT)(\varphi) = A(T(\varphi))$. Especially for distributions defined by functions this means that $AT_f(\varphi) = T_{Af}(\varphi) \neq T_f(A\varphi)$ in the $\mathbb{C}^d$ case below.

Some notable tempered distributions are "normal" distributions (over C_0^∞, the space of infinitely differentiable functions with compact support) with compact support and those defined by L^p functions and the integral expression (1.18).

Some handy aids for calculations are

$$\widehat{T^{(k)}} = (\mathrm{i}\xi)^k \widehat{T}$$
$$\widehat{T}^{(k)} = \widehat{(-\mathrm{i}\xi)^k T}$$
$$\widehat{\tau_a T} = e^{-\mathrm{i}a\xi} \widehat{T}$$
$$\tau_a \widehat{T} = \widehat{e^{\mathrm{i}a\xi} T}$$

.

Additionally the following formulas hold for tempered distributions defined by functions

$$f \in W^{k,p} \Rightarrow T_f^{(k)} = T_{f^{(k)}}$$
$$f \in L^2 \Rightarrow \widehat{T_f} = T_{\widehat{f}} \wedge \widetilde{T_f} = T_{\widetilde{f}}$$

Naturally some of these equalities also hold under weaker assumptions.

We can extend the notion of tempered distributions to C^d valued "functionals" by taking d functionals. Naturally such generalized tempered distribution can be defined by functions (but do not have to be), which are then written as being matrix valued with the usual matrix vector product replacing the semiconjugated dot product. Thus, if $G \in (\mathbb{C}^{d\times d})^{\mathbb{R}}$ of some suitable sorts, e.g L^p then

$$T_G(\varphi) = \int_{\mathbb{R}} G(\xi)\varphi(\xi)\mathrm{d}\xi \tag{1.19}$$

defines a "generalized" tempered distribution with all of the usual properties of tempered distributions (component-wise whenever applicable).

We point out that by using these definitions our normal calculus of tempered distributions can be applied to C^d valued tempered distributions without creating semantical problems.

For more on tempered distributions the interested readers are referred to [4].

Definition 1.1.9

Let $f \in L^2$. Then it has a Fourier transform $\widehat{f} \in L^2$ and inverse Fourier transform $\widetilde{f} \in L^2$ satisfying

$$\widehat{T_f} = T_{\widehat{f}} \qquad \widetilde{T_f} = T_{\widetilde{f}} \qquad \|\widehat{f}\|_{L^2} = \|f\|_{L^2} = \|\widetilde{f}\|_{L^2}$$

. For functions, whose Fourier transforms defined through their integral expressions exist, e.g. for $f \in L^1 \cap L^2$ this L^2 Fourier transform coincides with the "classical" Fourier transform defined through the integral expressions (1.17). Additionaly it also coincides with the L^2 Fourier transform obtained by some sorts of "improper" approach to the integral expression.

We recall that the Fourier transform transforms differentiation into multiplication with a polynomial function under certain conditions. If $f \in (\mathbb{C}^d)^{\mathbb{R}}$ is measurable and if

$$\forall 0 \leq j \leq k : \quad \xi \mapsto (\mathrm{i}\xi)^j f(\xi) \in L^2 \tag{1.20}$$

then $\widetilde{f} \in W^k$ and

$$\forall 0 \leq j \leq k : \quad \widetilde{f}^{(k)} = \widetilde{(\mathrm{i}\xi)^k f(\xi)} \tag{1.21}$$

.

The interested readers are again referred to [4].

2 Green's function(s)

In this chapter we follow Mallet-Paret's method of constructing a Green's function for a hyperbolic constant coefficient system as in [2] by taking the inverse Fourier transform of $\xi \mapsto \Delta_L^{-1}(\mathrm{i}\xi)$. Then we shall use the von Neumann series to obtain a Green's function for hyperbolic constant coefficient systems with small perturbations and retrieve a sufficient condition for such perturbations to be small enough.

2.1 Hyperbolic constant coefficient operators

Our course of action to obtaining a Green's function G_0 first takes us to shifting the path of integration in the inverse Fourier transform's integral expression to some $a_0 + \mathrm{i}\,\mathbb{R}$. This allows us to conclude exponential decay of all derivates of order $< k$ and thus yields $G_0 \in W^{k-1,p}$. By taking the convolution $x = G_0 * h$ it follows from commutation of convolution and differentiation under the right conditions that $x \in W^{k-1,p}$. Finally we infer that $x \in W^{k,p}$ and that x is a solution to $\Lambda_L x = h$ form analyzing the tempered distributions induced by the G_0 and x.

Proposition 2.1.1
Consider the characteristic equation for a hyperbolic constant coefficient operator L_0 as defined in 1.1.5.

$$0 = \Delta_{L_0}(\nu) = \nu^k - \sum_{i=1}^{N}\Big(\sum_{j=0}^{k-1} A_{i,j}\nu^j\Big)e^{\nu r_i}$$

Let

$$\begin{aligned} G_0 &: \mathbb{R} \to \mathbb{C}^{d\times d} \\ &\xi \mapsto \frac{1}{\sqrt{2\pi}}\Big(\widetilde{\Delta_{L^0}^{-1}(\mathrm{i}\nu)}\Big)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}\xi\nu}\Delta_{L_0}^{-1}(\mathrm{i}\nu)\mathrm{d}\nu \end{aligned} \tag{2.1}$$

. Then $G_0 \in W^{k-1,p}$ and $\exists K_0, a_0 > 0$ such that

$$\forall 0 \le j \le k-1 : \|G_0(\xi)^{(j)}\| \le K_0 e^{-a_0|\xi|} \tag{2.2}$$

. We point out that this holds for any $1 \le p \le \infty$.

Proof. Our agenda in this proof is to show existance of weak derivatives in L^2 and prove the exponetially decaying bounds. Since existance of weak derivatives is independent of p and the exponential growth boundaries establish L^p boundedness this suffices. To show the existance of weak derivatives we use the fact that the Fourier transform converts differentiation into multiplication with a polynomial. As the functions that are to be inversely Fourier transformed are holomorphic in some neighbourhood of the imaginary line we use Cauchy's integral theorem to shift the path of integration giving some $e^{\pm a\xi}$ while the remaining terms stay bounded.

Since $\xi \mapsto \Delta_{L_0}(\xi)$ is holomorphic and L_0 is hyperbolic we have some open neighbourhood of $\mathrm{i}\,\mathbb{R}$ where Δ_{L_0} stays invertible. If

$$a_1 := 1 + \sum_{i=1}^{N} \sum_{j=0}^{k-1} \|A_{i,j}\| e^{|r_i|}$$

and $\xi \in V := \{\nu \in \mathbb{C} \,||\mathrm{Re}\nu| < 1 \wedge |\mathrm{Im}\nu| > a_1\}$ then

$$\Delta_{L_0}(\xi) = \xi^k \Big(\mathcal{I} - \frac{1}{\xi^k} \sum_{i=1}^{N} \sum_{j=0}^{k-1} A_{i,j} \xi^j e^{\xi r_i}\Big)$$

where

$$\begin{aligned} \|\frac{1}{\xi^k} \sum_{i=1}^{N} \sum_{j=0}^{k-1} A_{i,j} \xi^j e^{\xi r_i}\| &\leq \frac{1}{|\xi|^k} \sum_{i=1}^{N} \sum_{j=0}^{k-1} \|A_{i,j}\| |\xi|^j e^{|\mathrm{Re}\xi||r_i|} \\ &\leq \frac{|\xi|^{k-1}}{|\xi|^k} \sum_{i=1}^{N} \sum_{j=0}^{k-1} \|A_{i,j}\| e^{|r_i|} \\ &< \frac{|\xi|^k}{|\xi|^k} = 1 \end{aligned}$$

. Hence $\Delta_{L_0}(\xi)$ invertible in V. As $\det A$ is simply a polynomial in the coefficients of A, we have that $\xi \mapsto \det \Delta_{L_0}(\xi)$ is holomorphic and hence on the compact set $W := \{\nu \in \mathbb{C} \,|\mathrm{Re}\nu \leq 1 \wedge \mathrm{Im}\nu \leq a_1\}$ this function is uniformly continuous.

$\xi \mapsto f(\xi) := |\det \Delta_{L_0}(\xi)|$ too is a uniformly continuous function on W and therefore on the compact set $\mathrm{i}[-a_1, a_1] \subset W$ it takes a minimum $\epsilon > 0$. Because of uniform continuity

$$\exists 1 > \delta > 0 : \forall \xi, \nu \in W : \quad |\xi - \nu| < \delta \Rightarrow |f(\xi) - f(\nu)| < \epsilon$$

In particular, if $\xi \in W$ and $|\mathrm{Re}\xi| < \delta$, then

$$f(\xi) \geq f(\mathrm{i}\mathrm{Im}\xi) - |f(\xi) - f(\mathrm{i}\mathrm{Im}\xi)| > \epsilon - \epsilon > 0$$

and hence $\Delta_{L_0}(\xi)$ is invertible. By setting $0 < a_0 < \delta$ and

$$U := \{\nu \in \mathbb{C} \,||\mathrm{Re}\nu| \leq a_0\} \tag{2.3}$$

we conclude that $\xi \mapsto \Delta_{L_0}(\xi)^{-1} \in (\mathbb{C}^{d\times d})^U$ is a well defined holomorphic function. Now assume $\xi \in U$.

Note that $\Delta_{L_0}(\mathrm{i}\nu) \in O(|\nu^k|)$ for $|\nu| \to \infty$. Hence $\forall 0 \leq j < k$ we derive $(\mathrm{i}\nu)^j \Delta_{L_0}^{-1}(\mathrm{i}\nu) \in O(|\nu|^{j-k})$. So $\forall 0 \leq j < k : \nu \mapsto (\mathrm{i}\nu)^j \Delta_{L_0}^{-1}(\mathrm{i}\nu) \in L^2$ and thus $G_0 \in W^{k-1}$. Therefore the weak derivatives of G_0 up to order $k-1$ exist and as they are (i.e. their existance as functions is) independent of p it suffices to establish L^p bounds for arbitrary $1 \le p \le \infty$ to show $G_0 \in W^{k-1,p}$ for any p.

First consider the case $j \le k-2$. If $C := 1 + \max\{\sum_{i=1}^N \|A_{i,j}\| | 0 \le j \le k-1\}$, which is well defined as $A_{i,j}$s are assumed to be uniformly bounded, and $|\mathrm{Im}\xi| > 2^{k-1}kC$ then

$$\begin{aligned}
\|(\xi)^j \Delta_{L^0}^{-1}(\xi)\| &= |\xi|^j \|\xi^k - \sum_{l=0}^{k-1}\sum_{i=1}^{N} A_{i,j}\xi^l e^{\xi r_i}\|^{-1} \\
&\le |\xi|^j \Big(|\xi|^k - \sum_{l=0}^{k-1} C|\xi|^l\Big)^{-1} \\
&\le |\xi|^j \Big(|\xi|^k - kC|\xi|^{k-1}\Big)^{-1} \\
&= (|\xi|^{k-j} - kC|\xi|^{k-1-j})^{-1} \\
&\le (|\mathrm{Im}\xi|^{k-j} - kC|1 + \mathrm{Im}\xi|^{k-j-1})^{-1} \\
&\le \Big(|\mathrm{Im}\xi|(|\mathrm{Im}\xi| - 2^{k-1}kC)|\mathrm{Im}\xi|^{k-2-j}\Big)^{-1} < \infty
\end{aligned}$$

. Thus $\xi \mapsto (\mathrm{i}\xi)\Delta_{L_0}(\mathrm{i}\xi)^{-1} \in O(|\mathrm{Im}\xi|^{-2})$ uniformly for $\xi \in U, |\mathrm{Im}\xi| \to \infty$.

Therefore $\forall a \in [-a_0, a_0] : \xi \mapsto (a+\mathrm{i}\xi)\Delta_{L_0}(a+\mathrm{i}\xi)$ is a holomorphic integrable function. Using Cauchy's integral theorem and the fact that $\xi \mapsto (\mathrm{i}\xi)\Delta_{L_0}(\mathrm{i}\xi)^{-1} \in O(|\mathrm{Im}\xi|^{-2})$ uniformly for $\xi \in U, |\mathrm{Im}\xi| \to \infty$ we get $\forall \xi \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} e^{\mathrm{i}\xi\nu}(\mathrm{i}\nu)^j \Delta_{L_0}^{-1}(\mathrm{i}\nu)\mathrm{d}\nu = \int_{-\infty}^{\infty} e^{\xi(a_0+\mathrm{i}\nu)}(a_0+\mathrm{i}\nu)^j \Delta_{L_0}^{-1}(a_0+\mathrm{i}\nu)\mathrm{d}\nu$$

, again for arbitrary $a \in [-a_0, a_0]$.

By changing the path of integration for $\xi \ge 0$ we obtain

$$\begin{aligned}
\|G_0^{(j)}(\xi)\| &= \|\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\xi\nu}(\mathrm{i}\nu)^j \Delta_{L_0}^{-1}(\mathrm{i}\nu)\mathrm{d}\nu\| \\
&= \|\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\xi(-a_0+\mathrm{i}\nu)}(-a_0+\mathrm{i}\nu)^j \Delta_{L_0}^{-1}(-a_0+\mathrm{i}\nu)\mathrm{d}\nu\| \\
&\le \frac{e^{-a_0\xi}}{2\pi}\int_{-\infty}^{\infty} \|(-a_0+\mathrm{i}\nu)^j \Delta_{L^0}^{-1}(-a_0+\mathrm{i}\nu)\|\mathrm{d}\nu \\
&\le \frac{e^{-a_0\xi}}{2\pi}\|\nu \mapsto (-a_0+\mathrm{i}\nu)^j \Delta_{L^0}^{-1}(-a_0+\mathrm{i}\nu)\|_{L^1}
\end{aligned}$$

. Likewise for $\xi < 0$

$$\|G_0^{(j)}(\xi)\| \le \frac{e^{a_0\xi}}{2\pi}\|\nu \mapsto (a_0+\mathrm{i}\nu)^j \Delta_{L^0}^{-1}(a_0+\mathrm{i}\nu)\|_{L^1}$$

. Hence $\forall 0 \le j \le k-2 : \forall 1 \le p \le \infty : G_0^{(j)} \in L^p$ and the bound for $G_0^{(k-1)}$ remains to be shown.

Let $\xi > 0$ and consider the map:

$$E_+ : \mathbb{R} \to \mathbb{C}^{d\times d}$$
$$\xi \mapsto \begin{cases} 0 & \xi < 0 \\ \sqrt{2\pi} e^{-\xi}\,\mathcal{I} & \xi \geq 0 \end{cases}$$

Then $E_+ \in L^1 \cap L^2$ and we can get its Fourier transform by evaluation of the (in this case "classicaly" well defined) integral expression for the Fourier transform. So for $\nu \in \mathbb{R}$

$$\widehat{E_+}(\nu) = \int_0^\infty e^{-\mathrm{i}\nu\xi} e^{-\xi}\,\mathcal{I}\,\mathrm{d}\xi = \frac{1}{\mathrm{i}\nu + 1}\,\mathcal{I}$$

. We see that $\widehat{E_+}$ is a holomorphic L^2 function in U. We now use a similar approach as above but not for $\nu \mapsto \nu^k \Delta_{L_0}^{-1}(\nu)$ (which is in general $\in L^2$ but $\notin L^1$ along the imaginary line) but

$$R_+(\nu) \coloneqq \nu \mapsto \nu^k \Delta_{L_0}^{-1}(\nu) - (\nu+1)^{-1}\,\mathcal{I}$$

. Again we start with an uniform quadratic decay in U for $|\mathrm{Im}\xi| \to \infty$. Let $\xi \in U$

$$C \coloneqq 2 + \max_{0\leq j\leq k-1} \sum_{i=1}^{N} \|A_{i,j}\|$$

and $|\mathrm{Im}\xi| > kC + 1$. Then

$$\begin{aligned}
\|R_+(\xi)\| &= \|\xi^{k-1}\Delta_{L_0}^{-1}(\xi) - (\xi+1)^{-1}\,\mathcal{I}\| \\
&= \|\xi^{k-1}\big(\xi^k\,\mathcal{I} - \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\xi^j e^{\xi r_i}\big)^{-1} - (\xi+1)^{-1}\,\mathcal{I}\| \\
&= \|\xi^{k-1}(\xi+1)\,\mathcal{I} - \xi^k\,\mathcal{I} + \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\xi^j e^{\xi r_i}\|\|\xi^{k+1}\,\mathcal{I} + \xi^k\,\mathcal{I} - \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\xi^j e^{\xi r_i}\|^{-1} \\
&\leq \Big(|\xi|^{k-1} + \sum_{j=1}^{k-1}\big(\sum_{i=1}^{N}\|A_{i,j}\|\big)|\xi|^j\Big)\Big(|\xi|^{k+1} - |\xi|^k - \sum_{j=0}^{k-1}\big(\sum_{i=1}^{N}\|A_{i,j}\|\big)|\xi|^j\Big)^{-1} \\
&\leq \big(kC|\xi|^{k-1}\big)\big(|\xi|^{k+1} - |\xi|^k - kC|\xi|^{k-1}\big)^{-1} \\
&\leq kC\big(|\xi|^2 - |\xi| - kC\big)^{-1} \\
&\leq kC\big(|\mathrm{Im}\xi|(|\mathrm{Im}\xi| - 2)\big)^{-1} < \infty
\end{aligned}$$

. Thus $R_+(\xi) \in O(|\mathrm{Im}\xi|^{-2})$ uniformly in U for $|\mathrm{Im}\xi| \to \infty$ and $\forall a \in [-a_0, a_0] : \nu \mapsto R_+(a + \mathrm{i}\nu) \in L^1$. As above we may shift the path of integration using Cauchy's integral theorem. This finally yields

$$\|G_0^{(k-1)}(\xi)\| = E_+(\xi) + \frac{e^{-a_0\xi}}{2\pi}\|\nu \mapsto R_+(-a_0+\mathrm{i}\nu)\|_{L^1} \leq \big(1 + \frac{1}{2\pi}\|\nu \mapsto R_+(-a_0+\mathrm{i}\nu)\|_{L^1}\big)e^{-a_0\xi}$$

.

Likewise using

$$E_- : \mathbb{R} \to \mathbb{C}^{d\times d}$$
$$\xi \mapsto \begin{cases} \sqrt{2\pi} e^{\xi} \mathcal{I} & \xi \leq 0 \\ 0 & \xi > 0 \end{cases}$$

the same line of argument gives

$$\|G_0^{(k-1)}(\xi)\| \leq E_-(\xi) + \frac{e^{a_0\xi}}{2\pi} \|\nu \mapsto R_-(a_0 + \mathrm{i}\nu)\|_{L^1} \leq \big(1 + \frac{1}{2\pi} \|\nu \mapsto R_-(a_0 + \mathrm{i}\nu)\|_{L^1}\big) e^{a_0\xi}$$

for $\xi < 0$.

Now it is evident that some $K > 0$ can be found satisfying

$$\forall 0 \leq j \leq k-1 : \forall \xi \in \mathbb{R} : \quad \|G_0^{(j)}(\xi)\| \leq K e^{-a_0|\xi|}$$

. So (2.2) is proven. It immediatly follows that $G_0 \in W^{k-1,p}$ for any $1 \leq p \leq \infty$ and the proof is finished. □

Theorem 2.1.2

Consider the operators from (1.6) and (1.8) for a hyperbolic constant coefficient system and conjugated $p, q \in [1, \infty]$ and let G_0 be as in Proposition 2.1.1. Then

1. Λ_{L_0} is one-to-one
2. Λ_{L_0} is an isomorphism and $\forall h \in L^p$

$$\big(\Lambda_{L_0}^{-1} h\big)(\xi) = (G_0 * h)(\xi) = \int_{\mathbb{R}} G_0(\xi - \nu) h(\nu) \mathrm{d}\nu \tag{2.4}$$

Proof. In this proof we show that the equation holds tempered distributionally for the tempered distribution induced by the function resulting from (2.4). Since $C_0^\infty \subseteq \mathfrak{S}$ as a set it follows that the equation also holds for the weak derivatives.

Let $\xi \in \mathbb{R}$, $\varphi \in \mathfrak{S}$.

1: First we prove injectivity. Suppose that $x \in W^{1,p}$ and $\Lambda_{L_0} x = 0$. We recall properties and calculus of distributional Fourier transforms and get

$$
\begin{aligned}
x^{(k)} &= \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\tau_{-r_i}x^{(j)} \\
\Rightarrow T_x^{(k)} &= \sum_{i=1}^{N}\sum_{j=0}^{k=1} A_{i,j}\tau_{-r_i}T_x^{(j)} \\
\Rightarrow \widehat{T_x^{(k)}} &= \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\widehat{\tau_{-r_i}T_x^{(j)}} \\
\Rightarrow (\mathrm{i}\xi)^k\widehat{T_x} &= \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}(\mathrm{i}\xi)^j e^{\mathrm{i}r_i}\widehat{T_x} \\
\Rightarrow \big(\mathrm{i}\xi - \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}(\mathrm{i}\xi)^j e^{\mathrm{i}\xi r_i}\big)\widehat{T_x} = 0
\end{aligned}
$$

. By our hyperbolicity condition $\forall\nu \in \mathbb{R} : \Delta_{L_0}(\mathrm{i}\nu) \neq 0$ and so $\widehat{T_x}$ is the zero distribution and thus $x = 0$ and 1 is proven.

2: Now to surjectivity:

As $G_0 \in L^2$ it induces a tempered distribution T_{G_0}. Consider the tempered distribution

$$
\Gamma := T_{G_0}^{(k)} - \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\tau_{-r_i}T_{G_0}^{(j)}
$$

. Again using the calculus of distributions yields

$$
\begin{aligned}
\widehat{\Gamma} &= \widehat{T_{G_0}^{(k)}} - \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\widehat{\tau_{-r_i}T_{G_0}^{(j)}} \\
&= (\mathrm{i}\xi)^k\widehat{T_{G_0}} - \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}(\mathrm{i}\xi)^j e^{\mathrm{i}\xi r_i}\widehat{T_{G_0}} \\
&= \Delta(\mathrm{i}\xi)T_{\widehat{G_0}} \\
&= \frac{1}{\sqrt{2\pi}}\Delta(\mathrm{i}\xi)T_{\Delta(\mathrm{i}\xi)^{-1}} \\
&= \frac{1}{\sqrt{2\pi}}T_{\mathcal{I}}
\end{aligned}
$$

. Thus $\widehat{\Gamma} = \widehat{\delta}$ where δ denotes the delta tempered distribution. Therefore

$$
T_{G_0}^{(k)} = \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\tau_{-r_i}T_{G_0}^{(j)} + \delta \tag{2.5}
$$

and by a simple calculation

$$T^{(k)}_{G_0^T} = \sum_{i=1}^{N}\sum_{j=0}^{k-1} \tau_{-r_i} T^{(j)}_{G_0^T} \circ A_{i,j}^T + \delta$$

.

Now we are "feature complete" for proving surjectivity as all necessary parts have been gathered. Let $x \coloneqq G_0 * h$. As $G_0 \in W^{k-1,1}$ it follows that $x \in W^{k-1,p}$. Because x is already $\in W^{k-1,p}$ it suffices to show that our equation holds tempered distributionally to prove that x is a solution. For any $1 \leq i \leq N, 0 \leq j \leq k-1$ by Tonelli's theorem, Young's inequality and Hölder's inequality

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} |\langle G_0^{(j)}(\xi + r_i - \nu)h(\nu), \overline{\varphi(\xi)}\rangle| \mathrm{d}\nu \mathrm{d}\xi &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \|G_0^{(j)}(\xi + r_i - \nu)h(\nu)\| \|\varphi(\xi)\| \mathrm{d}\nu \mathrm{d}\xi \\
&\leq \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \|G_0^{(j)}(\xi + r_i - \nu)\| \|h(\nu)\| \mathrm{d}\nu \Big) \|\varphi(\xi)\| \mathrm{d}\xi \\
&= \int_{\mathbb{R}} (\|G_0^{(j)}\| * \|h\|)(\xi + r_i) \|\varphi(\xi)\| \mathrm{d}\xi \\
&\leq \| \|G_0^{(j)}\| * \|h\| \|_{L^p} \|\varphi\|_{L^q} \\
&\leq \|G_0^{(j)}\|_{L^1} \|h\|_{L^p} \|\varphi\|_{L^q} < \infty
\end{aligned}$$

.

Thus we may use Fubini's theorem on the integrals in (2.5) and get

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\tau_{-r_i} T_x^{(j)}(\varphi) &= \sum_{i=1}^{N}\sum_{j=0}^{k-1} \int_{\mathbb{R}} \langle A_{i,j} x^{(j)}(\xi + r_i), \overline{\varphi(\xi)} \rangle \mathrm{d}\xi \\
&= \sum_{i=1}^{N}\sum_{j=0}^{k-1} \int_{\mathbb{R}} \langle A_{i,j} \int_{\mathbb{R}} G_0^{(j)}(\xi + r_i - \nu) h(\nu) \mathrm{d}\nu, \overline{\varphi(\xi)} \rangle \mathrm{d}\xi \\
&= \int_{\mathbb{R}} \langle h(\nu), \overline{\sum_{i=1}^{N}\sum_{j=0}^{k-1} \int_{\mathbb{R}} (G_0^{(j)}(\xi + r_i))^T A_{i,j}^T \varphi(\xi + \nu) \mathrm{d}\xi} \rangle \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \langle h(\nu), \overline{\sum_{i=1}^{N}\sum_{j=0}^{k-1} \tau_{-r_i} T^{(j)}_{G_0^T} \big(A_{i,j}^T \tau_{-\nu}\varphi \big)} \rangle \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \langle h(\nu), \overline{T^{(k)}_{G_0^T}\big(\tau_{-\nu}\varphi\big) - \delta\big(\tau_{-\nu}\varphi\big)} \rangle \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \langle h(\nu), \overline{(-1)^k \int_{\mathbb{R}} (G_0(\xi))^T \varphi^{(k)}(\xi + \nu) \mathrm{d}\xi - \varphi(\nu)} \rangle \mathrm{d}\nu \\
&= (-1)^k \int_{\mathbb{R}} \langle \int_{\mathbb{R}} G_0(\xi - \nu) h(\nu) \mathrm{d}\nu, \overline{\varphi^{(k)}(\xi)} \rangle \mathrm{d}\xi - \int_{\mathbb{R}} \langle h(\nu), \overline{\varphi(\nu)} \rangle \mathrm{d}\nu \\
&= (-1)^k \int_{\mathbb{R}} \langle x(\xi), \overline{\varphi^{(k)}(\xi)} \rangle \mathrm{d}\xi - \int_{\mathbb{R}} \langle h(\xi), \overline{\varphi(\xi)} \rangle \mathrm{d}\xi \\
&= T_x^{(k)}(\varphi) - T_h(\varphi)
\end{aligned}$$

. Hence

$$T_x^{(k)} = \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}\tau_{-r_i}T_x^j + T_h \tag{2.6}$$

from which it immediately follows that x is a solution and Λ_{L_0} is onto. $\square$

2.2 Perturbed hyperbolic constant coefficient operators

The next step is the the construction of a Green's function for small perturbations L_0+M of a hyperbolic constant coefficient operator L_0. We remark that by small we naturally mean sufficiently small since our matrix coefficients are already uniformly bounded by assumption.

Hence we assume $L(\xi) = L_0 + M(\xi)$, where L_0 is hyperbolic. Then by Theorem 2.1.2 Λ_{L_0} is invertible and by (1.14) we have

$$\Lambda_L = \Lambda_{L_0} - M' = \left(\mathcal{I} - M'\Lambda_{L_0}^{-1}\right)\Lambda_{L_0}$$

. Assuming $\|M'\Lambda_{L_0}^{-1}\|_{B(L^p)} < 1$ we can use the von Neumann series and obtain

$$\Lambda_L^{-1} = \Lambda_{L_0}^{-1}\sum_{i=0}^{\infty}\left(M'\Lambda_{L_0}^{-1}\right)^i \tag{2.7}$$

.

Our next steps are to prove technical lemma to facilitate deduction of conditions for $\|M'\Lambda_{L_0}^{-1}\|_{B(L^p)} < 1$ and then to use (2.7) to construct a Green's function for a lightly perturbed hyperbolical constant coefficient operator. We remark that, Λ_L is an isomorphism whenever the perturbation M fullfills such conditions.

Lemma 2.2.1
Let $K > 0, a > 0, K < a/2, \Psi : \mathbb{R} \to [0,\infty), \xi \mapsto Ke^{-a|\xi|}$. We define the i-fold convolutions of Ψ with itself by

$$i = 1: \quad \Psi^{*1} := \Psi \qquad\qquad i > 1: \quad \Psi^{*i} := \Psi * \Psi^{*i-1}$$

and set

$$a_1 = (a^2 - 2Ka)^{\frac{1}{2}} \qquad\qquad K_1 = \frac{Ka}{a_1}$$

. Then $\forall \xi \in \mathbb{R}$

$$\sum_{i=1}^{\infty}\Psi^{*i}(\xi) = K_1 e^{-a_1|\xi|} \tag{2.8}$$

.

Proof. Let $\nu \in \mathbb{R}$. For arbitrary $K > 0, a > 0$ the Fourier transform of Ψ is given by

$$\begin{aligned}
\widehat{\Psi}(\nu) &= \frac{K}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\mathrm{i}\nu\xi} e^{-a|\xi|} \mathrm{d}\xi \\
&= \frac{K}{\sqrt{2\pi}} \Big(\int_{-\infty}^{0} e^{\xi(-\mathrm{i}\nu + a)} \mathrm{d}\xi + \int_{0}^{\infty} e^{\xi(-\mathrm{i}\nu - a)} \mathrm{d}\xi \Big) \\
&= \frac{K}{\sqrt{2\pi}} \Big(\frac{1}{-\mathrm{i}\nu + a} - \frac{1}{-\mathrm{i}\nu - a} \Big) \\
&= \frac{1}{\sqrt{2\pi}} \frac{2aK}{a^2 + \nu^2}
\end{aligned}$$

. Since $K < a/2$ we have $\forall \nu \in \mathbb{R} : \widehat{\Psi}(\nu) < \frac{1}{\sqrt{2\pi}} \frac{a^2}{a^2+\nu^2} < \frac{1}{\sqrt{2\pi}}$ and thus using convergence of the geometric series we conclude

$$\begin{aligned}
\sum_{i=1}^{\infty} (\sqrt{2\pi})^{i-1} \widehat{\Psi}^i(\nu) &= \frac{1}{\sqrt{2\pi}} \Big(\frac{1}{1 - \widehat{\Psi}(\nu)} - 1 \Big) \\
&= \frac{1}{\sqrt{2\pi}} \frac{\widehat{\Psi}(\nu)}{1 - \widehat{\Psi}(\nu)} \\
&= \frac{1}{\sqrt{2\pi}} \frac{2Ka}{a^2 - 2Ka + \nu^2} \\
&= \frac{1}{\sqrt{2\pi}} \frac{2K_1 a_1}{a_1^2 + \nu^2}
\end{aligned}$$

, where the factor $\sqrt{2\pi}$ comes from the chosen normalization in the integral expressions for the Forier transform. As $\mathcal{F} : L^2 \to L^2$ and $\mathcal{F}^{-1} : L^2 \to L^2$ are continuous maps we end up with

$$\begin{aligned}
\sum_{i=1}^{\infty} \Psi^{*i}(\xi) &= \mathcal{F}^{-1} \Big(\sum_{i=1}^{\infty} (\sqrt{2\pi})^{i-1} \widehat{\Psi}^{*i} \Big)(\xi) \\
&= \mathcal{F}^{-1} \Big(\nu \mapsto \frac{1}{\sqrt{2\pi}} \frac{2K_1 a_1}{a_1^2 + \nu^2} \Big)(\xi) \\
&= K_1 e^{-a_1|\xi|}
\end{aligned}$$

□

Theorem 2.2.2
Assume that L_0 is hyperbolic. Then $\exists \epsilon > 0, K > 0, a > 0$ such that if

$$\begin{aligned}
M, M' \text{ as in 1.1.6} \qquad & \forall \xi \in \mathbb{R} : \ \|M(\xi)\|_{R^{k-1}} \le \epsilon \\
L(\xi) = L_0 + M(\xi) \qquad & \Lambda_L = \Lambda_{L_0} - M'
\end{aligned}$$

then

1. Λ_L is an isomorphism

2. For $0 \leq i \leq k-1$ there $\exists G_i : \mathbb{R}^2 \to \mathbb{C}^{d\times d}$ such that

$$\forall \xi \in \mathbb{R} : \forall \nu \in \mathbb{R} : \quad \|G_i(\xi,\nu)\| \leq Ke^{-a|\xi-\nu|} \tag{2.9}$$

and moreover $\forall h \in L^p, \forall \xi \in \mathbb{R}$

$$\left(\Lambda_L^{-1} h\right)^{(i)}(\xi) = \int_{\mathbb{R}} G_i(\xi,\nu)h(\nu)\mathrm{d}\nu \tag{2.10}$$

.

Proof. The agenda in this proof is to first construct viable kernels for all weak derivatives and then prove that the functions obtained from the convolution integrals (2.10) are indeed derivatives of a solution. We show the latter by proving convergence of the differentiated partial sums of the von Neumann series to the integral expressions.

Let $h \in L^p$, $\xi \in \mathbb{R}$. We assume that $\{\xi \in \mathbb{R} \,|\, \|M(\xi)\|_{R^{k-1}} \neq 0\}$ has positive measure. If this is not the case then there is nothing to show.

As L_0 is hyperbolic by Theorem 2.1.2 we have a Green's function G_{00} for L_0 satisfying $\forall 1 \leq j \leq k-1 : \forall \xi \in \mathbb{R} : \|G_{00}^{(j)}(\xi)\| \leq K_0 e^{-a_0|\xi|}$. Inductively define kernels

$$\begin{aligned} i = 1: \quad \Gamma_1(\xi,\nu) &:= \sum_{i=1}^{N}\sum_{j=0}^{k-1} B_{i,j}(\xi) G_{00}^{(j)}(\xi + r_i - \nu) \\ i > 1: \quad \Gamma_i(\xi,\nu) &:= \int_{\mathbb{R}} \Gamma_1(\xi,\mu)\Gamma_{i-1}(\mu,\nu)\mathrm{d}\mu \end{aligned}$$

. Then

$$\begin{aligned} \|\Gamma_1(\xi,\nu)\| &= \|\sum_{i=1}^{N}\sum_{j=0}^{k-1} B_{i,j}(\xi) G_{00}^{(j)}(\xi + r_i - \nu)\| \\ &= \|\big(M(\xi)\big)(\tau_{\nu-\xi}G_{00})\| \\ &\leq \|M(\xi)\|_{R^{k-1}} \|\tau_{\nu-\xi}G_{00}\|_{C_B^{k-1}([r_-,r_+])} \\ &\leq \sup_{\xi\in\mathbb{R}} \|M(\xi)\|_{R^{k-1}} \sup_{\mu\in[r_-,r_+]} K_0 e^{-a_0|\xi-\nu+\mu|} \\ &\leq \Big(K_0 \sup_{\xi\in\mathbb{R}} \|M(\xi)\|_{R^{k-1}} \max_{1\leq i\leq N} e^{a_0|r_i|}\Big) e^{-a_0|\xi-\nu|} \end{aligned} \tag{2.11}$$

. As the matrix coefficients are uniformly bounded we can set

$$\begin{aligned} K_{00} &:= K_0 \sup_{\xi\in\mathbb{R}} \|M(\xi)\|_{R^{k-1}} \max_{1\leq i\leq N} e^{a_0|r_i|} < \infty \\ \Psi(\xi) &:= K_{00} e^{-a_0|\xi|} \end{aligned}$$

and get $\forall i \in \mathbb{N}$

$$\|\Gamma_i(\xi,\nu)\| \leq \Psi^{*i}(\xi - \nu) \tag{2.12}$$

. This is obtained by induction

$i = 1$: by (2.11)

$i \to i+1$:

$$\begin{aligned}
\|\Gamma_i(\xi,\nu)\| &\leq \int_{\mathbb{R}} \|\Gamma_1(\xi,\mu)\Gamma_{i-1}(\mu,\nu)\| \mathrm{d}\mu \\
&\leq \int_{\mathbb{R}} \Psi^{*1}(\xi-\mu)\Psi^{*i-1}(\mu-\nu)\mathrm{d}\mu \\
&= \int_{\mathbb{R}} \Psi^{*1}((\xi-\nu)-(\mu-\nu))\Psi^{*i-1}(\mu-\nu)\mathrm{d}\mu \\
&= \int_{\mathbb{R}} \Psi^{*1}((\xi-\nu)-\mu)\Psi^{*i-1}(\mu)\mathrm{d}\mu \\
&= \Psi^{*i}(\xi-\nu)
\end{aligned}$$

.

Consideration of $M'\Lambda_{L_0}^{-1}$ gives

$$\begin{aligned}
\big(M'\Lambda_{L_0}^{-1}h\big)(\xi) &= M' \int_{\mathbb{R}} G_{00}(\xi-\nu)h(\nu)\mathrm{d}\nu \\
&= \sum_{i=1}^{N}\sum_{j=0}^{k-1} B_{i,j}(\xi) \int_{\mathbb{R}} G_{00}^{(j)}(\xi+r_i-\nu)h(\nu)\mathrm{d}\nu \\
&= \int_{\mathbb{R}} \Big(\sum_{i=1}^{N}\sum_{j=0}^{k-1} B_{i,j}(\xi)G_{00}^{(j)}(\xi+r_i-\nu)\Big)h(\nu)\mathrm{d}\nu \\
&= \int_{\mathbb{R}} \Gamma_1(\xi,\nu)h(\nu)\mathrm{d}\nu
\end{aligned} \tag{2.13}$$

.

Now assume $K_{00} < a_0/2$, that is

$$\begin{aligned}
&K_0 \sup_{\xi\in\mathbb{R}} \|M(\xi)\|_{R^{k-1}} \max_{1\leq i\leq N} e^{a_0|r_i|} < a_0/2 \\
\Leftrightarrow &\sup_{\xi\in\mathbb{R}} \|M(\xi)\|_{R^{k-1}} < \frac{a_0}{2K_0 \max_{1\leq i\leq N} e^{a_0|r_i|}}
\end{aligned} \tag{2.14}$$

.

Then by using Young's inequality on the very last integral in (2.13) we obtain

$$
\begin{aligned}
\|\big(M'\Lambda_{L_0}^{-1}\big)h\|_{L^p} &= \|\xi \mapsto \int_{\mathbb{R}} \Gamma_1(\xi,\nu)h(\nu)\mathrm{d}\nu\|_{L^p} \\
&\leq \|\xi \mapsto \int_{\mathbb{R}} \|\Gamma_1(\xi,\nu)h(\nu)\|\mathrm{d}\nu\|_{L^p} \\
&\leq \|\xi \mapsto \int_{\mathbb{R}} \Psi^{*1}(\xi-\nu)\|h(\nu)\|\mathrm{d}\nu\|_{L^p} \\
&= \|\Psi^{*1} * \|h\|\|_{L^p} \\
&\leq \|\Psi^{*1}\|_{L^1}\|h\|_{L^p} \\
&= \Big(K_{00}2\int_0^\infty e^{-a_0\xi}\mathrm{d}\xi\Big)\|h\|_{L^p} \\
&= \frac{2K_{00}}{a_0}\|h\|_{L^p} \\
&< \|h\|_{L^p}
\end{aligned}
\tag{2.15}
$$

. Thus we conclude $\|M'\Lambda_{L_0}^{-1}\|_{B(L^p)} < 1$ whenever the assumption $K_{00} < a_0/2$ hold. We recall (2.14) and take

$$
0 < \epsilon < \frac{a_0}{2K_0 \max_{1\leq i\leq N} e^{a_0|r_i|}}
\tag{2.16}
$$

. Hence by using the von Neumann series (2.7) the first part of our Theorem 1 is proven.

In order to prove the second part of the theorem we will examine the kernels defined for the Green's functions in more detail. We take $1 < i \in \mathbb{N}$, still assume $K_{00} < a_0/2$ and use Hölder's inequality and Tonelli's theorem and obtain

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} \|\Gamma_1(\xi,\mu)\Gamma_{i-1}(\mu,\nu)h(\nu)\|\mathrm{d}\mu\mathrm{d}\nu &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \Psi^{*1}(\xi-\mu)\Psi^{*i-1}(\mu-\nu)\mathrm{d}\mu\|h(\nu)\|\mathrm{d}\nu \\
&\leq \int_{\mathbb{R}} \Psi^{*i}(\xi-\nu)\|h(\nu)\|\mathrm{d}\nu \\
&\leq \|\Psi^{*i}\|_{L^q}\|h\|_{L^p} < \infty
\end{aligned}
$$

. Therefore Fubini's theorem is applicable and yields

$$
\begin{aligned}
\Big(\big(M'\Lambda_{L_0}^{-1}\big)^i h\Big)(\xi) &= \Big((M'\Lambda_{L_0}^{-1})(M'\Lambda_{L_0}^{-1})^{i-1}h\Big)(\xi) \\
&= \int_{\mathbb{R}} \Gamma_1(\xi,\mu)\Big(\int_{\mathbb{R}} \Gamma_{i-1}(\mu,\nu)h(\nu)\mathrm{d}\nu\Big)\mathrm{d}\mu \\
&= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} \Gamma_1(\xi,\mu)\Gamma_{i-1}(\mu,\nu)\mathrm{d}\mu\Big)h(\nu)\mathrm{d}\nu \\
&= \int_{\mathbb{R}} \Gamma_i(\xi,\nu)h(\nu)\mathrm{d}\nu
\end{aligned}
\tag{2.17}
$$

.

Let $0 \leq j \leq k-1, i \in \mathbb{N}$ and consider $(\mu,\nu) \mapsto G_{00}^{(j)}(\xi-\mu)\Gamma_i(\mu,\nu)h(\nu)$ where

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}\|G_{00}^{(j)}(\xi-\mu)\Gamma_i(\mu,\nu)h(\nu)\|\mathrm{d}\mu\mathrm{d}\nu &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} K_0 e^{-a_0|\xi-\mu|}\Big(\Psi^{*i}(\mu-\nu)\|h(\nu)\|\mathrm{d}\nu\Big)\mathrm{d}\mu \\ &\leq K_0\|\mu \mapsto e^{-a_0|\xi-\mu|}\|_{L^q}\|\Psi^{*i} * h\|_{L^p} \\ &\leq K_0\|\mu \mapsto e^{-a_0|\xi-\mu|}\|_{L^q}\|\Psi^{*i}\|_{L^1}\|h\|_{L^p} < \infty\end{aligned}$$

. Hence using Tonelli's and Fubini's theorems again yields

$$\begin{aligned}\Big(D^j\Lambda_{L_0}^{-1}\big(M'\Lambda_{L_0}^{-1}\big)^i h\Big)(\xi) &= \Big(\big(D^j\Lambda_{L_0}^{-1}\big)\big(M'\Lambda_{L_0}^{-1}\big)^i h\Big)(\xi) \\ &= \int_{\mathbb{R}} G_{00}^{(j)}(\xi-\mu)\Big(\int_{\mathbb{R}}\Gamma_i(\mu,\nu)h(\nu)\mathrm{d}\nu\Big)\mathrm{d}\mu \\ &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} G_{00}^{(j)}(\xi-\mu)\Gamma_i(\mu,\nu)\mathrm{d}\mu\Big)h(\nu)\mathrm{d}\nu\end{aligned} \tag{2.18}$$

.

Hence Lemma 2.2.1 gives

$$\sum_{i=1}^{\infty}\|\Gamma_i(\xi,\nu)\| \leq \sum_{i=1}^{\infty}\Psi^{*i}(\xi-\nu) \leq K_1 e^{-a_1|\xi-\nu|} \tag{2.19}$$

where

$$a_1 \coloneqq (a_0^2 - 2K_{00}a_0)^{\frac{1}{2}} \qquad\qquad K_1 = \frac{K_{00}a_0}{a_1}$$

.

Thus we can set

$$G_i(\xi,\mu) \coloneqq G_{00}^{(i)}(\xi-\nu) + \int_{\mathbb{R}} G_{00}^{(i)}(\xi-\mu)\Big(\sum_{j=1}^{\infty}\Gamma_j(\mu,\nu)\Big)\mathrm{d}\mu \tag{2.20}$$

for $0 \leq i \leq k-1$. These are well defined $W^{k-1-i,p}$ functions by Proposition 2.1.1 and Equation (2.19). Furthermore these functions already formally satisfy (2.10). Using $a_1 < a_0$ we get

$$\begin{aligned}\|G_i(\xi,\nu)\| &\leq K_0 e^{-a_0|\xi-\nu|} + \int_{\mathbb{R}} K_0 e^{-a_0|\xi-\mu|}K_1 e^{-a_1|\mu-\nu|}\mathrm{d}\mu \\ &\leq K_0 e^{-a_1|\xi-\nu|} + K_0K_1\Big(\frac{2}{a_0+a_1} + \frac{1}{|a_0-a_1|} + \frac{1}{|a_1-a_0|}\Big)e^{-a_1|\xi-\nu|}\end{aligned}$$

, the second inequality following from explicit calculation of the integral. Specifically if

$\xi \leq \nu$ then

$$
\begin{aligned}
&\int_{-\infty}^{\infty} e^{-a_0|\xi-\mu|-a_1|\mu-\nu|}\mathrm{d}\mu \\
&= \int_{-\infty}^{\xi} e^{-a_0(\xi-\mu)-a_1(\nu-\mu)}\mathrm{d}\mu + \int_{\xi}^{\nu} e^{a_0(\xi-\mu)-a_1(\nu-\mu)}\mathrm{d}\mu + \int_{\nu}^{\infty} e^{a_0(\xi-\mu)+a_1(\nu-\mu)}\mathrm{d}\mu \\
&= e^{-a_0\xi-a_1\nu}\int_{-\infty}^{\xi} e^{(a_0+a_1)\mu}\mathrm{d}\mu + e^{a_0\xi-a_1\nu}\int_{\xi}^{\nu} e^{(-a_0+a_1)\mu}\mathrm{d}\mu + e^{a_0\xi+a_1\nu}\int_{\nu}^{\infty} e^{-(a_0+a_1)\mu}\mathrm{d}\mu \\
&= \frac{1}{a_0+a_1}e^{-a_1(\nu-\xi)} + \frac{1}{a_1-a_0}e^{-a_0(\nu-\xi)} + \frac{1}{a_0-a_1}e^{-a_1(\nu-\xi)} + \frac{1}{a_0+a_1}e^{-a_0(\nu-\xi)} \\
&\leq \Big(\frac{2}{a_0+a_1} + \frac{1}{|a_0-a_1|} + \frac{1}{|a_1-a_0|}\Big)e^{-a_1|\xi-\nu|}
\end{aligned}
$$

. The (almost) same calculation gives the same inequality for $\xi > \nu$, namely

$$
\exists K > 0 : \forall 0 \leq i \leq k : \quad \|G_i(\xi,\nu)\| \leq K e^{-a_1|\xi-\mu|} \tag{2.21}
$$

.

To complete the proof it remains to be shown that if $x = \Lambda_L^{-1}h$ then indeed $\forall 0 \leq i \leq k-1 : x^{(i)}(\xi) = \int_{\mathbb{R}} G_i(\xi,\nu)h(\nu)\mathrm{d}\nu$. To this end let $0 \leq i \leq k-1$ and set

$$
x_{i,j} = D^i\Lambda_{L_0}^{-1}\big(\sum_{l=0}^{j}(M'\Lambda_{L_0}^{-1})^l\big)h
$$

. We recall that $K_{00} < a_0/2$ implies $\|M'\Lambda_{L_0}^{-1}\|_{B(L^p)} < 1$ and conclude that $\forall 0 \leq i \leq k-1$:

$$
\|x_{i,j} - x^{(i)}\|_{L^p} \leq \|D^i\Lambda_{L_0}^{-1}\|_{B(L^p)}\Big(\sum_{l=j+1}^{\infty}\|(M'\Lambda_{L_0}^{-1})\|_{B(L^p)}^l\Big)\|h\|_{L^p} \to 0 \quad (j \to \infty)
$$

. Let y_i denote the functions resulting from the right hand side of (2.10), that is the integral expressions of a solution's derivatives. By Young's inequality $\forall 0 \leq i \leq k-1$

$$\begin{aligned}
\|x_{i,j} - y_i\|_{L^p} &= \|\xi \mapsto \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} G_{00}^{(i)}(\xi - \mu) \sum_{l=j+1}^{\infty} \Gamma_l(\mu, \nu) \mathrm{d}\mu \Big) h(\nu) \mathrm{d}\nu \|_{L^p} \\
&\leq \|\xi \mapsto \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} K_0 e^{-a_0|\xi - \mu|} \sum_{l=j+1}^{\infty} \Psi^{*l}(\mu - \nu) \mathrm{d}\mu \Big) h(\nu) \mathrm{d}\nu \|_{L^p} \\
&\leq \|\xi \mapsto \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} K_0 e^{-a_0|\xi - \nu - \mu|} \sum_{l=j+1}^{\infty} \Psi^{*l}(\mu) \mathrm{d}\mu \Big) h(\nu) \mathrm{d}\nu \|_{L^p} \\
&= K_0 \|\xi \mapsto \int_{\mathbb{R}} \Big(e^{-a_0|\xi|} * \sum_{l=j+1}^{\infty} \Psi^{*l}(\xi) \Big)(\xi - \nu) h(\nu) \mathrm{d}\nu \|_{L^p} \\
&\leq K_0 \|\xi \mapsto e^{-a_0|\xi|} * \sum_{l=j+1}^{\infty} \Psi^{*l}(\xi)\|_{L^1} \|h\|_{L^p} \\
&\leq K_0 \|\xi \mapsto e^{-a_0|\xi|}\|_{L^1} \| \sum_{l=j+1}^{\infty} \Psi^{*l}\|_{L^1} \|h\|_{L^p} \\
&\leq K_0 \|\xi \mapsto e^{-a_0|\xi|}\|_{L^1} \|h\|_{L^p} \sum_{l=j+1}^{\infty} \|\Psi\|_{L^1}^{l} \\
&\to 0 \quad (j \to \infty)
\end{aligned}$$

so $\forall 0 \leq i \leq k-1 : \|x^{(i)} - y_i\|_{L^p} = 0$ and the proof is finished. □

Intuitively one might assume that solving Equation (1.1) is more of a hassle when stepping from $k = 1$ to $k \in \mathbb{N}$ but simulateously the hyperbolicity conditions is not the same for equations of different order.

3 Towards a Fredholm Alternative

In this chapter we gradually develop some technical aids to prove finite dimensionality of the kernel of Λ_L and end up with Fredholmness of Λ_L and a formulated Fredholm Alternative theorem.

3.1 Finite dimensionality of the kernel(s)

Proposition 3.1.1
Assume that L as in (1.6) is asymptotically hyperbolic at ∞. Then there exist $K, K', a > 0$ such that $\forall x \in W^{k,p} : \forall h \in L^p : \Lambda_L x = h \Rightarrow \forall \xi \geq 0 : \forall 0 \leq j \leq k-1 :$

$$\begin{aligned} \|x^{(j)}(\xi)\| &\leq K e^{-a|\xi|} \|x\|_{W^{k-1,\infty}} + K \int_{\mathbb{R}} e^{-a|\xi - \nu|} \|h(\nu)\| \mathrm{d}\nu \\ &\leq K e^{-a|\xi|} \|x\|_{W^{k-1,\infty}} + K' \|h\|_{L^p} \end{aligned} \tag{3.1}$$

. If the equation is asymptotically hyperbolic at $-\infty$ the same holds $\forall \xi \leq 0$. If it is asymptotically hyperbolic (at both ∞ and $-\infty$) this inequality holds $\forall \xi \in \mathbb{R}$ and additionally there $\exists K'' > 0$ such that

$$\|x\|_{W^{k,p}} \leq K''(\|x\|_{W^{k-1,\infty}} + \|h\|_{L^p}) \tag{3.2}$$

. Note that all constants K, K', K'', a only depend on L.

Proof. Unlike in the last Theorem 2.2.2 we do not have uniformly small bounds for M. By by asymptotic hyperbolicity we know that they will be samll enough for some large ξ. So we cut off the “large” parts of M, put them into the inhomogeneity and apply the exponential bounds of the last theorem.

We first note that the continuous embedding $W^{k,p} \to W^{k-1,\infty}$ implies that the factor $\|x\|_{W^{k-1,\infty}}$ is finite and note further that the second inequality(ies) directly follow(s) from the first by applying Hölder's inequality.

Assume asymptotic hyperbolicity of L at ∞. Hence $\exists L_0 : \forall \xi \in \mathbb{R} : \exists M(\xi) \in R^{k-1}$ such that $L(\xi) = L_0 + M(\xi)$, $\|M(\xi)\|_{R^{k-1}} \to 0 \quad (\xi \to \infty)$ and L_0 is hyperbolic.

Let ϵ, a, K_1 be the constants from Theorem 2.2.2 for L_0. Assume $\xi \geq 0$ and fix some $\tau > 0$ such that $\forall \xi \geq \tau : \|M(\xi)\|_{R^{k-1}} \leq \epsilon$. Set

$$\begin{aligned} \alpha_+ &: \mathbb{R} \to \mathbb{R} \\ &\xi \mapsto \begin{cases} 0 & \xi < \tau \\ 1 & \xi \geq \tau \end{cases} \end{aligned}$$

and

$$L_1(\xi) \coloneqq L_0 + \alpha(\xi)M(\xi) \qquad\qquad M_1(\xi) \coloneqq (1-\alpha(\xi))M(\xi)$$

. Obviously $\forall x \in W^{k,p} : \xi \mapsto M_1\tau_\xi x + h(\xi) \in L^p$ so our functional differential equation can be rewritten to

$$x^{(k)}(\xi) = L_1\tau_{-\xi}x + \big(M_1\tau_{-\xi}x + h(\xi)\big)$$

. By definition $\forall \xi \in \mathbb{R} : \|\alpha_+(\xi)M(\xi)\|_{R^{k-1}} \leq \epsilon$ so using (2.10) and the notation introduced therein we conclude

$$\begin{aligned} x^{(i)}(\xi) &= \int_{\mathbb{R}} G_i(\xi,\nu)\Big(M_1(\nu)\tau_{-\nu}x + h(\nu)\Big)\mathrm{d}\nu \\ &= \int_{-\infty}^{\tau} G_i(\xi,\nu)\big(M_1(\nu)\tau_{-\nu}x\big)\mathrm{d}\nu + \int_{\mathbb{R}} G_i(\xi,\nu)h(\nu)\mathrm{d}\nu \end{aligned}$$

for $\forall \xi \in \mathbb{R} : \forall 0 \leq i \leq k-1$.

Hence

$$\begin{aligned} \|x^{(i)}(\xi)\| &\leq \int_{-\infty}^{\tau} \|G_i(\xi,\nu)\|\|M_1(\xi)\tau_{-\nu}x\|\mathrm{d}\nu + \int_{\mathbb{R}} \|G_i(\xi,\nu)h(\nu)\|\mathrm{d}\nu \\ &\leq \int_{-\infty}^{\tau} K_1 e^{-a|\xi-\nu|}\|M_1(\nu)\|_{R^{k-1}}\|\tau_{-\nu}x\|_{C_B^{k-1}([r_-,r_+])}\mathrm{d}\nu + \int_{\mathbb{R}} K_1 e^{-a|\xi-\nu|}\|h(\nu)\|\mathrm{d}\nu \\ &\leq K_1 \sup_{\xi\in\mathbb{R}}\|M(\xi)\|_{R^{k-1}}\|x\|_{W^{k-1,\infty}} \int_{\infty}^{\tau} e^{-a|\xi-\nu|}\mathrm{d}\nu + \int_{\mathbb{R}} K_1 e^{-a|\xi-\nu|}\|h(\nu)\|\mathrm{d}\nu \end{aligned}$$

. We note existance of the sup by assumption of uniform boundedness of A_i. We explicitly calculate the first integral to finish our proof of (3.1). First consider $\xi < \tau$ and recall $\tau > 0$ hence $e^{\pm a\tau} \neq 0$

$$\begin{aligned} \int_{-\infty}^{\tau} e^{-a|\xi-\nu|}\mathrm{d}\nu &= \int_{-\infty}^{\xi} e^{-a(\xi-\nu)}\mathrm{d}\nu + \int_{\xi}^{\nu} e^{a(\xi-\nu)}\mathrm{d}\nu \\ &= e^{-a\xi}\frac{1}{a}e^{a\xi} + e^{a\xi}\frac{1}{a}(e^{-a\xi} - e^{-a\tau}) \\ &= \frac{2}{a} - \frac{1}{a}e^{-a(\tau-\xi)} \leq \frac{3}{a} \leq \frac{3e^{-a\tau}}{a}e^{-a|\xi|} \end{aligned}$$

and for $\xi \geq \tau$

$$\int_{-\infty}^{\tau} e^{-a|\xi-\nu|}\mathrm{d}\nu = \frac{1}{a}e^{-a(\xi-\tau)} = \frac{e^{a\tau}}{a}e^{-a|\xi|}$$

. Hence $\exists K > 0$, such that (3.1) holds.

A similar line of argument using the limiting equation at $-\infty$, $\tau < 0$ such that $\forall \xi \leq \tau : \quad \|M(\xi)\|_{R^{k-1}} \leq \epsilon$ and

$$\alpha_-(\xi) = \begin{cases} 1 & \xi \leq \tau \\ 0 & \xi > \tau \end{cases}$$

proves the inequality for asymptotic hyperbolicity at $-\infty$ and $\xi \leq 0$.

Thus if L is asymptotically hyperbolic at both $\pm\infty$ then (3.1) holds $\forall \xi \in \mathbb{R}$.

To prove the $W^{k,p}$ bound, we apply Young's inequality to the convolution integrals in (3.1).

$$\forall 0 \leq i \leq k-1: \quad \|x^{(i)}\|_{L^p} \leq K\|\xi \mapsto e^{-a|\xi|}\|_{L^p}\|x\|_{W^{k-1,\infty}} + K\|\xi \mapsto e^{-a|\xi|}\|_{L^1}\|h\|_{L^p}$$

Setting $K''' := K(\|\xi \mapsto e^{-a|\xi|}\|_{L^p} + \|\xi \mapsto e^{-a|\xi|}\|_{L^1})$ we obtain

$$\forall 0 \leq i \leq k-1: \quad \|x^{(i)}\|_{L^p} \leq K'''(\|x\|_{W^{k-1,\infty}} + \|h\|_{L^p}) \tag{3.3}$$

. Additionaly, as L' is bounded, we conclude

$$\begin{aligned} \|x^{(k)}\|_{L^p} &= \|L'x + h\|_{L^p} \\ &\leq \|L'\|_{B(W^{k-1,p},L^p)}\|x\|_{W^{k-1,p}} + \|h\|_{L^p} \\ &\leq \|L'\|_{B(W^{k-1,p},L^p)}kK'''\Big(\|x\|_{W^{k-1,\infty}} + \|h\|_{L^p}\Big) + \|h\|_{L^p} \\ &\leq \Big(1 + \|L'\|_{B(W^{k-1,p},L^p)}kK'''\Big)\Big(\|x\|_{W^{k-1,\infty}} + \|h\|_{L^p}\Big) \end{aligned} \tag{3.4}$$

and the proof is finished. □

Remark 3.1.2

Suppose L is asymptotically hyperbolic $1 \leq p \leq \infty$ and take $x \in W^{k,p}$ with $\Lambda_L x = 0$, that is $x \in ker\Lambda_L$ with $\Lambda_L : W^{k,p} \to L^p$. Setting $h = 0$ and using Proposition 3.1.1 yields $x(\xi) \in O(e^{-a|\xi|})$ as $|\xi| \to \infty$. Hence $\forall 1 \leq p \leq \infty : x \in W^{k,p}$ so we setting

$$\mathfrak{K}_L^p := \{x \in W^{k,p} | \Lambda_L x = 0\} \tag{3.5}$$

gives (an) indentical and well defined space(s), which we will denote by $\mathfrak{K}_L$.

Lemma 3.1.3

Assume L is asymptotically hyperbolic, $(x_n) \in (W^{k,p})^{\mathbb{N}}$ bounded, $(h_n) \in (L^p)^{\mathbb{N}}$ with $\forall n \in \mathbb{N} : \Lambda_L x_n = h_n$, and $h_n \to h_* \in L^p$ in L^p. Then there exists a subsequence x_{n_m} converging to some $x_* \in W^{k,p}$ in $W^{k,p}$ with $\Lambda_L x_* = h_*$.

Proof. Our goal is to use the Arzelà-Ascoli theorem yielding a limit in $W^{k-1,p}$ and to prove it to be actually in $W^{k,p}$ and satisfying Equation (1.6).

Thus we first check the conditions for using the Arzelà-Ascoli theorem. This will yield a convergence of a subsequence x_{n_k} to some limiting function in $x_* \in W^{k-1,p}$, but with uniform convergence only on compact sets. We then show this convergence to be in $W^{k-1,\infty}$. Using Proposition 3.1.1 it will follow, that x_{n_k} is in fact a Cauchy sequence in $W^{k,p}$ and hence that it converges to a limit x'_*. By continuity such x'_* has to satisfy the differential equation and incidentally x_* and x'_* will coincide. As we are concurrently examining convergence in different spaces in this proof, we shall try to always explicitly state the space of convergence.

We start by proving sufficient conditions for use of the Arzelà-Ascoli theorem. Since the sequence is bounded in $W^{k,p}$, it is also bounded in $W^{k-1,\infty}$, that is $\exists C > 0 : \forall n \in \mathbb{N} : \|x_n\|_{W^{k-1,\infty}} < C$

If $p = \infty$ it immediatly follows from boundedness in $W^{k,\infty}$ that $\forall 0 \leq i \leq k-1 :$ the sequence $(x_n^{(i)})$ is equicontinuous.

If $p < \infty$ by Sobolev embedding $x \in W^{k-1,\infty}$ so $\forall : 0 \leq i \leq k-2 :$ the sequence $(x_n^{(i)})$ is equicontinuous. For $i = k$ we need some more work: Let $\operatorname{sgn}(\xi)$ denote the sign of ξ and define

$$H_n(\xi) = \int_0^{|\xi|} h_n(\operatorname{sgn}(\xi)\nu)\mathrm{d}\nu \qquad\qquad H_*(\xi) = \int_0^{|\xi|} h_*(\operatorname{sgn}(\xi)\nu)\mathrm{d}\nu$$

. Then $\forall \xi \in \mathbb{R}$:

$$\begin{aligned}
\|H_n(\xi) - H_*(\xi)\| &= \|\int_0^{|\xi|} h_n(\operatorname{sgn}(\xi)\nu) - h_*(\operatorname{sgn}(\xi)\nu)\mathrm{d}\nu\| \\
&\leq \left(\int_0^{|\xi|} \Big(\|\xi\| \|h_n(\operatorname{sgn}(\xi)\nu) - h_*(\operatorname{sgn}(\xi)\nu)\| \Big) \Big(\frac{1}{|\xi|}\mathrm{d}\nu \Big) \right)^{\frac{p}{p}} \\
&\leq |\xi| \Big(\int_0^{|\xi|} \|h_n(\operatorname{sgn}(\xi)\nu) - h_*(\operatorname{sgn}(\xi)\nu)\|^p \big(\frac{1}{|\xi|}\mathrm{d}\nu \big) \Big)^{\frac{1}{p}} \\
&\leq |\xi|^{1-\frac{1}{p}} \|h_n - h_*\|_{L^p}
\end{aligned}$$

by Jensen's inequality. Hence $H_n \to H_*$ uniformly on compacts interval and as such is equicontinuous on compact intervals.

To show this we take a compact interval $K \subset \mathbb{R}$ and $\xi \in K, \epsilon > 0$. As $H_n \to H_*$ uniformly in K we obtain

$$\exists N \in \mathbb{N} : \forall n \geq N : \forall \nu \in K : \|H_n(\nu) - H_*(\nu)\| < \frac{\epsilon}{3}$$

in K. Forall $k < N : H_n$ is continuous, so

$$\forall n < N : \exists \delta_n > 0 : \forall \nu \in K : |\xi - \nu| < \delta \Rightarrow \|H_n(\xi) - H_n(\nu)\| < \frac{\epsilon}{3} < \epsilon$$

. Additionally, as H_* is continuous, such a δ_* exists for H_*. Setting $\delta := \min\{\delta_1, \cdots, \delta_{N-1}, \delta_*\}$ we get $\forall \nu \in K$ with $|\xi - \nu| < \delta$ and $\forall n \geq N$

$$\|H_n(\xi) - H_n(\nu)\| \leq \|H_n(\xi) - H_*(\xi)\| + \|H_*(\xi) - H_*(\nu)\| + \|H_*(\nu) - H_n(\nu)\| < \epsilon$$

. Hence

$$\forall \nu \in K : \forall n \in \mathbb{N} : \|H_n(\xi) - H_n(\nu)\| < \epsilon$$

. Moreover

$$\|(x_n^{(k-1)} - H_n)^{(1)}\|_{L^\infty} = \|L' x_n\|_{L^\infty} \leq \|L'\|_{R^{k-1}} \|x_n\|_{W^{k-1,\infty}} \leq \|L'\|_{R^{k-1}} C$$

so the sequence $(x_n^{(k-1)})$ is equicontinuous on each compact interval.

To sum up $\forall 0 \le i \le k-1$ the sequence $(x_n^{(i)})$ is uniformly bounded and equicontinuous on each compact interval (regardless of p) and thus satisfies the conditions of the Arzelà-Ascoli theorem on each compact interval.

Fix one such compact inteval K and consider all functions restricted to K. Applying the Arzelà-Ascoli theorem to $(x_n|_K)$ yields a subsequence $(x_{n_k}|_K)$ converging to some $x_* \in C_B^0(K)$ uniformly. This subsequence naturally has all the properties of $(x_n|_K)$ with additionally $x_{n_k}|_K \to x_*$. For convenience we now denote this subsequence by (x_n). Applying the Arzelà-Ascoli theorem to $(x_n^{(1)}|_K)$ yields a subsequence $(x_{n_k}^{(1)}|_K)$ converging to some $g \in C_B^0(K)$ uniformly. As the convergence is uniform we have necessarily that $g = x_*^{(1)}$. Again denoting this subsequence by $(x_n|_K)$ for convenience and thus proceeding till $k-1$ we end up with $(x_n|_K)$ converging to $x_{*,K} \in C_B^{k-1}(K)$ with convergence in $C_B^{k-1}(K)$.

As $\mathbb{R}$ is σ-compact we can find a subsequence converging on $\mathbb{R}$ by a classical diagonalization argument. Consider a decompostion $\mathbb{R} = \bigcup_{i=1}^{\infty} K_i$ of $\mathbb{R}$ into compact intervals K_i. The argument in the last paragraph yields a subsequence $(x_{n_{1,k}})_{k\in\mathbb{N}}$ converging to some $y_1 \in C_B^{k-1}(K_1)$ uniformly in K_1. $(x_{n_{1,k}})$ has all the properties of x_n. We now can find a subsequence $(x_{n_{2,k}})_{k\in\mathbb{N}}$ of $(x_{n_{1,k}})_{k\in\mathbb{N}}$ converging to some $y_2 \in C_B^{k-1}(K_1)$. After inductively proceeding in this manner and setting $n_k := n_{k,k}$ we have $\forall i \in \mathbb{N} : x_{n_k}|_{K_i} \to y_i \in C_B^{k-1}(K_i)$ with convergence in those $C_B^{k-1}(K_i)$. As each of those x_{n_k} is already in $C_B^{k-1}(\mathbb{R})$ it follows that the pointwise limit $x_* = \lim_{k\to\infty} x_{n_k}$ exists, that $x_* \in C_B^{k-1}(\mathbb{R})$ and that the convergence $x_{n_k} \to x_*$ is actually in $C_B^{k-1}(K)$ for any a compact interval K. Recalling (x_n) being bounded in $W^{k,p}$ we additionally conclude $x_* \in W^{k-1,p} \cap W^{k-1,\infty}$. We now abuse notation and write (x_n) for the obtained converging subsequence.

Next we show this convergence to be in $W^{k-1,\infty}$. Proposition 3.1.1 is applicable to x_n, h_n for any $n \in \mathbb{N}$. We again stress the fact, that the constants in 3.1.1 do only depend on L. Hence $\forall n \in \mathbb{N}, \forall 0 \le i \le k-1, \forall \xi \in \mathbb{R}$:

$$\|x_n^{(i)}(\xi) - x_*^{(i)}(\xi)\| \le Ke^{-a|\xi|}\|x_n^{(i)} - x_*^{(i)}\|_{W^{k-1,\infty}} + K'\|h_n - h_*\|_{L^p}$$

so for any $\tau > 0$

$$\begin{aligned} \sup_{|\xi|\ge\tau} \|x_n^{(i)}(\xi) - x_*^{(i)}(\xi)\| &\le Ke^{-a\tau}\|x_n^{(i)} - x_*^{(i)}\|_{W^{k-1,\infty}} + K'\|h_n - h_*\|_{L^p} \\ &\le Ke^{-a\tau}(C + \|x_*\|_{W^{k-1,\infty}}) + K'\|h_n - h_*\|_{L^p} \end{aligned} \tag{3.6}$$

. As we already know that on the compact interval $[-\tau, \tau] : \forall 0 \le i \le k-1 : x_n^{(i)} \to x_*^{(i)}$ uniformly we conclude

$$\limsup_{n\to\infty} \|x_n - x_*\|_{W^{k-1,\infty}} \le Ke^{-a\tau}(C + \|x_*\|_{W^{k-1,\infty}}) \tag{3.7}$$

. As $\tau > 0$ is arbitrary we have convergence in $W^{k-1,\infty}$.

Hence (x_n) is a Cauchy sequence in $W^{k-1,\infty}$ and naturally (h_n) is one as well in L^p. Fix $\epsilon > 0$. Then taking $K'' > 0$ from Proposition 3.1.1 we have that

$$\exists N_x \in \mathbb{N} : \forall n, m \geq N_x : \quad \|x_n - x_m\|_{W^{k-1,\infty}} < \frac{\epsilon}{2K''}$$
$$\exists N_h \in \mathbb{N} : \forall n, m \geq N_h : \quad \|h_n - h_m\|_{L^p} < \frac{\epsilon}{2K''}$$

Hence applying (3.2) to $x_n - x_m$ and setting $N = \max\{N_x, N_h\}$ we have

$$\forall n, m \geq N : \quad \|x_n - x_m\|_{W^{k,p}} \leq K''(\|x_n - x_m\|_{W^{k-1,\infty}} + \|h_n - h_m\|_{L^p}) < \epsilon \tag{3.8}$$

. so (x_n) is a Cauchy sequence. As $W^{k,p}$ is a Banach space the sequence is convergent to some $x'_* \in W^{k,p}$ in $W^{k,p}$ and necessarily x_* and x'_* conincide. By continuity of Λ_L we finally obtain

$$\Lambda_L x_* = \Lambda_L \lim_{n\to\infty} x_n = \lim_{n\to\infty} \Lambda_L x_n = \lim_{n\to\infty} h_n = h \tag{3.9}$$

, which finishes the proof. □

Corollary 3.1.4
Assume L is asymptotically hyperbolic. Then the kernel $\mathfrak{K}_L$ of the operator Λ_L is finite dimensional, i.e. $\forall 1 \leq p \leq \infty : \mathfrak{K}_L$ is a finite dimensional subspace of $W^{k,p}$. As already mentioned in remark 3.1.2 the kernel is independent from p by Proposition 3.1.1 so this formulation is permitted.

Proof. We proceed by proving compactness of the unit ball in the kernel. Since this only holds in finite dimensional spaces it would finish the proof. Let $1 \leq p \leq \infty$.

Take any sequence $(x_n) \in (\mathfrak{K}_L)^{\mathbb{N}}$ with $\forall n \in \mathbb{N} : \|x_n\|_{L^p} \leq 1$. Set $h_n \equiv 0$ as the zero sequence in L^p. By Lemma 3.1.3 there exists a subsequence x_{n_k} converging to x_* in $W^{k,p}$, with $\Lambda_L x_* = 0$. Hence $x_* \in \mathfrak{K}_L$. Thus the unit ball in $\mathfrak{K}_L \subset W^{k,p}$ is compact and the proof is finished. □

Before we start with formulating a Fredholm Alternative we list a small technical aid, which can be and is directly taken from the original work [2] and is included for sake of completeness. Note that directly copying the proof is only possible as we were able to prove sufficiently strong extenstions to the original theorems in the preceding sections.

Corollary 3.1.5
Assume L is asymptotically hyperbolic. Then $\forall 1 \leq p \leq \infty : \Lambda_L(W^{k,p}) \subseteq L^p$ is closed.

Proof. Let $(h_n) \in \Lambda_L(W^{k,p})^{\mathbb{N}}, h_* \in L^p$ with $h_n \to h_*$ in L^p. We need to show $h_* \in \Lambda_L(W^{k,p})$.

By Corollary 3.1.4 $\mathfrak{K}_L$ is finite dimensional. Thus it is complemented in $W^{k,p}$, that is $\exists E \subset W^{k,p}$ closed linear subspace such that $W^{k,p} = \mathfrak{K}_L \oplus E$. Certainly $\Lambda_L(E) = \Lambda_L(W^{k,p})$ so $\exists (x_n) \in E^{\mathbb{N}} : \forall n \in \mathbb{N} : \Lambda_L x_n = h_n$.

Assume (x_n) is bounded in $W^{k,p}$. By Lemma 3.1.3 we obtain some $x_* \in W^{k,p}$ such that $\Lambda_L x_* = h_*$. Hence $h_* \in \Lambda_L(W^{k,p})$.

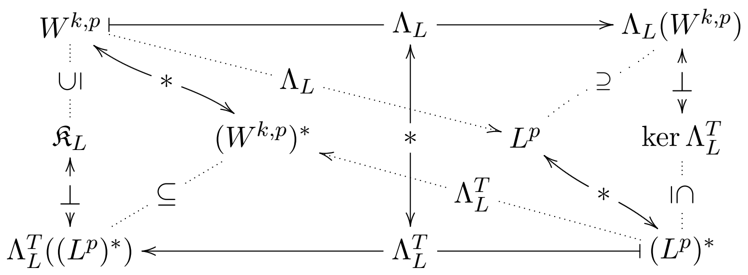

Figure 3.1: "Classical" dual correspondencies

If (x_n) is unbounded, we take some subsequence (x_{n_k}) with $\|x_n\|_{W^{k,p}} \to \infty \quad (n \to \infty)$. For convenience we now write (x_n) for this subsequence. $\forall n \in \mathbb{N}$ set

$$y_n := x_n \|x_n\|^{-1}_{W^{k,p}} \qquad\qquad i_n := h_n \|x_n\|^{-1}_{W^{k,p}}$$

. Then $\forall n \in \mathbb{N} : \Lambda_L y_n = i_n \wedge \|y_n\|_{W^{k,p}} = 1$ and $i_n \to 0$ in L^p if $n \to \infty$. With Lemma 3.1.3 we obtain, after passing to some subquence, that $y_n \to y_*$ in $W^{k,p}$ with $\Lambda_L y_* = 0$. Hence $y_* \in \mathfrak{K}_L$ and $y_* \in E$. Thus $y_* \in \ker \Lambda_L \cap E = \{0\}$. However $\forall n \in \mathbb{N} : \|y_n\|_{W^{k,p}} = 1$, so $\|y_*\|_{W^{k,p}} = 1$ which is a contradiction. Therefore (x_n) is bounded and by the first part of the proof $x_* \in \Lambda_L(W^{k,p})$. Hence $\Lambda L(W^{k,p})$ is closed. □

3.2 The Fredholm Alternative

Assume L were asymptotically hyperbolic, the associated operator Λ_L were Fredholm and $1 \leq p \leq \infty$. Then the "classical" Fredholm Alternative would yield dual correspondencies as in Figure 3.2 and Definition 1.1.3. Unfortunately the duals of the Sobolev spaces are generally not very nice to handle so we would prefer some special asymptotically hyperbolic linear operator $L^*(L)$ superseding any need to directly consider those duals. We will show that this can indeed be done, yielding a duality structure as illustrated in 3.2 with some appropriately defined L^* such that

$$\Lambda_L(W^{k,p}) = (\mathfrak{K}_{L^*})_{\perp L^p} \tag{3.10}$$

.

Definition/Lemma 3.2.1
Let $L \in (R^{k-1})^{\mathbb{R}}$ with matrix coefficients $A_{i,j} : \mathbb{R} \to \mathbb{C}^{d \times d}$ uniformly bounded and measurable as in 1.1.4. Then the family of operators $L^* \in (R^{k-1})^{\mathbb{R}}$ defined by

$$\begin{aligned} L^*(\xi) : C_B^{k-1}([-r_+, -r_-]) &\to \mathbb{C}^d \\ \varphi &\mapsto (-1)^k \sum_{i=1}^{N} \sum_{j=0}^{k-1} (-1)^j (A_{i,j}(\xi - r_i))^T \varphi^{(j)}(-r_i) \end{aligned} \tag{3.11}$$

where $\xi \in \mathbb{R}$, is called the *adjoint* ([2]) or *quasidual* (family) of L.

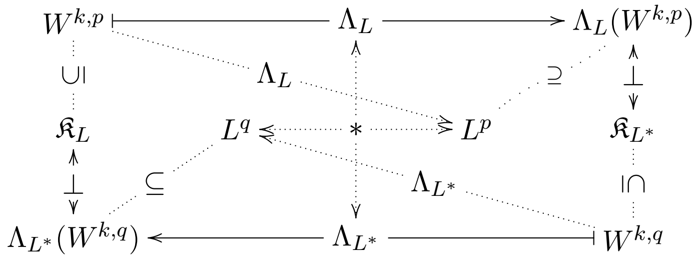

Figure 3.2: "Dual" correspondencies for L and pseudodual L^*

Formulating our differential equation (1.1) for L^* yields the *adjoint equation* or *quasidual equation*

$$y^{(k)}(\xi) = L^*(\xi)\tau_{-\xi}y = (-1)^k \sum_{i=1}^{N}\sum_{j=0}^{k-1}(-1)^j (A_{i,j}(\xi - r_i))^T y^{(j)}(\xi - r_i) \tag{3.12}$$

Then the following hold for $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

1. $(L^*)^* = L$
2. L has constant coefficients iff L^* has constant coefficients
3. L is asymptotically autonomous iff L^* is asymptotically autonomous
4. L is hyperbolic iff L^* is hyperbolic
5. L is asymptotically hyperbolic iff L^* is asymptotically hyperbolic
6. if L is asymptotically hyperbolic then $\mathfrak{K}_{L^*} \subseteq \ker \Lambda_L^T$

Proof. 1, 2, 3 follow directly from definition.

Ad 4: Let $\xi \in \mathbb{C}$, L be constant. Then

$$\begin{aligned}
\Delta_{L^*}(\xi) &= \xi^k \mathcal{I} - (-1)^k \sum_{i=1}^{N}\sum_{j=0}^{k-1}(-1)^j A_{i,j}^T \xi^j e^{-\xi r_i} \\
&= (-1)^k \Big((-\xi)^k \mathcal{I}^T - \sum_{i=1}^{N}\sum_{j=0}^{k-1}(-\xi)^j A_{i,j}^T e^{-\xi r_i}\Big) \\
&= (-1)^k (\Delta_L(-\xi))^T
\end{aligned}$$

. Hence $\forall \nu \in \mathbb{R}$

$$\det \Delta_{L^*}(\mathrm{i}\nu) = (-1)^{kd} \det \Delta_L(-\mathrm{i}\nu) \tag{3.13}$$

so L is hyperbolic iff L^* is hyperbolic.

5 follows directly from 3 and 4.

Ad 6: Using integration by parts for Sobolev functions we obtain $\forall x \in W^{k,p}, y \in \mathfrak{K}_{L^*} \subset W^{k,q} \subset (L^p)^*$

$$\begin{aligned}
(\Lambda_L^T y)(x) &= \int_{\mathbb{R}} \langle y(\nu), \overline{(\Lambda_L x)(\nu)} \rangle \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \langle y(\nu), \overline{x^{(k)}(\nu) - \sum_{i=1}^{N}\sum_{j=0}^{k-1} A_{i,j}(\xi) x^{(j)}(\nu + r_i)} \rangle \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \langle (-1)^k y^{(k)}(\nu) - \sum_{i=1}^{N}\sum_{j=0}^{k-1} (-1)^j A_{i,j}^T(\nu - r_i) y^{(j)}(\nu - r_i), \overline{x(\nu)} \rangle \mathrm{d}\nu \\
&= (-1)^k \int_{\mathbb{R}} \langle y^{(k)}(\nu) - (-1)^k \sum_{i=1}^{N}\sum_{j=0}^{k-1} (-1)^j A_{i,j}^T(\nu - r_i) y^{(j)}(\nu - r_i), \overline{x(\nu)} \rangle \mathrm{d}\nu \\
&= (-1)^k \int_{\mathbb{R}} \langle (\Lambda_{L^*} y)(\nu), \overline{x(\nu)} \rangle \mathrm{d}\nu \\
&= (-1)^k \int_{\mathbb{R}} \langle 0, \overline{x(\nu)} \rangle \mathrm{d}\nu = 0 \qquad \square
\end{aligned}$$

.

Lemma 3.2.2
Let $1 \le p, q \le \infty$, p, q conjugated, L asymptotically hyperbolic. Then

$$\Lambda_L(W^{k,p}) = (\mathfrak{K}_{L^*})_{\perp L^p} \tag{3.14}$$

Proof. As our previous results are rather strong we can again use almost the exact wording as in the original work [2]. By Corollary 3.1.5 $\Lambda_L(W^{k,p})$ is a closed linear subspace. We recall some properties of annihilators and dual operators:

$$\Lambda_L(W^{k,p}) = (\Lambda_L(W^{k,p})^{\perp})_{\perp} = (\ker \Lambda_L^T)_{\perp} \tag{3.15}$$

As always with L^p spaces the case $1 \le p < \infty$ is rather simple:
It is well known that C_0^∞ is dense in all L^p spaces. Hence $W^{k,q}$ lies dense in L^q so it suffices to show that $(\ker \Lambda_L^T \cap W^{k,q}) \subseteq \mathfrak{K}_{L^*}$. Then

$$\ker \Lambda_L^T = \overline{\ker \Lambda_L^T \cap W^{k,q}} \subseteq \overline{\mathfrak{K}_{L^*}} \subseteq \overline{\ker \Lambda_L^T} = \ker \Lambda_L^T$$

by 3.2.1,6 would yield

$$(\mathfrak{K}_{L^*})_{\perp L^p} = (\overline{\mathfrak{K}_{L^*}}^{w*})_{\perp L^p} \subseteq (\overline{\mathfrak{K}_{L^*}})_{\perp L^p} = (\ker \Lambda_L^T)_{\perp} \subseteq (\mathfrak{K}_{L^*})_{\perp L^p}$$

where $w*$ denotes closure in the weak-$*$ topology on $(L^p)^*$.

Assume $y \in \ker \Lambda_L^T \cap W^{k,q}, \varphi \in \mathfrak{S}$. Thus clearly $\varphi \in W^{k,p}$. Applying Λ_L^T to y yields

$$\begin{aligned}
0 &= (\Lambda_L^T y)(\varphi) \\
&= \int_{\mathbb{R}} \langle y(\nu), \overline{(\Lambda_L \varphi)(\nu)} \rangle \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \langle y(\nu), \overline{\varphi^{(k)}(\nu) - \sum_{i=1}^{N} \sum_{j=0}^{k-1} A_{i,j}(\nu) \varphi^{(j)}(\nu + r_i)} \rangle \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \langle y(\nu), \overline{\varphi^{(k)}(\nu)} \rangle \mathrm{d}\nu - \sum_{i=1}^{N} \sum_{j=0}^{k-1} \int_{\mathbb{R}} \langle A_{i,j}^T(\nu - r_i) y(\nu - r_i), \overline{\varphi^{(j)}(\nu)} \rangle \\
&= (-1)^k \Bigg(\Big((-1)^k \int_{\mathbb{R}} \langle y(\nu), \overline{\varphi^{(k)}(\nu)} \rangle \mathrm{d}\nu \Big) \\
&- (-1)^k \sum_{i=1}^{N} \sum_{j=0}^{k-1} (-1)^j \Big((-1)^j \int_{\mathbb{R}} \langle A_{i,j}^T(\nu - r_i) y(\nu - r_i), \overline{\varphi^{(j)}(\nu)} \rangle \mathrm{d}\nu \Big) \Bigg)
\end{aligned}$$

. Therefore y satisfies the quasidual equation tempered distributionally. As $y \in W^{k,q}$ it follows that $y \in \mathfrak{K}_{L^*}$ and the proof is finished for finite p.

Now assume $p = \infty$. We use a more direct approach as for the finite case. We claim that any $h \in L^\infty$ can be written as $h = h_1 + h_2$ such that $h_1 \in \Lambda_L(W^{k,p})$ and h_2 has compact support.

Assume this were proven and $h \in (\ker \Lambda_{L^*})_\perp$. Since $h = h_1 + h_2$ and by 3.2.1,6 $h_1 \in (\ker \Lambda_L^T)_\perp \subseteq (\mathfrak{K}_{L^*})_{\perp L^p}$ we get $h_2 \in (\mathfrak{K}_{L^*})_{\perp L^p}$ as this is clearly a closed linear space. As $h_2 \in L^\infty$ and h_2 has compact support it follows that $\forall 1 \leq p' < \infty : h_2 \in L^{p'}$ as well and moreover $h_2 \in (\mathfrak{K}_{L^*})_{\perp L^{p'}}$. Fix any such p'.

By the first (finite p) part of the Lemma we thus have $h_2 \in \Lambda_L(W^{k,p'})$ so $\exists x \in W^{k,p'}$: $\Lambda_L x = h_2$. Using the Sobolev embedding we get $x \in W^{k-1,\infty}$ and hence by L^∞ boundedness of h_2 and using boundedness of the differential operator $L' \in B(W^{k-1,\infty}, L^\infty)$ we obtain that $x \in W^{k,\infty}$. Hence $h_2 \in \Lambda_L(W^{k,\infty})$ so $h = h_1 + h_2 \in \Lambda_L(W^{k,\infty})$. Together with 3.2.1, 6 finally

$$(\mathfrak{K}_{L^*})_{\perp L^p} \subseteq \Lambda_L(W^{k,\infty}) = (\ker \Lambda_L^T)_\perp \subseteq (\mathfrak{K}_{L^*})_{\perp L^p}$$

.

To obtain such a decomposition $h = h_1 + h_2$ as above, it is sufficient to find $x \in W^{k,\infty}$ such that Equation (1.1) holds for large $|\xi|$, say for $|\xi| \geq \tau$. Then setting $h_1 = \Lambda_L x$ and $h_2 = h - h_1$ we have $h_1 \in \Lambda_L(W^{k,\infty})$ and $h_2 \in L^\infty$ with $\mathrm{supp}(h_2) \subseteq [-\tau, \tau]$. We employ a construction used in the proof of Proposition 3.1.1.

L is asymptotically hyperbolic. Therefore $\exists L_+, L_- \in R^{k-1}, M_+, M_- \in (R^{k-1})^{\mathbb{R}}$ such that

$$\begin{aligned}\forall \xi \in \mathbb{R}: \quad & L(\xi) = L_+ + M_+(\xi) = L_- + M_-(\xi) \\ & \|M_+(\xi)\|_{R^{k-1}} \to 0 \quad (\xi \to \infty) \\ & \|M_-(\xi)\|_{R^{k-1}} \to 0 \quad (\xi \to -\infty)\end{aligned}$$

. Let $\tau_0 > 0$ and set

$$\alpha_+(\xi) = \begin{cases} 0 & \xi < \tau_0 \\ 1 & \xi \geq \tau_0 \end{cases} \qquad \alpha_-(\xi) = \begin{cases} 1 & \xi \leq -\tau_0 \\ 0 & \xi > -\tau_0 \end{cases}$$

$$L_1(\xi) = L_+ + \alpha_+(\xi) M_+(\xi) \qquad L_2(\xi) = L_- + \alpha_-(\xi) M_-(\xi)$$

. By Theorem 2.2.2 we can choose τ_0 sufficiently large such that $\Lambda_{L_1} \in B(W^{k,\infty}, L^\infty)$ and $\Lambda_{L_2} \in B(W^{k,\infty}, L^\infty)$ are isomorphisms. Then with $x_+ = \Lambda_{L_1}^{-1} h$ and $x_- = \Lambda_{L_2}^{-1} h$ we have

$$\forall \xi \geq \tau_0: \quad x_+^{(k)} = L(\xi)\tau_{-\xi} x_+ \qquad \forall \xi \leq -\tau_0: \quad x_-^{(k)} = L(\xi)\tau_{-\xi} x_-$$

. Now if $\beta \in C^k(\mathbb{R}, \mathbb{R}) : \beta|_{(\infty,0]} = 0, \beta|_{[1,\infty)} = 1$ the function defined by

$$x(\xi) := \beta(\xi) x_+(\xi) + \beta(-\xi) x_-(\xi) \tag{3.16}$$

satisfies Equation (1.1) for $|\xi| \geq (2 + \tau_0 + \max_{1 \leq i \leq N} |r_i|)$ and the proof is finished. □

Summarizing some of our results we get

Theorem 3.2.3 (A Fredholm Alternative)
Assume all our assumptions hold and that L is asymptotically hyperbolic, $1 \leq p, q \leq \infty$ and that p, q are conjugated. Then

1. Λ_L is Fredholm
2. $\ker \Lambda_L$ is independent of p
3. $\Lambda_L(W^{k,p}) = (\mathfrak{K}_{L^*})_{\perp L^p}$ and $\Lambda_{L^*}(W^{k,q}) = (\mathfrak{K}_L)_{\perp L^q}$
4. $\dim\ker \Lambda_{L^*} = \operatorname{codim} \Lambda_L(W^{k,p}), \quad \dim\ker \Lambda_L = \operatorname{codim} \Lambda_{L^*}(W^{k,q}), \quad \operatorname{ind} \Lambda_L = -\operatorname{ind} \Lambda_{L^*}$
5. If L is a hyperbolic constant coefficient operator then Λ_L is an isomorphism

Proof. Ad 1: By Corollary 3.1.4 : $\dim\ker \Lambda_L < \infty$ and $\dim\ker \Lambda_{L^*} < \infty$. By Lemma 3.2.2 : $\operatorname{codim} \Lambda_L(W^{k,p}) = \operatorname{codim}(\ker \Lambda_{L^*})_\perp = \dim\ker \Lambda_{L^*} < \infty$. Thus Λ_L is Fredholm
Ad 2: see remark 3.1.2 and Proposition 3.1.1
Ad 3: by Definition/Lemma 3.2.1 and 3.2.2,1
Ad 4: by 3.2.2 and 3.2.1,1:

$$\operatorname{ind} \Lambda_L = \dim \mathfrak{K}_L - \operatorname{codim} \Lambda_L(W^{k,p}) = \operatorname{codim} \Lambda_{L^*}(W^{k,q}) - \dim \mathfrak{K}_{L^*} = -\operatorname{ind} \Lambda_{L^*}$$

Ad 5: see Theorem 2.1.2 □

4 An Application

In the orignal work [2], Mallet-Paret not only introduces the beforementioned but also several results on calculating the Fredholm index. To make up for this obivous lack of content we dwelve into [3].

Obtaining (formal) solutions to linear equations is fairly easy along the lines of 2.1.2. Hence one might consider applications to nonlinear equations.

We faithfully reproduce parts of [3] which i.a. showcases such an application to first order equations and $d = 1$ by using [2] to ascertain conditions for use of the implicit function theorem. Of course the original work contains more and more interesting findings so again we refer the interested reader to the orginal papers for further reading and references.

4.1 Preliminaries

Consider the following nonlinear autonomous equation with shifts r_i as in 1.1.1 and let $p = \infty$. Unless stated otherwise we are now examining real functions only, but still as part of complex function spaces. Consider

$$-cx^{(1)}(\xi) = F(x(\xi + r_1), x(\xi + r_2), x(\xi + r_N), \rho) \tag{4.1}$$

for $\xi \in \mathbb{R}$, $x \in W^{1,\infty}(\mathbb{R}, \mathbb{R}) \subseteq W^{1,\infty}(\mathbb{R}, \mathbb{C})$, $c \in \mathbb{R}\setminus\{0\}$, $\rho \in \overline{V}$, where $overlineV$ is the closure of an open subset V of some Banach space, and additionally the following properties hold.

(i) $F : \mathbb{R}^N \times \overline{V} \to \mathbb{R}, (u, \rho) \mapsto F(u, \rho)$ is C^1. $D_u F : \mathbb{R}^N \times \overline{V} \to \mathbb{R}^N$ is locally lipschitz in u.

(ii) $\forall \rho \in \overline{V} : \exists U(\rho) \subseteq \{2, \cdots, N\} : U(\rho) \neq \emptyset$ such that

  a) $\forall j \in U(\rho) : \forall u \in \mathbb{R}^N : (\frac{\partial}{\partial u_j} F)(u, \rho) > 0$

  b) $\forall j \in \{2, \cdots, N\} \setminus U(\rho) : F(u, \rho)$ independent of u_j

  .

(iii) Setting

$$\begin{aligned} \Phi : \mathbb{R} \times \overline{V} &\to \mathbb{R} \\ (x, \rho) &\mapsto F(x, \cdots, x, \rho) \end{aligned}$$

this function satisfies $\forall \rho \in \overline{V} : \exists q(\rho) \in [-1, 1]$:

$$\begin{aligned}
&\forall x \in (-\infty, -1) \cup (q(\rho), 1) : && \Phi(x, \rho) > 0 \\
&\forall x \in (-1, q(\rho)) \cup (1, \infty) : && \Phi(x, \rho) < 0
\end{aligned}$$

$$\Phi(-1, \rho) = \Phi(q(\rho), \rho) = \Phi(1, \rho) = 0$$

and $\forall \rho \in V : q(\rho) \in (-1, 1)$.

(iv) For this $q(\rho)$ additionally

$$\begin{aligned}
D_1\Phi(-1, \rho) &< 0 && \text{if } q(\rho) > -1 \\
D_1\Phi(1, \rho) &< 0 && \text{if } q(\rho) < 1 \\
D_1\Phi(q(\rho), \rho) &> 0 && \text{if } q(\rho) \in (-1, 1)
\end{aligned}$$

.

We specifically seek real solutions joining the equilibria ± 1, that is solutions $x \in \mathbb{R}^{\mathbb{R}}$ satisfying the boundary conditions

$$\lim_{\xi \to -\infty} x(\xi) = -1 \qquad \lim_{\xi \to \infty} x(\xi) = 1 \tag{4.2}$$

As $c \neq 0$ the equation implies that solutions to (4.1) are at least C^2 smooth and that x and $x^{(1)}$ are uniformly continuous.

Using (iv) and assuming $q(\rho) \in (-1, 1)$ we can apply the implicit function theorem and obtain that $\rho \mapsto q(\rho)$ is C^1. For convenience we additionally introduce

$$W := \{\rho \in \overline{V} | -1 < q(\rho) < 1\} \tag{4.3}$$

.

First we present some technical aids. Let $G : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy

$$G : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}, (\xi, u) \mapsto G(\xi, u) \text{ is continuous and locally lipschitz in } u \tag{4.4a}$$

$$\forall \xi \in \mathbb{R} : \forall u \in \mathbb{R}^N : \forall j \in \{2, \cdots, N\} : \quad (\frac{\partial}{\partial u_j} G)(\xi, u) \geq 0 \tag{4.4b}$$

as appear in (4.1) at specific values of $\rho \in \overline{V}$. Note that local lipschitz continuity implies existance of $\frac{\partial}{\partial u_i}$ in a weak sense.

Let $c \in \mathbb{R} \setminus \{0\}$, $J \subseteq \mathbb{R}$ an interval, $J^\# := J + \{r_i | 1 \leq i \leq N\}$ and $x : \mathbb{R} \to \mathbb{R}$, satisfying

$$x|_{J^\#} \in C(J^\#, \mathbb{R}) \tag{4.5a}$$

$$x|_J \in C^1(J, \mathbb{R}) \tag{4.5b}$$

$$\forall \xi \in J : -cx^{(1)}(\xi) = G(\xi, x(\xi + r_1), \cdots, x(\xi + r_N)) \tag{4.5c}$$

be called a solution to (4.1) on J, as might appear from (4.1) for specific values of ρ or linearizatins about particular solutions. If $c = 0$ by a solution we only mean a function $x : \mathbb{R} \to \mathbb{R}$ (possibly discontinuous) satisfying

$$\forall \xi \in J : G(\xi, x(\xi + r_1), \cdots, x(\xi + r_N)) = 0 \tag{4.6}$$

and no further regularity properties.

Lemma 4.1.1
Assume G as in (4.4), $c \in \mathbb{R} \setminus \{0\}$, x_1, x_2 as in (4.5) and assume further that

$$\forall \xi \in \mathbb{R} : \quad x_1(\xi) \geq x_2(\xi) \tag{4.7a}$$

$$\exists \tau \in \mathbb{R} : \quad x_1(\tau) = x_2(\tau) \tag{4.7b}$$

. If $c > 0$ then $\forall \xi \geq \tau : x_1(\xi) = x_2(\xi)$. If $c < 0$ then $\forall \xi \leq \tau : x_1(\xi) = x_2(\xi)$.

Proof. Set $y \coloneqq x_1 - x_2$. Then by (4.7) $y \geq 0$ and $y(\tau) = 0$. We recall $r_1 = 0$ and by (4.4b) obtain

$$\begin{aligned} -cy^{(1)}(\xi) &= G(\xi, x_1(\xi + r_1), x_1(\xi + r_2), \cdots, x_1(\xi + r_N)) - G(\xi, x_2(\xi + r_1), \cdots, x_2(\xi + r_N)) \\ &= G(\xi, x_2(\xi) + y(\xi), x_1(\xi + r_2), \cdots, x_1(\xi + r_N)) - G(\xi, x_2(\xi), \cdots, x_2(\xi + r_N)) \end{aligned}$$

Setting

$$H(\xi, y) \coloneqq -\frac{1}{c} G(\xi, x_2(\xi) + y, x_1(\xi + r_2), \cdots$$

it follows that $\forall \xi \in \mathbb{R} : H(\xi, 0) \leq 0$ and that y is a solutions to the initial value problem $y^{(1)}(\xi) = H(\xi, y(\xi)), \xi \geq \tau, y(\tau) = 0$. Hence it follows that $y \leq 0$. Assume this were not true. Then $\tau_0 \coloneqq \sup\{\tau_1 \geq \tau | \forall \xi \leq \tau_1 : y(\xi) \leq 0\}$ would exist. As H is continuous and locally Lipschitz in the second argument, the Picard-Lindelöf theorem implies that for some $\epsilon > 0$ this initial value problem possesses a unique solution on $[\tau_0, \tau_0 + \epsilon]$. This solution must coincide with y on $[\tau_0, \tau_0 + \epsilon]$ and can be obtained by the limit of the Picard iteration.

Hence $\forall \xi \in [\tau_0, \tau_0 + \epsilon] : y(\xi) = \lim_{n \to \infty} y_n(\xi)$ where

$$y_0(\xi) \coloneqq y(\tau_0) = 0$$

By induction we have $\forall \xi \in [\tau_0, \tau_0 + \epsilon] : \forall n \in \mathbb{N} : y_n(\xi) = 0$ as $y \geq 0$ and for $n \in \mathbb{N}$ and $\xi \in [\tau_0, \tau_0 + \epsilon]$

$$y_n(\xi) = \underbrace{y_{n-1}(\xi)}_{=0} + \underbrace{\int_{\tau_0}^{\xi} H(\xi, \underbrace{y_{n-1}(\xi)}_{=0} \, \mathrm{d}\xi}_{\leq 0} \leq 0$$

. Thus $y|_{[\tau_0, \tau_0 + \epsilon]} = 0$ which is a contradiction to the definition of τ_0 so $\forall \xi \geq \tau : y(\xi) = 0$.

Likewise if $c < 0$ then $\forall \xi \leq \tau : y(\xi) = 0$. □

Lemma 4.1.2
Assume G as in (4.4), $c \in \mathbb{R}$, x_1, x_2 as in (4.5) for some interval $J \subseteq \mathbb{R}$ and additionally that (4.4b) is strict, that is

$$\forall \xi \in \mathbb{R} : \forall u \in \mathbb{R}^N : \forall j \in \{2, \cdots, N\} : (\frac{\partial}{\partial u_j} G)(\xi, u) > 0 \tag{4.8}$$

. If $\exists \tau \in J$:

$$[\tau + r_-, \tau + r_+] \subseteq J \tag{4.9a}$$
$$\forall \xi \in [\tau + r_-, \tau + r_+] : \quad x_1(\xi) = x_2(\xi) \tag{4.9b}$$

then one of the two following holds.

$$\forall \xi \in J^\# \cap (-\infty, \tau + r_+] : \quad x_1(\xi) = x_2(\xi) \tag{4.10a}$$
$$r_- = 0 \wedge c = 0 \tag{4.10b}$$

The analogous result holds for $[\tau + r_-, \infty)$ and r_+.

Proof. As J contains a closed interval of length $r_+ - r_-$, $J^\#$ is a closed interval too. Now assume (4.10a) does not hold. Hence by continuity $\exists \tau_0 \in J^\# \cap (\infty, \tau + r_-]$

$$\forall \xi \in [\tau_0, \tau + r_+] : x_1(\xi) = x_2(\xi)$$
$$\forall \epsilon > 0 : \exists \xi \in J^\# \cap (\tau_0 - \epsilon, \tau_0) : x_1(\xi) \neq x_2(\xi)$$

. The second property implies that there exists some open (in $\mathbb{R}$) neighbourhood $U \subseteq J^\#$ of τ_0 and the same applies to $\tau_0 - r_- \in (\tau_0, \tau) \subseteq J^\#$.

Assume further that $r_- < 0$. Without loss of generality let $r_- = r_2$. Let $\epsilon > 0$: $\tau_0 - r_- - \epsilon \in J^\#$ and $\forall j \neq 2 : \tau_0 - r_- - \epsilon + r_j > \tau_0$. Consider $(\xi \in \tau_0 - r_2 - \epsilon, \tau_0 - r_2) \subseteq J$. By assumption $\forall j \neq 2 : x_1(\xi + r_j) = x_2(\xi + r_j)$. As $\tau > \tau + r_2 - r_2 > \tau_0 - r_2 > \xi > \xi + r_2 > \tau_0 - r_2 - \epsilon > \tau_0$ there exists some open neighbourhood $U \subseteq (\tau_0, \tau) \subseteq J$, of ξ. Hence if $c \neq 0$ if follows that $x_1^{(1)}(\xi) = x_2^{(1)}(\xi)$. Recall that if $c = 0$ then for $j \in 1, 2 : G(\xi, x_j(\xi + r_1), \cdots, x_j(\xi + r_N)) = 0$. Thus $\forall c \in \mathbb{R}$:

$$\begin{aligned} G(\xi, x_1(\xi + r_1), \cdots, x_1(\xi + r_N)) &= G(\xi, x_2(\xi + r_1), \cdots, x_2(\xi + r_N)) \\ &= G(\xi, x_1(\xi + r_1), x_2(\xi + r_2), x_1(\xi + r_3), \cdots, x_1(\xi + r_N)) \end{aligned}$$

. Using the strict inequality (4.8) it follows that $x_1(\xi + r_-) = x_2(\xi + r_-)$ where $\xi + r_- < \tau_0 - r_- + r_- = \tau_0$, which contradicts the definition of τ_0. Hence $r_- = 0$.

It remains to be shown that $r_- = 0 \Rightarrow c = 0$. Assume $r_- = 0, c \neq 0$. As G is locally lipschitz, solutions to (4.5) and (4.9b) are uniquely determined in backwards direction, i.e. $J^\# \cap (-\infty, \tau + r_+]$. So (4.10a).

The proof for $\xi \geq \tau + r_-$ follows similar lines. □

4.2 Linear Equations

In the next part we reproduce some results on linear equations.

We consider linear equations, which may arise from linearizations about particular solutions (compare to (1.1), 1.1.4) in spaces of functions mapping to $\mathbb{C}$ as in the preliminaries of this thesis, even though the better part of the remaining thesis deals exclusively with real solutions.

Consider

$$-cx^{(1)}(\xi) = L(\xi)\big(\tau_{-\xi}x\big) + h(\xi) = \sum_{i=1}^{N} A_i(\xi)x(r_i+\xi) + h(\xi) \tag{4.11}$$

where all the asumptions made in the preliminary section hold and additionally $A_i = A_{i,0}$ are continuous. For sake of convenience introduce

$$\Lambda_{c,L} : W^{1,\infty} \to L^\infty$$

$$(\Lambda_{c,L}x)(\xi) \coloneqq (-c\Lambda_{-\frac{1}{c}L}x)(\xi) = -cx^{(1)}(\xi) - \sum_{i=1}^{N} A_i(\xi)x(\xi+r_i) \tag{4.12}$$

$$\mathfrak{K}_{c,L} \coloneqq \ker \Lambda_{c,L} \tag{4.13}$$

$$\mathfrak{R}_{c,L} \coloneqq \Lambda_{c,L}(W^{1,\infty}) \tag{4.14}$$

We recall that by Proposition 3.1.2 the kernel is independent of the $W^{k,p}$ space $\Lambda_{c,L} = -c\Lambda_{-\frac{1}{c}L}$ is defined on and our Fredholm Alternative theorem 3.2.3 gives

$$\mathfrak{R}_{c,L} = \big(\mathfrak{K}_{c,L^*}\big)_{\perp L^\infty} = \{h \in L^\infty | \forall y \in \mathfrak{K}_{c,L^*} : \int_{\mathbb{R}} h(\xi)y(\xi)\mathrm{d}\xi = 0\} \tag{4.15}$$

.

As a special case also consider the homogeneous constant coefficient equation

$$-cx^{(1)}(\xi) = L_0\big(\tau_{-\xi}x\big) = \sum_{i=1}^{N} A_i x(r_i+\xi) \tag{4.16}$$

for which in accordance to (1.9) and (1.7) we introduce

$$\Delta_{c,L_0}(\xi) \coloneqq -c\Delta_{-\frac{1}{c}L_0}(\xi) = -c\xi - \sum_{i=1}^{N} A_i e^{\xi r_i} \tag{4.17}$$

. We remark that using this definition Δ_{c,L_0} and $\Delta_{-\frac{1}{c}L_0}$ both can be used to determine hyperbolicity of the system (4.16).

For such a constant coefficient system we define its *eigenvalues* to be the point spectrum of $-\frac{1}{c}L_0'$ (see (1.5)) that is

$$\sigma_{c,L_0} \coloneqq \sigma_p(-\frac{1}{c}L_0') = \big(\Delta_{-\frac{1}{c}L_0'}\big)^{-1}(\{0\}) = \big(\Delta_{c,L_0}^{-1}\big)(\{0\}) = \{\xi \in \mathbb{C} \,|\, -c\xi - \sum_{i=1}^{N} A_i e^{\xi r_i} = 0\} \tag{4.18}$$

To any such $\lambda \in \sigma_{c,L_0}$ corresponds a finite dimensional set of *eigensolutions* x of the form $e^{\lambda\xi}p(\xi)$ with some polynomial function p, that is functions satisfying

$$- c\lambda x(\xi) = -c\big(-\frac{1}{c}\sum_{i=1}^{N} A_i x(\xi + r_i)\big) = \sum_{i=1}^{N} A_i x(\xi + r_i) = -cx^{(1)}(\xi) \tag{4.19}$$

Later we will impose additional restrictions on A_i stemming from the original problem (4.1). To avoid writing them down several times we will list them here.

From (ii) we will later obtain

$$\exists\alpha_1 \in \mathbb{R}, \exists\alpha_2, \cdots, \alpha_N \in (0, \infty), \exists\beta_1, \cdots, \beta_N \in \mathbb{R} : \forall\xi \in \mathbb{R} : \alpha_i \leq A_i(\xi) \leq \beta_i \tag{4.20}$$

and

$$A_1 \in \mathbb{R}, \quad A_2, \cdots, A_N \in (0, \infty) \tag{4.21}$$

for constant coefficient systems. In some instances we will impose these restrictions only for some interval $[\tau, \infty)$ or $(-\infty, \tau]$, e.g. when studying asymptotic behaviour in only one direction.

Set

$$A_{\Sigma,L_0} \coloneqq \sum_{i=1}^{N} A_i = -\Delta_{c,L_0}(0) \tag{4.22}$$

. Then the linear constant coefficient equivalent to (iv) is

$$A_{\Sigma,L_0} < 0 \tag{4.23}$$

.

For further use with asymptotically hyperbolic systems We write L_+, L_- for the limiting constant coefficient operators at $\infty, -\infty$. We recall and (re)define

$$\begin{aligned}
L(\xi)\varphi &= \sum_{i=1}^{N} A_i(\xi)\varphi(r_i) \\
L_+\varphi &= \sum_{i=1}^{N} A_{i,+}\varphi(r_i) \\
L_-\varphi &= \sum_{i=1}^{N} A_{i,-}\varphi(r_i) \\
A_{\Sigma,+} &\coloneqq A_{\Sigma,L_+} \\
A_{\Sigma,-} &\coloneqq A_{\Sigma,L_-} \\
\lim_{\xi\to\infty} A_i(\xi) &= A_{i,+} \\
\lim_{\xi\to-\infty} A_i(\xi) &= A_{i,-}
\end{aligned} \tag{4.24}$$

as in 1.1.4 and 1.1.6.

First we show some results on solutions and eigenvalues of constant coefficient systems, as obtaind from the limiting equations in asymptotically hyperbolic linear systems.

Lemma 4.2.1
Let $A_i \in \mathbb{R}$, $b \in \mathbb{R}$, $c \in \mathbb{R}\backslash\{0\}$, $\epsilon > 0$, $F \subset \sigma_{c,L_0}$ a finite set of eigenvalues, $W^{1,\infty}(\mathbb{R},\mathbb{R}) \ni y \neq 0$ a nontrivial solution to the linear Equation (4.11) given by a finite sum of eigensolutions to eigenvalues $\lambda \in F$. Let $\forall\lambda \in F : \mathrm{Re}\lambda = -b$ and $x \in \mathbb{R}^{\mathbb{R}}$ satisfy

$$x - y \in O(\xi \mapsto e^{-(b+\epsilon)\xi}) \quad (\xi \to \infty) \tag{4.25}$$

. If $\forall\lambda \in F : \mathrm{Im}\lambda \neq 0$, then there are arbitrarily large ξ such that $x(\xi) > 0$ and $x(\xi) < 0$, i.e. $\forall\tau > 0 : \exists\xi_1, \xi_2 \geq \tau : x(\xi_1) > 0, x(\xi_2) < 0$. If $F = \{-b\}$ then $\exists\tau > 0 : \forall\xi \geq \tau : x(\xi) \neq 0$. The analogous results holds for $\xi \to -\infty$.

Proof. First assume $\forall\lambda \in F : \mathrm{Im}\lambda \neq 0$. Recalling $x \in \mathbb{R}^{\mathbb{R}}$ we can, without loss of generality, assume that all functions are $\mathbb{R}$ valued . Hence

$$\begin{aligned} y(\xi) = \mathrm{Re}y(\xi) &= \mathrm{Re}\sum_{j=1}^{M} e^{\lambda_j\xi}p_j(\xi) \\ &= \sum_{j=1}^{M} e^{-b\xi}cos(\mathrm{Im}\lambda_j\xi)p_j(\xi) \\ &= e^{-b\xi}\xi^L\Big(\sum_{j=1}^{M} c_j cos(\mathrm{Im}\lambda_j\xi) + \xi^{-L}\sum_{j=1}^{M} R_j(\xi)cos(\mathrm{Im}\lambda_j\xi)\Big) \end{aligned}$$

for $\lambda_j \in F; M \in \mathbb{N}, L = \max\deg p_j$ the maximal degree of the real nontrivial polynomial functions p_j where $\xi \mapsto e^{\lambda_j\xi}p_j(\xi)$ are eigensolutions to eigenvalues $\lambda_j \in F$ and R_j are residual functions in the decomposition $p_j(\xi) = c_j\xi^L + R_j(\xi))$ with $\deg R_j < L$. By assumption of y being nontrivial $q(\xi) = \sum_{j=1}^{M} c_j cos(\mathrm{Im}\lambda_j\xi)$ defines a quasiperiodic function of mean value 0. In particular

$$\liminf_{\xi\to\infty} q(\xi) < 0 < \limsup_{\xi\to\infty} q(\xi)$$

. For $\xi \to \infty$ we now have that

$$\xi^{-L}e^{b\xi}x(\xi) - q(\xi) \in O(\xi \mapsto \xi^{-1})$$

As q takes both signs for large ξ so does $\xi \mapsto \xi^{-M}e^{b\xi}x(\xi)$ and thus x.

If $F = \{-b\}$ then $y(\xi) = e^{-b\xi}p(\xi)$ for some real polynomial function p. Hence taking the leading coefficient of p to be c_L with $L = \deg p$ we have

$$\lim_{\xi\to\infty} \xi^{-J}e^{b\xi}x(\xi) = c_L \neq 0$$

So $x(\xi) \neq 0$ for large ξ. □

Next we reproduce some lemmata on the existance of certain eigenvalues which will later enable us to give exponential bounds for solutions to the linear Equation (4.11).

Lemma 4.2.2
Assume that A_i as in (4.21), $a \in \mathbb{R}, c \in \mathbb{R}\setminus\{0\}$.
If $\Delta_{c,L_0}(a) > 0$ then $\forall \mu \in \mathbb{R} : \Delta_{c,L_0}(a + \mathfrak{i}\mu) \neq 0$
If $\Delta_{c,L_0}(a) = 0$ then $\forall \mu \in \mathbb{R}\setminus\{0\} : \Delta_{c,L_0}(a + \mathfrak{i}\mu) \neq 0$

Proof. Suppose $\Delta_{c,L_0}(a) \geq 0$ and $\mu \in \mathbb{R} : \Delta_{c,L_0}(a + \mathfrak{i}\mu) = 0$. Then recalling $r_1 = 0$ we have both

$$c(a + \mathfrak{i}\mu) + A_1 = -\sum_{i=2}^{N} A_i e^{r_i(a+\mathfrak{i}\mu)}$$

$$ca + A_1 \leq -\sum_{i=2}^{N} A_i e^{r_i a}$$

. Hence as $\forall \lambda \in \mathbb{C} : |\mathrm{Re}\lambda| \leq |\lambda|$ we get

$$-(ca + A_1) \leq |ca + A_1| \leq |c(a + \mathfrak{i}\mu) + A_1| = |\sum_{i=2}^{N} A_i e^{r_i a} e^{\mathfrak{i} r_i \mu}| \leq \sum_{i=2}^{N} A_i e^{r_i a} 1 \leq -(ca + A_1) \tag{4.26}$$

. Thus all the inequalities are actually equalities so it follows that $\Delta_{c,L_0}(a) = 0$.
Suppose $\Delta_{c,L_0}(a) = 0$. Then $\Delta_{c,L_0}(a) \geq 0$ still holds and (4.26) gives

$$|ca + A_1|^2 = |c(a + \mathfrak{i}\mu) + A_1|^2 = |ca + A_1|^2 + |c\mathfrak{i}\mu|^2$$

. So $c\mu = 0$. As $c \neq 0$ it follows that $\mu = 0$. □

Proposition 4.2.3
Assume again that $c \neq 0$, A_i as in (4.21) and additonally that (4.23) holds, that is

$$A_\Sigma := \sum_{i=1}^{N} A_i < 0 \tag{4.27}$$

and consider the corresponding operator L_0.

Then Λ_{c,L_0} is hyperbolic and there exist at most one real positive eigenvalue $\lambda^+ \in (0, \infty)$ and one real negative eigenvalue $\lambda^- \in (-\infty, 0)$. As a convention we write $\lambda^- = -\infty$, respectively $\lambda^+ = \infty$, whenever one of those real eigenvalues does not exist.

These eigenvalues, whenever they actually exist, are simple, that is $\Delta^{(1)}_{c,L_0}(\lambda^\pm) \neq 0$, and depend C^1 smoothly on both c and the coefficients A_i.

Furthermore whenever λ^- exists, resp. λ^+ exists

$$\frac{\partial}{\partial c}\lambda^\pm < 0 \tag{4.28}$$

Additionally

$$\forall \xi \in (\lambda^-, \lambda^+) : \quad \Delta_{c,L_0}(\xi) > 0 \tag{4.29}$$

and

$$\forall \lambda \in \sigma_{c,L_0} \setminus \{\lambda^-, \lambda^+\} : \quad \mathrm{Re}\lambda \in (-\infty, \lambda^-) \cup (\lambda^+, \infty) \tag{4.30}$$

with the conventions $(-\infty, -\infty) = (\infty, \infty) = \emptyset$.

Proof. (4.22) implies that $\Delta_{c,L_0}(0) = -A_\Sigma > 0$. Hence by Lemma 4.2.2 $\forall \xi \in \mathbb{R}$: $\Delta_{c,L_0}(\mathrm{i}\xi) \neq 0$ so L_0 is hyperbolic.

Consider $f : \mathbb{R} \to \mathbb{R}, \xi \mapsto \Delta_{c,L_0}(\xi)$. Then $f \in C^2(\mathbb{R}, \mathbb{R})$, $f(0) > 0$ and $\forall \xi \in \mathbb{R}$

$$f^{(2)}(\xi) = -\sum_{i=2}^{N} \underbrace{A_i}_{>0} r_i^2 e^{r_i \xi} < 0$$

. The mean value theorem implies f and thus Δ_{c,L_0} possessing at most one zero in each $(-\infty, 0)$ and $(0, \infty)$.

By continuity $\forall \xi \in (\lambda^-, \lambda^+) : \Delta_{c,L_0}(\xi) > 0$.

By Lemma 4.2.2 we know

$$\forall a, \nu \in \mathbb{R} : \Delta_{c,L_0}(a + \mathrm{i}\nu) = 0 \Rightarrow \Delta_{c,L_0}(a) \leq 0$$

. Hence

$$\begin{aligned} \mathrm{Re}(\sigma_{c,L_0} \setminus \{\lambda^-, \lambda^+\}) &\subseteq \Delta_{c,L_0}^{-1}((-\infty, 0)) \\ &= (-\infty, \lambda^-) \cup (\lambda^+, \infty) \end{aligned}$$

with Δ_{c,L_0} being interpreted as $\in \mathbb{R}^{\mathbb{R}}$.

Writing $L_0 = L_0(A_1, \cdots, A_N)$ and setting

$$H(\xi, c, A_1, \cdots, A_N) \coloneqq \Delta_{c,L_0(A_1,\cdots,A_N)}(\xi)$$

we get

$$\Big(\frac{\partial}{\partial \xi} H\Big)(\lambda^\pm, c, A_1, \cdots, A_N) = \Delta^{(1)}_{c,L_0(A_1,\cdots,A_N)}(\lambda^\pm) \lessgtr 0 \tag{4.31}$$

whenever those eigenvalues exist, for example by applying the mean value theorem again. This implies that these real eigenvalues are simple.

$H : \mathbb{R}^{N+2} \to \mathbb{R}$ being C^1, $H(\lambda^\pm, c, A_1, \cdots, A_N) = 0$ and $\Big(\frac{\partial}{\partial \xi} H\Big)(\lambda^\pm, c, A_1, \cdots, A_N) \lessgtr 0$ enable us to use the implicit function theorem yielding $\lambda^\pm : \mathbb{R}^{N+1} \to \mathbb{R}, (\gamma, \alpha_1, \cdots, \alpha_N) \mapsto \lambda^\pm(\gamma, \alpha_1, \cdots, \alpha_N)$ is C^1 in some neighbourhood of $(c, A_1, \cdots, A_N)$.

Furthermore by implicit differentiation, fixing $A_1, \cdots, A_N$ and setting $\lambda^\pm(c) = \lambda^\pm(c, A_1, \cdots, A_N)$ we have

$$\begin{aligned} \Big(\frac{\partial}{\partial c}\lambda^\pm\Big)(c) &= -\bigg(\Big(\frac{\partial}{\partial \xi} H\Big)(\lambda^\pm(c), c, A_1, \cdots, A_N)\bigg)^{-1} \Big(\frac{\partial}{\partial c} H\Big)(\lambda^\pm(c), c, A_1, \cdots, A_N) \\ &= \Delta^{(1)}_{c,L_0(A_1,\cdots,A_N)}(\lambda^\pm(c))^{-1} \lambda^\pm(c) \\ &< 0 \end{aligned}$$

by definitions of $\lambda^\pm$ and (4.31) $\square$

The next lemma gives some insight into conditions for existance of these eigenvalues. We will occasionally speak of finite and infinite real eigenvalues instead of existing and nonexisting real eigenvalues as already set forth in preceding conventions.

Lemma 4.2.4
Assume that (4.20) hold for some $A_i \in \mathbb{R}$. Let $c \in \mathbb{R} \setminus \{0\}$. Then

$$\begin{aligned} \lambda^+ = \infty \Leftrightarrow r_+ = 0 \wedge c < 0 \\ \lambda^- = -\infty \Leftrightarrow r_- = 0 \wedge c > 0 \end{aligned} \tag{4.32}$$

.

Proof. $\Rightarrow$: As existance of eigenvalues boils down to zeroes of $\Delta_{c,L}$ we study conditions for $\Delta_{c,L}(\xi) \neq 0$, that is $\Delta_{c,L}(\xi) > 0$ by continuity and $\Delta_{c,L}(0) > 0$.

$$0 < \Delta_{c,L_0}(\xi) = -c\xi - \sum_{i=1}^{N} A_i e^{r_i \xi} = -c\xi - A_1 - \underbrace{\sum_{i=2}^{N} A_i e^{r_i \xi}}_{>0} \tag{4.33}$$

If λ^+ does not exist, then $\forall \xi > 0 : \Delta_{c,L_0}(\xi) > 0$. Then the last summand in (4.33) is bounded for $\xi \to \infty$. Hence $r_+ = 0$. Additionally $\exists \tau > 0 : \forall \xi \geq \tau : -c\xi > 0$ must necessarily hold, so $c < 0$. If λ^- does not exist, then $\forall \xi < 0 : \Delta_{c,L_0}(\xi) > 0$. Then the last summand in (4.33) is bounded for $\xi \to -\infty$. Hence $r_- = 0$. Additionally $\exists \tau < 0 : \forall \xi \leq \tau : -c\xi > 0$ must necessarily hold, so $c > 0$. $\Leftarrow$: Consider Δ_{c,L_0} as $\in C^1(\mathbb{R}, \mathbb{R})$. Calculating its derivative we obtain

$$\Delta^{(1)}_{c,L_0}(\xi) = -c - \sum_{i=2}^{N} A_i r_i e^{r_i \xi} \tag{4.34}$$

So if $c < 0, r_+ = 0$ then $\forall \xi > 0 : \Delta_{c,L_0}(\xi) \geq 0$. If $c > 0, r_- = 0$ then $\forall \xi < 0 :$ $\Delta^{(1)}_{c,L_0}(\xi) \leq 0$. By using continuity and an application of the mean value theorem the proof is finished. $\square$

Now we obtain our first results on asymptotics of solutions to (4.11).

Proposition 4.2.5
Consider (4.11) on $[\tau, \infty)$ for some $\tau \in \mathbb{R}$. Assume that $c \neq 0$, A_i, α_i, β_i satisfy (4.20) on $[\tau, \infty)$. Assume $x : J^{\#} \to [0, \infty)$ is a solution to (4.11) on $J := [\tau, \infty)$ as in (4.5).

Then $\exists a, b \in \mathbb{R}, R \in [0, \infty)$ such that

$$\forall \xi \geq \tau + R : \quad ax(\xi) \leq x^{(1)}(\xi) \leq bx(\xi) \tag{4.35}$$

. The analogous result holds for $(-\infty, \tau]$.

Proof. The case $(-\infty, \tau]$ can be handled by a change of variable $\xi \to -\xi$ and considering the linear equation for $-c$ and $\tilde{A}_i(\xi) = A_i(-\xi), \tilde{r}_i = -r_i$, which fulfills all requirements of this Proposition.

Hence consider the case $[\tau, \infty)$. Assume $c > 0$.

We have $\forall \xi \geq \tau$:

$$\begin{aligned}
x^{(1)}(\xi) &= \underbrace{-\frac{1}{c}}_{<0} \sum_{i=1}^{N} A_j(\xi) x(\xi + r_i) \\
&= -\frac{1}{c} A_1(\xi) x(\xi) - \underbrace{\frac{1}{c} \sum_{i=2}^{N} A_i(\xi) x(\xi + r_i)}_{\geq 0} \qquad (4.36) \\
&\leq -\frac{1}{c} A_1(\xi) x(\xi) \leq \big(-\frac{1}{c}\alpha_1\big) x(\xi)
\end{aligned}$$

. Setting $b \coloneqq -\frac{1}{c}\alpha_1$ establishes the right hand inequality in (4.35)

For the left hand inequality first consider the case $r_- < 0$. Let $y(\xi) = e^{-b\xi}x(\xi)$. Then $\forall \xi \geq \tau : y^{(1)}(\xi) \leq 0$. Set $\epsilon = \frac{1}{2}|\max\{r_i | r_i < 0\}|$. Fix $\xi_1 > \tau - r_- + \epsilon$. For any $\xi \in [\tau - r_-, \xi_1]$ we get $\forall i \in 1, \cdots, N : \xi + r_i \geq \tau$. By applying the bounds (4.20) and (4.36) we obtain

$$\begin{aligned}
y^{(1)}(\xi) &= -be^{-b\xi} x(\xi) - \frac{1}{c} e^{-b\xi} \sum_{i=1}^{N} A_i(\xi) x(\xi + r_i) \\
&= \underbrace{(-\frac{1}{c} A_1(\xi) - b)}_{\leq -c^{-1}\alpha_1 - b = b - b = 0} \; y(\xi) - \frac{1}{c} \sum_{i=2}^{N} A_i(\xi) e^{br_i} y(\xi + r_i) \\
&\leq -\frac{1}{c} \sum_{i=1}^{N} \underbrace{\alpha_i e^{br_i} y(\xi + r_i)}_{\geq 0 \text{ for all } i} \qquad (4.37) \\
&\leq -\frac{1}{c} \sum_{r_i < 0} \alpha_i e^{br_i} \underbrace{y(\xi + r_i)}_{\geq y(\xi_1 + r_i) \geq y(\max\{\xi_1 + r_i | r_i < 0\}) = y(\xi_1 - 2\epsilon)} \\
&\leq -\frac{1}{c} \Big(\sum_{r_i < 0} \alpha_i e^{br_i}\Big) y(\xi_1 - 2\epsilon)
\end{aligned}$$

. Integrating (4.37) over $[\xi_1 - \epsilon, \xi_1] \subseteq [\tau - r_-, \xi_1]$ yields

$$\begin{aligned}
-y(\xi_1 - \epsilon) \leq y(\xi_1) - y(\xi_1 - \epsilon) &= \int_{\xi_1 - \epsilon}^{\xi_1} y^{(1)}(\xi) \leq -\frac{1}{c} \Big(\sum_{r_i < 0} \alpha_i e^{br_i}\Big) \int_{\xi_1 - \epsilon}^{\xi_1} y(\xi_1 - 2\epsilon) \\
&= -\frac{1}{c} \epsilon \underbrace{\left(\sum_{r_i < 0} \alpha_i e^{br_i}\right)}_{>0} y(\xi_1 - 2\epsilon)
\end{aligned}$$

. Thus setting $C \coloneqq c\epsilon^{-1}\Big(\sum_{r_i<0} \alpha_i e^{br_i}\Big)^{-1}$ we obtain

$$y(\xi_1 - 2\epsilon) \leq Cy(\xi_1 - \epsilon)$$

. So if $\delta \in [0,\epsilon]$ and $\xi \geq \tau - r_-$ then $\xi + \epsilon$ satisfies the conditions imposed on ξ_1 making the last inequality applicable to $\xi + \epsilon$. Hence by using the last inequality and y's monotonicity

$$\forall \xi \geq \tau - r_- : \forall \delta \in [0,\epsilon] : \quad y(\xi - \delta) \leq y(\xi - \epsilon) = y\big((\xi+\epsilon) - 2\epsilon\big) \leq Cy\big((\xi+\epsilon) - \epsilon\big) = y(\epsilon) \tag{4.38}$$

. Assume $\xi \geq \tau - r_-$ and $r < 0$ such that $\xi + r \geq \tau - r_-$. Then using $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}, \xi \mapsto \max\{z \in \mathbb{Z} \,|\, z \leq \xi\}$ we observe $\xi \geq \xi - \big(1 + \lfloor \frac{|r|}{\epsilon} \rfloor\big)\epsilon \geq \xi - \epsilon$ so repeated application of (4.38) yields

$$\forall \xi \geq \tau - r_- : \forall r \geq \tau - r_- - \xi : \quad y(\xi + r) \leq y(\xi - \big(1 + \lfloor \frac{|r|}{\epsilon} \rfloor\big)\epsilon) \leq C^{1+\lfloor \frac{|r|}{\epsilon} \rfloor} y(\xi) \tag{4.39}$$

. If $\xi \geq \tau - 2r_-$ then $\forall i \in \{1, \cdots, N\} : \xi + r_i \geq \tau - r_-$. Using the second line of (4.37) and (4.39) we get

$$\begin{aligned} y^{(1)}(\xi) &= \big(-\frac{1}{c}A_1(\xi) - b\big)y(\xi) - \frac{1}{c}\sum_{r_i>0} A_i(\xi)\underbrace{y(\xi + r_i)}_{\leq y(\xi)} - \frac{1}{c}\sum_{r_i<0} A_i(\xi) \underbrace{y(\xi + r_i)}_{\leq C^{1+\lfloor \frac{|r_i|}{\epsilon} \rfloor} y(\xi)} \\ &\geq \Big(-\frac{1}{c}\big(\sum_{r_i \geq 0} \beta_i - \sum_{r_i<0} \beta_i C^{1+\lfloor \frac{|r_i|}{\epsilon} \rfloor}\big) - b\Big)y(\xi) \end{aligned} \tag{4.40}$$

for such ξ and hence setting $a \coloneqq -\frac{1}{c}\big(\sum_{r_i \geq 0} \beta_i - \sum_{r_i<0} \beta_i C^{1+\lfloor \frac{|r_i|}{\epsilon} \rfloor}\big)$ yields

$$\forall \xi \geq \tau - 2r_- : x^{(1)}(\xi) \geq bx(\xi) + (a - b)x(\xi) = ax(\xi) \tag{4.41}$$

.

If $r_- = 0$ then, with $y(\xi) = e^{-b\xi}x(\xi)$ as before, we again have $\forall \xi \geq \tau : y^{(1)}(\xi) \leq 0$ and $\forall : i \in 1, \cdots, N : y(\xi + r_i) \leq y(\xi)$. So $\forall \xi \geq \tau$

$$\begin{aligned} y^{(1)}(\xi) &= -by(\xi) - \frac{1}{c}\sum_{i=1}^{N} A_i(\xi)e^{kr_i}y(\xi + r_i) \\ &\geq \Big(-b - \frac{1}{c}\sum_{i=1}^{N} \alpha_i e^{kr_i}\Big)y(\xi) \end{aligned} \tag{4.42}$$

. Therefore

$$\forall \xi \geq \tau : x^{(1)}(\xi) \geq ax(\xi) \tag{4.43}$$

.

The proof for $c < 0$ is very similar but with inequalities the other way round and β_is instead of α_is. □

Proposition 4.2.6
Assume $c \neq 0$, A_i as in (4.11) satisfying (4.20) and the associated operator family L being asymptotically autonomous. Let $A_\pm, A_{\Sigma,\pm}$ as in (4.24), which we assume to satisfy

$A_{\Sigma,\pm} < 0$ as in (4.23). Further assume the limiting operators being approached at an exponential rate, that is $\exists k > 0$:

$$\|M_\pm(\xi)\| = \|A(\xi) - A_\pm\| \in O(e^{-k\xi}) \tag{4.44}$$

for $\xi \to \pm\infty$.

Then for these A_i, c Equation (4.11) is asymptotically hyperbolic. Additionally there exist four (by previous convention possibly infinite) quantities $\lambda_\pm^\pm$ satisfying

$$-\infty \le \lambda_\pm^- < 0 < \lambda_\pm^+ \le \infty$$

such that, if finite, $\lambda_+^\pm$ are real eigenvalues of L_+ and $\lambda_-^\pm$ are real eigenvalues of L_-, defined as in (4.18). They are uniquely determined by the property of being real positive/negative eigenvalues of the limiting equations. Whenever finite, they, i.e. every one that is finite is, are simple.

If, similar to the last Proposition, the assumptions only hold for $[\tau, \infty)$, $\tau \in \mathbb{R}$ and moreover a bounded function x satisfies Equation (4.11) in this interval $[\tau, \infty)$) exists then the following holds.

If $\lambda_+^- > -\infty$ then $\exists \epsilon > 0 : \exists C_+ \in \mathbb{R}$

$$x(\xi) - C_+ e^{\lambda_+^- \xi} \in O(e^{(\lambda_+^- - \epsilon)\xi}) \tag{4.45}$$

for $\xi \to \infty$. Additionally the asymptotic bound obtained by formally differentiating (4.45) holds, with different in the residal terms constants though. If $\exists \tau \in \mathbb{R} : \forall \xi \ge \tau :$ $x(\xi) \ge 0$ and $x|_{[\tau,\infty)} \ne 0$ then $C_+ > 0$.

If $\lambda_+^- = -\infty$, that is $\forall \xi < 0 : \Delta_{c,L_+} \ne 0$, then $\forall \xi \ge \tau : x(\xi) = 0$.

The analogous results holds for $(-\infty, \tau])$. In particular if $\lambda_-^+ < \infty$ then $\exists C_- \in \mathbb{R}$: $\exists \epsilon > 0$:

$$x(\xi) - C_- e^{\lambda_-^+ \xi} \in O(e^{(\lambda_-^+ + \epsilon)\xi}) \tag{4.46}$$

for $\xi \to -\infty$.

Proof. If all the assumptions hold for some $[\tau, \infty)$, $\tau \in \mathbb{R}$, then $L_+ = \lim_{\xi \to \infty} L(\xi)$ satisfies

$$\forall i \in 2, \cdots, N : \quad \lim_{\xi \to \infty} A_i(\xi) \ge \alpha_i > 0$$

by assumed continuity of the coefficient functions A_i. As $A_{\Sigma,+} = \sum_{i=1}^N A_{i,+} < 0$ by (4.23) we may use Proposition 4.2.3 and obtain existance of $\lambda_+^\pm$ with described properties.

The same conclusions hold for $(-\infty, \tau], \tau \in \mathbb{R}$, if conditions (4.20) and (4.23) hold for L_-. This proves the first few conclusions of the Proposition.

For further use we recall $\mathrm{Re}(\sigma_{c,L_+} \setminus \{\lambda_+^\pm\}) \subset (-\infty, \lambda_-^-) \cup (\lambda_-^+, \infty)$ from Proposition 4.2.3.

For the exponential bounds we again only present the proof for assumptions holding in $[\tau, \infty), \tau \in \mathbb{R}$, the proof for $(-\infty, \tau]$ being very similar. Hence suppose x solves (4.11) on $[\tau, \infty)$ for some $\tau \in \mathbb{R}$ as defined in (4.5).

Assume $\lambda_+^- < \infty$. As x is a bounded solution to (4.11) it is a $O(e^{0\xi})$ solution to

$$x^{(1)}(\xi) = -\frac{1}{c}\sum_{i=1}^{N} A_i(\xi)x(\xi + r_i) = -\frac{1}{c}L(\xi)\tau_{-\xi}x \tag{4.47}$$

on $[\tau, \infty)$. This equation is asymptotically hyperbolic and hence asymptotically autonomous at ∞ and additionally features exponential apporach of the limiting equations. By Proposition 7.2 of [2] either $\exists\lambda \in \sigma_{c,L_+} = \sigma_p(-\frac{1}{c}L_+) : \mathrm{Re}\lambda \in (-\infty, 0], \exists\epsilon > 0$ and some nontrivial eigensolution y to λ such that $x(\xi) - y(\xi) \in O(e^{(\mathrm{Re}\lambda-\epsilon)\xi})$ for $\xi \to \infty$ or $\forall b \in \mathbb{R} : \lim_{\xi\to\infty} e^{b\xi}x(\xi) = 0$. If $\forall b \in \mathbb{R} : \lim_{\xi\to\infty} e^{b\xi}x(\xi) = 0$ then (4.45) holds trivially. In the other case we have aldready obtained (4.45) from Proposition 7.2 as $\mathrm{Re}\lambda \leq \lambda_+^-$.

Suppose further that $\forall\xi \geq \tau : x(\xi) \geq 0$ and x does not vanish identically. As $x \geq 0$ in $[\tau, \infty)$ we obviously have $C_+ \geq 0$. Assume $C_+ = 0$. As x does not vanish identically, Proposition 4.2.5 implies that $\exists\lambda \in \sigma_{c,L_+} \setminus \{\lambda_+^-\} : \mathrm{Re}\lambda < \lambda_+^-$ such that $x(\xi) - y(\xi) \in O(e^{(\mathrm{Re}\lambda-\epsilon)\xi})$ for $\xi \to \infty, \epsilon > 0$ and some eigensolution y to λ. All eigenvalues of Λ_{c,L_0} except $\lambda_+^\pm$ have nonzero imaginary part. Hence by Lemma 4.2.1 $\forall\tau \in \mathbb{R} : \exists\xi \geq \tau : x(\xi) < 0$ which is a contradiction.

For the bound obtained by formal differentiation we only have to plug (4.45) into (4.11). As it is a bit cumbersome to write this down while using formally correct Landau notation we only outline the steps. We recall $f(\xi) = e^{\lambda_+^-\xi}$ being an eigensolution and obtain

$$\begin{aligned}
\text{“} -cx^{(1)}(\xi) &= \sum_{i=1}^{N} A_i(\xi)x(\xi + r_i) \\
&= \sum_{i=1}^{N} A_i(\xi)\Big(C_+e^{\lambda_+^-(\xi+r_i)} + O(e^{(\lambda_+^--\epsilon)(\xi+r_i)})\Big) \\
&= C_+\sum_{i=1}^{N} A_i(\xi)\underbrace{e^{\lambda_+^-(\xi+r_i)}}_{f(\xi+r_i)} + O\Big(\underbrace{\big(\sum_{i=1}^{N} A_i(\xi)e^{(\lambda_+^--\epsilon)r_i}\big)}_{\text{bounded}}\, e^{(\lambda_+^--\epsilon)\xi}\Big) \\
&= \lambda_+^-C_+e^{\lambda_+^-\xi} + O(e^{(\lambda_+^--\epsilon)\xi}) \text{”}
\end{aligned}$$

.

If $\lambda_+^- = \infty$ then by Lemma 4.2.4 $r_- = 0, c > 0$. We gain use propostion 7.2 from [2]. The first of the two alternatives is not possible as $\mathrm{Re}\sigma_{c,L_+} \cap (-\infty, 0) \subset (-\infty, \lambda_+^-) = \emptyset$. Hence the second option holds. By Proposition 4.2.5 this is only possible if

$$\exists R > 0 : \forall\xi \geq \tau + R : x(\xi) = 0 \tag{4.48}$$

. Hence $\exists R > 0 : \forall\tau + R = \tau + R + r_- \leq \xi \leq \tau + R + r_+ : x(\xi) = 0$. Obviously $x \equiv 0$ is a solution to (4.11), $c \neq 0$ and $[\tau, \infty)^\# = [\tau, \infty)$ as $r_- = 0$. We apply Lemma 4.1.2 and finally obtain

$$\forall\xi \in [\tau, \tau + R + r_+] \cup [\tau + R, \infty) = [\tau, \infty) : \quad x(\xi) = 0 \tag{4.49}$$

. □

At this point it might be usefull to remind the reader that, as claimed in remark 3.1.2 the kernel of $\Lambda_{c,L} : W^{1,p} \to L^p$ is independent of p, allowing notation such as both $\mathfrak{K}_{c,L} = \ker(\Lambda_{c,L} : W^{1,\infty} \to L^\infty)$ and $\mathfrak{K}_{c,L^*} = \ker(\Lambda_{c,L} : W^{1,1} \to L^1)$ with p implied by context (here $p = \infty$ and Fredholm Alternative (1,∞ conjugated).

Proposition 4.2.7

Assume all of the prerequisites of the last Proposition hold and additionally that some nontrivial solution $p \in W^{1,\infty}, p \geq 0$ to (4.11) on $\mathbb{R}$ exists.

Then (4.11) is asymptotically hyperbolic

$$\lambda_+^- \in (-\infty, 0) \qquad \lambda_-^+ \in (0, \infty) \tag{4.50}$$

exist and $\Lambda_{c,L} : W^{1,\infty} \to L^\infty$ is Fredholm with

$$\dim \mathfrak{K}_{c,L} = \dim \mathfrak{K}_{c,L^*} = \operatorname{codim} \mathfrak{R}_{c,L} = 1 \tag{4.51a}$$

$$\operatorname{ind}(\Lambda_{c,L}) = 0 \tag{4.51b}$$

. $p \in \mathfrak{K}_{c,L}$ is strictly positive, i.e. $\forall \xi \in \mathbb{R} : p(\xi) > 0$.

Proof. Proposition 4.2.6 yields asymptotic hyperbolicity and existance of $\lambda_\pm^\pm \in \mathbb{R} \cup \{-\infty, \infty\}$. As x does not vanish identically λ_+^- and λ_-^+ are finite, otherwise x would vanish on at least one of the intervals $[\tau, \infty)$ or $(-\infty, \tau]$ for some $\tau \in \mathbb{R}$ by Proposition 4.2.6.

If $p(\tau) = 0$ for some $\tau \in \mathbb{R}$ then by Lemma 4.1.1 $\forall \xi \geq \tau : p(\xi) = 0$ or $\forall \xi \leq \tau : p(\xi) = 0$. Then 4.1.2 would force p to vanish on the remaining domain, i.e. make p vanish identically on $\mathbb{R}$, which is a contradiction.

Theorem 3.2.3 asserts Fredholmness of $\Lambda_{c,L}$ and also of Λ_{c,L^*} and additonally that the correspondencies between those two hold. In particular, as $\operatorname{ind}(\Lambda_{c,L}) = \dim \mathfrak{K}_{c,L} - \operatorname{codim} \mathfrak{R}_{c,L}$ and $\dim \mathfrak{K}_{c,L^*} = \operatorname{codim} \mathfrak{R}_{c,L}$ hold, it suffices to show $\dim \mathfrak{K}_{c,L} = 1$ and $\operatorname{ind}(\Lambda_{c,L}) = 0$.

First we will establish $\dim \mathfrak{K}_{c,L} = 1$. $0 \neq p \in \mathfrak{K}_{c,L}$ implies $\dim \mathfrak{K}_{c,L} \geq 1$. Assume $\dim \mathfrak{K}_{c,L} > 1$. Then $\exists y \in \mathfrak{K}_{c,L}$ linearly independent from p. By Proposition 4.2.6 there are $C_+(p), C_-(p) \in (0, \infty)$, $C_+(y), C_-(y) \in \mathbb{R}$, $\epsilon > 0$ such that

$$p(\xi) = \begin{cases} C_-(p) e^{\lambda_-^+ \xi} + O(e^{(\lambda_-^+ + \epsilon)\xi}) & \xi \to -\infty \\ C_+(p) e^{\lambda_+^- \xi} + O(e^{(\lambda_+^- - \epsilon)\xi}) & \xi \to \infty \end{cases}$$

$$y(\xi) = \begin{cases} C_-(y) e^{\lambda_-^+ \xi} + O(e^{(\lambda_-^+ + \epsilon)\xi}) & \xi \to -\infty \\ C_+(y) e^{\lambda_+^- \xi} + O(e^{(\lambda_+^- - \epsilon)\xi}) & \xi \to \infty \end{cases}$$

with the usual conventions for calcuations with Landau notation. Set $x = y - \frac{C_+(y)}{C_+(p)} p$. This x satisfies

$$x(\xi) = \begin{cases} C_-(x) e^{\lambda_-^+ \xi} + O(e^{(\lambda_-^+ + \epsilon)\xi}) & \xi \to -\infty \\ O(e^{(\lambda_+^- - \epsilon)\xi}) & \xi \to \infty \end{cases} \tag{4.52}$$

. Without loss of generality let $C_-(x) \leq 0$, otherwise consider $-x$.

Because of linear independence x is not the zero function. Hence there are arbitrarily large $\xi \in \mathbb{R}$ for which $x(\xi) \neq 0$. We claim further that there exist arbitrarily large $\xi \in \mathbb{R}$ for which $x(\xi) > 0$. Assume the contrary were the case. Then $\forall \xi \in \mathbb{R} : x(\xi) \leq 0$. Applying Proposition 4.2.6 to $-x \geq 0$ yields $\exists C > 0$ such that for $\xi \to \infty$

$$x(\xi) = -Ce^{\lambda_+^- \xi} + O(e^{(\lambda_+^- - \epsilon)\xi})$$

which is a contradiction to our construction of x. Assume $\xi_0 > 0 : x(\xi_0) > 0$.

Let $\mu \geq 0$ and consider $p - \mu x$. As p is bounded $\exists \mu_0 > 0 : (p - \mu_0 x)(\xi_0) < 0$. Our asymptotic bounds imply

$$(p - \mu x)(\xi) = \begin{cases} \underbrace{\Big(C_-(p) - \mu C_-(x)\Big)}_{>0} e^{\lambda_-^+ \xi} + O(e^{(\lambda_-^+ + \epsilon)\xi}) & \xi \to -\infty \\ \underbrace{C_+(p)}_{>0} e^{\lambda_+^- \xi} + O(e^{(\lambda_+^- - \epsilon)\xi}) & \xi \to \infty \end{cases}$$

. We know that the residual terms in this formulas become arbitrarily smaller than the leading terms, so $\exists \tau > \xi_0 > 0 : \forall |\xi| > \tau : \forall \mu \geq 0 : (p - \mu x)(\xi) > 0$. x is continuous and hence $x_+ = \max\{x(\xi) | \xi \in [-\tau, \tau]\}$ exists. As the set $S := \{\mu \in [0, \mu_0] | \forall \xi \in \mathbb{R} : p(\xi) - \mu x(\xi) \geq 0\} \ni 0$ is nonempty and bounded above by μ_0, $\mu_* := \sup S$ is well defined.

By definition $\forall \xi \in \mathbb{R} : p(\xi) - \mu_* x(\xi) \geq 0$ and $\exists \xi_1 \in \mathbb{R} : (p - \mu_* x)(\xi_1) = 0$. The first is obvious from definition of μ_*. Assuming the second to be wrong would imply $(p - \mu_* x)|_{[-\tau,\tau]} > 0$. This continuous function would take a minimum p_- in the compact set $[-\tau, \tau]$. Now if $x_+ \leq 0$ then $\forall \delta > 0 : (p - (\mu_* + \delta)x)|_{[-\tau,\tau]} > 0$. If $x_+ > 0$ then taking $0 < \delta < \frac{p_-}{x_+}$ we would obtain $\forall \xi \in [-\tau, \tau]$:

$$(p - (\mu_* + \delta)x)(\xi) = (p - \mu_* x)(\xi) - \delta x(\xi) > p_- - \delta x_+ > 0$$

As $(p - \mu x)|_{(-\infty,-\tau] \cup [\tau,\infty)} > 0$ anyway this would give $\mu_* < \mu_* + \delta \in S$ and hence lead to a contradiction. Thus $\exists \xi_1 \in \mathbb{R} : (p - \mu * x)(\xi_1) = 0$.

However then Lemma 4.1.1 would imply $p - \mu_* x = 0$ on $(-\infty, \xi_1]$ or $[\xi_1, \infty)$ contradicting existance of τ. Hence y cannot exist and the proof, that $\mathfrak{K}_{c,L}$ is one dimensional, is thus complete.

To show $\mathrm{ind}(\Lambda_{c,L}) = 0$ we note that by setting $L^\rho = ((1 - \rho)L_- + (1 + \rho)L_+)/2$ the corresponding constant coefficient system is hyperbolic for all $\rho \in [-1, 1]$. An Application of Theorem B of [2] thus finishes the proof. □

We finish our section on linear equations by gathering all introduced parts and formulating Theorem 4.1 from [2], which will later be the basis for application of the implicit function theorem.

Theorem 4.2.8

We follow notation and assumptions already set forth in 1.1.1, 1.1.4, 1.1.6, (4.11), (4.13), (4.14), (4.12), (4.20) and (4.23) i.a. . Let all of these assumptions hold, that is, the assumptions of the last proposition.

Then our linear equation is asymptotically hyperbolic, and $\Lambda_{c,L} \in B(W^{1,\infty}, L^\infty)$ is Fredholm with

$$\dim \mathfrak{K}_{c,L} = \dim \mathfrak{K}_{c,L^*} = \operatorname{codim} \mathfrak{R}_{c,L} = 1 \qquad \operatorname{ind}(\Lambda_{c,L}) = 0$$

$p \in \mathfrak{K}_{c,L}$ satisfies $p > 0$ and there exists some $p^* \in \mathfrak{K}_{c,L^*}$ satisfying $p^* > 0$ as well.

Proof. Asymptotic hyperbolicity has been established in Proposition 4.2.6. By 3.2.3 it followed that $\Lambda_{c,L}$ is Fredholm. In 4.2.7 we have proven the results on kernel dimension, Fredholm index, positivity of p and finitness of the eigenvalues λ_+^-,λ_-^+.

All that remains to be shown are the existance and positivity of some p^* in $\mathfrak{K}_{c,L^*}$. As with p in Proposition 4.2.7) it suffices to show $p^* \geq 0$ and $p \neq 0$ for one nontrivial $p^* \in \mathfrak{K}_{c,L^*}$ as strict positivity follows through Lemmata 4.1.1 and 4.1.2 exactly as for p. We already know that $\mathfrak{K}_{c,L^*}$ is one dimensional. Hence nontrivial elements exist in the kernel. If one of them satisfied $p^* \geq 0$, then all real functions $q \in \mathfrak{K}_{c,L^*}$ would satisy $q \geq 0$ or $q \leq 0$. Thus assume existance of some $p^* \in \mathfrak{K}_{c,L^*}$, $\xi_1, \xi_2 \in \mathbb{R}$ such that

$$p^*(\xi_1) > 0 \qquad p^*(\xi_2) < 0 \tag{4.53}$$

We seek a contradiction.

Lemma 4.1.1 implies that p^* cannot vanish on any interval of length $r_+ - r_-$, if p^* is truly nontrivial. Hence assume $|\xi_1 - \xi_2| < r_+ - r_-$. It follows next, that there exists a nontrivial continuous function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} p^*(\xi)h(\xi)\mathrm{d}\xi = 0 \tag{4.54}$$

.

We briefly discuss one possible way ob obtaining such h. p^* is continuous so $\exists \epsilon > 0 : p^*|_{[\xi_1-\epsilon,\xi_1+\epsilon]} > 0 \wedge p^*|_{[\xi_2-\epsilon,\xi_2+\epsilon]} < 0$. Now set

$$h(\xi) = \begin{cases} 1 - \frac{|\xi-\xi_1|}{\epsilon} & \xi \in [\xi_1 - \epsilon, \xi_1 + \epsilon] \\ \frac{h(\xi_1-\xi_2+\xi)p^*(\xi_1-\xi_2+\xi)}{|p^*(\xi)|} & \xi \in [\xi_2 - \epsilon, \xi_2 + \epsilon] \\ 0 & \xi \in \mathbb{R} \setminus ([\xi_1 - \epsilon, \xi_1 + \epsilon] \cup [\xi_2 - \epsilon, \xi_2 + \epsilon] \end{cases}$$

Then h is continuous, nonnegative and compactly supported and

$$\begin{aligned} \int_{\mathbb{R}} p^*(\xi)h(\xi)\mathrm{d}\xi &= \int_{\xi_1-\epsilon}^{\xi_1+\epsilon} p^*(\xi)h(\xi)\mathrm{d}\xi + \int_{\xi_2-\epsilon}^{\xi_2+\epsilon} h(\xi_1 - \xi_2 + \xi)p^*(\xi_1 - \xi_2 + \xi) \underbrace{\frac{p^*(\xi)}{|p^*(\xi)|}}_{=-1} \mathrm{d}\xi \\ &= \int_{\xi_1-\epsilon}^{\xi_1+\epsilon} p^*(\xi)h(\xi)\mathrm{d}\xi - \int_{\xi_1-\epsilon}^{\xi_1+\epsilon} p^*(\xi)h(\xi)\mathrm{d}\xi = 0 \end{aligned}$$

.

By choosing ϵ small enough we can assume $\exists \tau_1 < \tau_2 \in \mathbb{R}$ such that h is supported in $[\tau_1, \tau_2]$ and $\tau_2 - \tau_1 < r_+ - r_-$. As $\dim \mathfrak{K}_{c,L^*} = 1$, (4.54) and Theorem 3.2.3 imply that $h \in \mathfrak{R}_{c,L} = \left(\mathfrak{K}_{c,L^*}\right)_{\perp L^\infty}$. Hence $\exists x \in W^{1,\infty} : \Lambda_{c,L}x = h$.

As h is compactly supported x satisfies the homogeneous equation for large $|\xi|$. Proposition 4.2.6 assures that x and p enjoy estimates (4.45) and (4.45) for some $\epsilon > 0$ and quantities $C_+(x), C_-(x) \in \mathbb{R}$, $C_+(p), C_-(p) > 0$. Hence $\exists \tau_0 > 0 : \exists \mu_0 > 0 : \forall |\xi| \geq \tau_0 : \forall \mu \geq \mu_0 : (x + \mu p)(\xi) > 0$. On the compact set $[-\tau_0, \tau_0]$ the continuous function p takes a minimum. This fact and x's boundedness imply that $\exists \mu_1 > 0 : \forall |\xi| \leq \tau_0 : \forall \mu \geq \mu_1 : (x + \mu p)(\xi) > 0$ as well (compare to proof of Proposition 4.2.7). Similarily we conclude that $\exists \mu < 0 : (x + \mu p)(\xi) < 0$. Thus

$$\mu_* := \min\{\mu \in \mathbb{R} \,|\, x + \mu p \geq 0\} \tag{4.55}$$

is a well defined element of $\mathbb{R}$.

Consider the function $y = x + \mu_* p \in \Lambda_{c,L}^{-1}(\{h\})$. Of course $y \geq 0$, y solves $\Lambda_{c,L} y = h$, y solves the homogeneous equation for $\xi \notin [\tau_1, \tau_2]$ and by Proposition 4.2.6 enjoys the estimates

$$y(\xi) = \begin{cases} C_+(y) e^{\lambda_-^+ \xi} + O(e^{(\lambda_-^+ + \epsilon)\xi}) & \xi \to -\infty \\ C_-(y) e^{\lambda_+^- \xi} + O(e^{(\lambda_+^- - \epsilon)\xi}) & \xi \to \infty \end{cases} \tag{4.56}$$

with $C_+(y) \geq 0$,$C_-(y) \geq 0$ and some $\epsilon > 0$. As $y \geq 0$, Proposition 4.2.6 further implies either that $C_+(y) > 0$ or that y vanishes for large ξ, i.e. $\exists \tau_3 \in \mathbb{R} : \forall \xi \geq \tau_3 : y(\xi) = 0$, and similarly either that $C_-(y) > 0$ or that y vanishes for small ξ, i.e. $\exists \tau_4 \in \mathbb{R} : \forall \xi \leq \tau_4 : y(\xi) = 0$.

We claim that y cannot vanish for both small and large ξ. Assuming otherwise would yield $\exists \tau_+ > \tau_2 : \forall \xi \in [\tau_+ + r_-, \infty) \supseteq [\tau_+ + r_-, \tau_+ + r_+] : y(\xi) = 0$. Then by setting $J = [\tau_2 + r_-, \infty)$ Lemma 4.1.2 would imply that $\forall \xi \in (-\infty, \tau_+ + r_+] \cap J^\# = [\tau_2 + r_-, \tau_+ + r_+] : x(\xi) = 0$. Hence $\forall \xi \in [\tau_2 + r_-, \infty) : x(\xi) = 0$. Additionally $\exists \tau_- < \tau_1 : \forall \xi \in (-\infty, \tau_- + r_+] \supseteq [\tau_- + r_-, \tau_- + r_+] : y(\xi) = 0$. Setting $J = (-\infty, \tau_1 + r_+]$ we could use Lemma 4.1.2 again, yielding $\forall \xi \in (-\infty, \tau_1 + r_+] : x(\xi) = 0$. But $\tau_2 - \tau_1 < r_+ - r_-$ so $\tau_1 + r_+ > \tau_2 + r_-$. Hence $(-\infty, \tau_1 + r_+] \cup [\tau_2 + r_-, \infty) = \mathbb{R}$ and $y = 0$. Thus $0 \neq h = \Lambda_{c,L} y = \Lambda_{c,L} 0 = 0$, which would be a contradiction.

We again have $\exists \xi_0 : y(\xi_0) = 0$. If not then $y > 0$ and hence Proposition 4.2.6 would imply that $C_+(y) > 0$ and $C_-(y) > 0$. The rest of the argument is similar to the last proof's but a little bit more cumbersome to write down. The following formulas are but conclusions obtained directly from the definition of Landau notation. As

$$y(\xi) = \begin{cases} C_-(y) e^{\lambda_-^+ \xi} + a_-(\xi) & \xi \leq -\tau_5 \\ C_+(y) e^{\lambda_+^- \xi} + a_+(\xi) & \xi \geq \tau_5 \end{cases}$$

$$p(\xi) = \begin{cases} C_-(p) e^{\lambda_-^+ \xi} + b_-(\xi) & \xi \leq -\tau_5 \\ C_+(p) e^{\lambda_+^- \xi} + b_+(\xi) & \xi \geq \tau_5 \end{cases}$$

with

$$\forall \xi \leq -\tau_5 : |a_-(\xi)| \leq D_1 e^{(\lambda_-^+ + \epsilon)\xi} \qquad \forall \xi \geq \tau_5 : |a_+(\xi)| \leq D_2 e^{(\lambda_+^- - \epsilon)\xi}$$
$$\forall \xi \leq -\tau_5 : |b_-(\xi)| \leq D_3 e^{(\lambda_-^+ + \epsilon)\xi} \qquad \forall \xi \geq \tau_5 : |b_+(\xi)| \leq D_4 e^{(\lambda_+^- - \epsilon)\xi}$$

for some $\epsilon > 0$,$\tau_5 > 0$ and various constants $D_1, \cdots, D_4 \geq 0$ by our asymptotic bounds, we could then conclude that if $\tau_6 \geq \tau_5$ such that

$$\forall |\xi| \geq \tau_6 : \quad \max\{D_i | i \in \{1, \cdots, 4\}\} e^{-\epsilon|\xi|} < \frac{1}{2} \min\{C_-(y), C_+(y), C_-(p), C_+(p)\}$$

, which really does exist, then $\forall |\xi| \geq \tau_6 : \forall 0 \leq \delta < \delta_0 := \min\{\frac{C_-(y)}{3C_-(p)}, \frac{C_+(y)}{3C_+(p)}\}$:

$$\begin{aligned}
(x + (\mu_* - \delta)p)(\xi) &= (y - \delta p)(\xi) \\
&= \begin{cases} C_-(y)e^{\lambda_-^+ \xi} + a_-(\xi) - \delta\big(C_-(p)e^{\lambda_-^+ \xi} + b_-(\xi)\big) & \xi \leq -\tau_6 \\ C_+(y)e^{\lambda_+^- \xi} + a_+(\xi) - \delta\big(C_+(p)e^{\lambda_+^- \xi} + b_+(\xi)\big) & \xi \geq \tau_6 \end{cases} \\
&\geq \begin{cases} \big(C_-(y) - D_1 e^{\epsilon\xi}\big)e^{\lambda_-^+ \xi} - \delta\big(C_-(p) + D_3 e^{\epsilon\xi}\big)e^{\lambda_-^+ \xi} & \xi \leq -\tau_6 \\ \big(C_+(y) - D_2 e^{-\epsilon\xi}\big)e^{\lambda_+^- \xi} - \delta\big(C_+(p) + D_4 e^{-\epsilon\xi}\big)e^{\lambda_+^- \xi} & \xi \geq \tau_6 \end{cases} \\
&> \begin{cases} \left(\frac{1}{2}C_-(y) - \frac{C_-(y)}{3C_-(p)}\frac{3}{2}C_-(p)\right)e^{\lambda_-^+ \xi} & \xi \leq -\tau_6 \\ \left(\frac{1}{2}C_+(y) - \frac{C_+(y)}{3C_+(p)}\frac{3}{2}C_+(p)\right)e^{\lambda_+^- \xi} & \xi \geq \tau_6 \end{cases} \\
&> 0
\end{aligned} \tag{4.57}$$

. Furthermore the continuous functions y and p would take a minimum $y_- > 0$ and a maximum $p_+ > 0$ in the compact set $[-\tau_6, \tau_6]$. Finally by setting $\delta_1 := \frac{y_-}{p_+} > 0$ we would obtain $\forall 0 \leq \delta < \min\{\delta_0, \delta_1\}$:

$$\forall \xi \in \mathbb{R} : \quad (x + (\mu_* - \delta)p)(\xi) = (y - \delta p)(\xi) > 0 \tag{4.58}$$

which would be a contradiction to the definition of μ_*.

Without loss of generality we now assume that $C_+(y) > 0$ as the proof when $C_-(y) > 0$ runs along the same lines. $C_+(y) > 0$ implies $y(\xi) > 0$ for large ξ. As $y(\xi_0) = 0$, $y^{-1}(\{0\}) \subseteq \mathbb{R}$ is not empty. Moreover this set is bounded, as $y(\xi) > 0$ for large ξ, and closed, as y is continuous, so

$$\xi_0 := \sup\{\xi \in \mathbb{R} \,| y(\xi) = 0\} \in y^{-1}(\{0\}) \tag{4.59}$$

is well defined.

Certainly $y(\xi_0) = 0$ and $\forall \xi > \xi_0 : y(\xi) > 0$. We recall that h, y and A_is are continuous and conclude that y is C^1. Hence ξ_0 is a minimum of a C^1 function and it necessarily follows that $y^{(1)}(\xi_0) = 0$.

On the other hand, if, without loss of generality, $r_2 = r_+ > 0$ and hence $\xi_0 + r_+ > \xi_0$, we obtain

$$\begin{aligned}
-cy^{(1)}(\xi_0) &= \sum_{i=1}^{N} A_i(\xi_0) y(\xi_0 + r_i) + h(\xi_0) \\
&= A_1(\xi_0) \underbrace{y(\xi_0)}_{=0} + \sum_{i=2}^{N} \underbrace{A_i(\xi_0)}_{>0} \underbrace{y(\xi_0 + r_i)}_{\geq 0} \\
&\geq A_2(\xi_0) y(\xi_0 + r_2) > 0
\end{aligned}$$

, which is false.

If $r_+ = 0$ then $\lambda_-^+ < \infty$ implies $c > 0$ by Lemma 4.2.4 and necessarily $r_- < 0$. If $C_-(y) > 0$ then $y(\xi) > 0$ for sufficiently small ξ and an argument at $\inf\{\xi \in \mathbb{R} \,|y(\xi) = 0\}$ analogous to the last one yields a contradiction. If $C_-(y) \leq 0$ then y vanishes identically for sufficiently small ξ, that is $\exists \tau_7 \in \mathbb{R} : \forall \xi \leq \tau_7 : y(\xi) = 0$. Set

$$\xi_1 = \sup\{\tau \in \mathbb{R} \,|\forall \xi \leq \tau : y(\xi) = 0\} \tag{4.60}$$

which is well defined as the set in question is bounded above by ξ_0. Now take $0 < \epsilon_1 < \min\{|r_i||1 \leq i \leq N\}$ and consider Equation (4.11) on the interval $[\xi_1, \xi_1 + \epsilon_1]$. By definition of ξ_1 it holds that $\forall \xi \in [\xi_1, \xi_1 + \epsilon_1] : \forall 2 \leq i \leq N : \xi + r_i < \xi_1$ so $\forall \xi \in [\xi_1, \xi_1 + \epsilon_1] : \forall 2 \leq i \leq N : y(\xi + r_i) = 0$. Thus (4.11) dissolves into

$$-cy^{(1)}(\xi) = A_1(\xi)y(\xi) + h(\xi) =: H(\xi, y(\xi))$$

on $[\xi_1, \xi_1 + \epsilon_1]$. As A_1 is bounded we have H continuous and Lipschitz continuous in y. Hence by the Picard-Lindelöf existance theorem $y|_{[\xi_1, \xi_1+\epsilon_1]}$ is the pointwise limit of the Picard iteration, that is $\forall \xi \in [\xi_1, \xi_1 + \epsilon_1] : y(\xi) = \lim_{n\to\infty} y_n(\xi)$ where

$$\begin{aligned} y_0(\xi) &= y(\xi_1) = 0 \\ y_n(\xi) &= y_{n-1}(\xi) - \frac{1}{c}\int_{\xi_1}^{\xi} A_1(\xi)y_{n-1}(\xi) + h(\xi)\mathrm{d}\xi \end{aligned}$$

. A simple induction using $y \geq 0$ yields $\forall n \in \mathbb{N} : \forall \xi \in [\xi_1, \xi_1 + \epsilon_1] : y_n(\xi) = 0$. Hence $y|_{[\xi_1, \xi_1+\epsilon_1]} = 0$. This is a contradiction and completes the proof. ☐

4.3 Nonlinear Equations

We finish our chapter on applications by paving the way for and proving Proposition 6.4 from [3].

Lemma 4.3.1
Let $c \neq 0$, $\rho \in \overline{V}$, and $x \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$ be a solution to (4.1). Let

$$\mu_- := \inf\{x(\xi)|\xi \in \mathbb{R}\} \qquad \mu_+ := \sup\{x(\xi)|\xi \in \mathbb{R}\} \tag{4.61}$$

Then

$$\mu_- \in [-1, q(\rho)] \cup \{1\} \qquad \mu_+ \in \{-1\} \cup [q(\rho), 1] \tag{4.62}$$

The same conclusion holds for

$$\begin{aligned} \mu_{-,\infty} &:= \liminf_{\xi\to\infty} x(\xi) & \mu_{+,\infty} &:= \limsup_{\xi\to\infty} x(\xi) \\ \mu_{-,-\infty} &:= \liminf_{\xi\to-\infty} x(\xi) & \mu_{+,-\infty} &:= \limsup_{\xi\to-\infty} x(\xi) \end{aligned}$$

Proof. We first remark that $\mu_\pm$ and $\mu_{\pm,\pm\infty}$ really exist as $x \in W^{1,\infty}(\mathbb{R},\mathbb{R})$.

By (i) we have

$$\mu_- \in [-1, q(\rho)] \cup [1,\infty) \Leftrightarrow \Phi(\mu_-,\rho) \leq 0$$
$$\mu_+ \in (-\infty,-1] \cup [q(\rho),1] \Leftrightarrow \Phi(\mu_+,\rho) \geq 0$$

As $\mu_- \leq \mu_+$ it follows that if

$$\Phi(\mu_-,\rho) \leq 0 \leq \Phi(\mu_+,\rho) \tag{4.63}$$

then

$$\mu_- \in [-1,q(\rho)] \cup \{1\} \qquad\qquad \mu_+ \in \{-1\} \cup [q(\rho),1]$$

which is what we need to show for the first third of the theorem. Hence we attempt to prove (4.63).

We only show $\Phi(\mu_+,\rho) \geq 0$, the proof for the other inequality being similar. Let $(\xi_n) \in \mathbb{R}^{\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} x(\xi_n) = \mu_+$. Without loss of generality we may assume that this sequence satisfies $\forall n \in \mathbb{N} : |(|x(\xi_n) - \mu_+) < 1$. For any $2 \leq i \leq N$ the sets $\{x(\xi_n + r_i)|n \in \mathbb{N}\}$ are bounded as $\forall 1 \leq i \leq N : \forall n \in \mathbb{N} :$

$$|x(\xi_n + r_i)| \leq |x(\xi_n)| + |r_i|\|x^{(1)}\|_{L^\infty} \leq |\mu_+| + 1 + \max\{|r_i||1 \leq i \leq N\}\|x^{(1)}\|_{L^\infty} < \infty \tag{4.64}$$

Hence, after passing to a subsequence, we may assume that $\exists \mu_2, \cdots, \mu_N \in (-\infty, \mu_+]$, such that $\forall 1 \leq i \leq N : \lim_{n\to\infty} x(\xi_n + r_i) = \mu_i$. Hence $\lim_{n\to\infty} x^{(1)}(\xi_n) = \lim_{n\to\infty} F(x(\xi_n), x(\xi_n + r_2), \cdots, x(\xi_n + r_N), \rho)$ exists.

We claim that $\lim_{n\to\infty} x^{(1)}(\xi_n) = 0$. If (ξ_n) is a bounded function then there exists a convergent subsequence $(\xi_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \xi_{n_k} = \xi_0 \in \mathbb{R}$ and necessarily $x(\xi_0) = \mu_+$. Hence $x^{(1)}(\xi_0) = 0$ and $\lim_{k\to\infty} x^{(1)}(\xi_{n_k}) = 0$. Thus $\lim_{n\to\infty} x^{(1)}(\xi_n) = \lim_{k\to\infty} x^{(1)}(\xi_{n_k}) = 0$. Now assume (ξ_n) is unbounded. Hence after passing to a subsequence we may assume $|\xi_n| \to \infty$. Assume $\lim_{n\to\infty} x^{(1)}(\xi_n) = c \neq 0$. We consider the case $c > 0$, the proof for $c < 0$ being very similar. Then setting $\epsilon = \frac{c}{3}$ we may pass to a subsequence of x_n such that $\forall n \in \mathbb{N} : x^{(1)}(\xi_n) \geq 2\epsilon$. By uniform continuity of $x^{(1)}$ there exists $\delta_0 > 0 : \forall \xi_1, \xi_2 \in \mathbb{R} : |\xi_1 - \xi_2| < \delta_0 \Rightarrow |x^{(1)}(\xi_1) - x^{(1)}(\xi_2)| < \epsilon$. In particular

$$\forall 0 \leq \delta < \delta_0 : \forall n \in \mathbb{N} : x^{(1)}(\xi_n + \delta) > x^{(1)}(\xi_n) - \epsilon \geq \epsilon \tag{4.65}$$

Fixing $\delta_1 \in (0, \delta_0)$ we obtain $\forall n \in \mathbb{N}$

$$\frac{x(\xi_n + \delta) - x(\xi_n)}{\delta} \geq \min\{x^{(1)}(\xi)|\xi \in [\xi_n + \delta, \xi_n]\} \Rightarrow x(\xi_n + \delta) \geq x(\xi_n) + \delta\epsilon$$

using the mean value theorem. But this leads to a contradicicition as

$$\lim_{n\to\infty} x(\xi_n + \delta) \geq \lim_{n\to\infty} x(\xi_n) + \delta\epsilon = \mu_+ + \delta\epsilon > \mu_+ \tag{4.66}$$

.

Thus, having already passed to appropriate subsequences, $\lim_{n\to\infty} x^{(1)}(\xi_n) = 0$. Therefore, by using (ii), $\mu_i \le \mu_+$ and continuity of F, we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} x^{(1)}(\xi_n) \\ &= \lim_{n\to\infty} F(x(\xi_n), x(\xi_n + r_2), \cdots, x(\xi_n + r_N), \rho) \\ &= F(\lim_{n\to\infty} x(\xi_n), \lim_{n\to\infty} x(\xi_n + r_2), \cdots, \lim_{n\to\infty} x(\xi_n + r_N), \rho) \\ &= F(\mu_+, \mu_2, \cdots, \mu_N, \rho) \\ &\le F(\mu_+, \mu_+, \cdots, \mu_+, \rho) \\ &= \Phi(\mu_+, \rho) \end{aligned} \tag{4.67}$$

.

We now proceed to prove (4.62) for $\mu_{+,\infty}$, the proof for $\mu_{-,\infty}$ and $\mu_{\pm,-\infty}$ being similar. Let $(\xi_n) \in \mathbb{R}^{\mathbb{N}}$ be a sequence such that $\xi_n \to \infty$ and $x(\xi_n) \to \mu_{+,\infty}$ for $n \to \infty$. Set $y_n = \tau_{-\xi_n} y$, i.e. $\forall \xi \in \mathbb{R} : y_n(\xi) = y(\xi + \xi_n)$. Each of those y_n is a solution to (4.1) and hence uniformly continuous. Thus y_n are uniformly bounded and equicontinuous, so by using the Arzelà-Ascoli theorem we may pass to an uniformly convergent subsequence on each compact interval. As in the proof of 3.1.3 we conclude that, after passing to a subsequence, y_n converges pointwise to some $y \in W^{1,\infty}$, with this convergence being uniform on each compact interval. Of course this argument also applies to $y^{(1)}$ so we have no problems with smoothness. Thus y is also a solution to (4.1) by continuity of F and y satisfies $\mu_- \le y \le \mu_{+,\infty}$ and in fact $\mu_{+,\infty} = \lim_{\xi\to\infty} x(\xi) = \sup\{x(\xi) | \xi \in \mathbb{R}\}$. Therefore, from the first part of this proof, we have hat $\mu_{+,\infty} \in \{-1\} \cup [q(\rho), 1]$ as claimed. □

Corollary 4.3.2
Let $c \neq 0$, $\rho \in \overline{V}$, and $P \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$ be a solution to (4.1) satisfying the boundary conditions (4.2).

Then

$$\forall \xi \in \mathbb{R} : \quad -1 < P(\xi) < 1 \tag{4.68}$$

Proof. Lemma 4.3.1 implies that $\forall \xi \in \mathbb{R} : -1 \le P(\xi) \le 1$. Assume $\exists \tau \in \mathbb{R} : P(\xi) = 1$. The constant 1 function is a solution to (4.1) so by Lemma 4.1.1 $P|_{[\tau,\infty)} = 1$ or $P|_{(-\infty,\tau]} = 1$. In either case Lemma 4.1.2 would imply $\forall \xi \in \mathbb{R} : P(\xi) = 1$. Then P cannot satisfy $\lim_{\xi\to-\infty} P(\xi) = -1$ and we have a contradiction.

The same arguments can be applied for -1. □

Theorem 4.3.3
Let $c \neq 0$, $\rho \in W$, and $P \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$ be a solution to (4.1) satisfying the boundary conditions (4.2).

Then $\exists C_\pm > 0$, $\exists \epsilon > 0$, such that

$$P(\xi) = \begin{cases} -1 + C_- e^{\lambda^u \xi} + O(e^{(\lambda^u + \epsilon)\xi}) & \xi \to -\infty \\ 1 - C_+ e^{\lambda^s \xi} + O(e^{(\lambda^s - \epsilon)\xi}) & \xi \to \infty \end{cases} \tag{4.69}$$

where $\lambda^u \in (0, \infty)$ is the unique positive eigenvalue of the linearization about $x = -1$, i.e. λ_-^+ obtained from a linearization about $x = -1$, and λ^s is the unique negative eigenvalue of the linearization about $x = 1$, i.e. λ_+^- obtained from a linearization about $x = 1$.

The formulas obtained by formally differentiating (4.69) also hold.

Proof. Again we only consider the case $\xi \to \infty$, as the proof(s) of the results for $\xi \to -\infty$ are similar.

Consider $y = 1 - P$. Let $\xi \in \mathbb{R}$. For any two $v, w \in \mathbb{R}^N$ we have

$$\begin{aligned} F(v,\rho) - F(w,\rho) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} F(tv + (1-t)w, \rho)\mathrm{d}t \\ &= \sum_{i=1}^N \Big(\int_0^1 \big(\frac{\partial}{\partial u_i} F\big)(tv + (1-t)w, \rho)\mathrm{d}t \Big)(v_i - w_i) \end{aligned} \tag{4.70}$$

Hence setting

$$A_i(\xi) := \int_0^1 \big(\frac{\partial}{\partial u_i} F\big)(t + (1-t)P(\xi + r_1), \cdots, t + (1-t)P(\xi + r_N))\mathrm{d}t \tag{4.71}$$

we conclude that y solves the linear equation

$$-cy^{(1)} = \sum_{i=1}^N A_i(\xi) y(\xi + r_i) \tag{4.72}$$

Without loss of generality we may assume $U(\rho) = \{2, \cdots, N\}$. As P satisfies the boundary conditions (4.2) we have $\forall 1 \le i \le N$:

$$\begin{aligned} A_{i,+} &= \lim_{\xi\to\infty} A_i(\xi) \\ &= \lim_{\xi\to\infty} \int_0^1 \big(\frac{\partial}{\partial u_i} F\big)(t + (1-t)P(\xi + r_1), \cdots, t + (1-t)P(\xi + r_N), \rho)\mathrm{d}t \\ &= \int_0^1 \big(\frac{\partial}{\partial u_i} F\big)(t + (1-t)\lim_{\xi\to\infty} P(\xi + r_1), \cdots, t + (1-t)\lim_{\xi\to\infty} P(\xi + r_N), \rho)\mathrm{d}t \\ &= \int_0^1 \big(\frac{\partial}{\partial u_i} F\big)(1, \cdots, 1, \rho)\mathrm{d}t \\ &= \big(\frac{\partial}{\partial u_i} F\big)(1, \cdots, 1, \rho) \end{aligned}$$

using dominated convergence and continuity. Hence for $\xi \to \infty$ the matrix coefficients are exactly those obtained from a linearization around the solution $x = 1$. Moreover (iv) implies that

$$A_{\Sigma,+} = \sum_{i=1}^N A_{i,+} = \Big(\sum_{i=1}^N \big(\frac{\partial}{\partial u_i} F\big)\Big)(1, \cdots, 1, \rho) = D_1\Phi(1, \rho) < 0 \tag{4.73}$$

. An application of Proposition 4.2.3 yields hyperbolicity of the linear equation (4.72) at ∞. Hence this equation is asymptotically hyperbolic at ∞. We now obtain $\exists a > 0 : y \in O(e^{-a\xi})$ for $\xi \to \infty$ from Proposition 3.1.1 with $h = 0$.

D_1F is locally Lipschitz by (i) so this function is Lipschitz continuous on compact sets. By Corollary 4.3.2 we already know that $-1 < P(\xi) < 1$. Hence $\{t + (1-t)P(\xi) | \xi \in \mathbb{R}, t \in [0,1]\} \subseteq \mathbb{R}$ is bounded and hence relatively compact. Thus, if L is the Lipschitz constant of D_1F on this set's closure, then $\forall \xi \in \mathbb{R}$:

$$\begin{aligned}
|A_i(\xi) - A_{i,+}| &= |\int_0^1 (\frac{\partial}{\partial u_i} F)(t + (1-t)P(\xi), \rho)\mathrm{d}t - (\frac{\partial}{\partial u_i} F)(1, \rho)| \\
&= |\int_0^1 (\frac{\partial}{\partial u_i} F)(t + (1-t)P(\xi), \rho) - (\frac{\partial}{\partial u_i} F)(1, \rho)\mathrm{d}t| \\
&\leq \int_0^1 |(\frac{\partial}{\partial u_i} F)(t + (1-t)P(\xi), \rho) - (\frac{\partial}{\partial u_i} F)(1, \rho)|\mathrm{d}t \\
&\leq \int_0^1 C|t + (1-t)P(\xi) - 1|\mathrm{d}t \qquad (4.74) \\
&= C \int_0^1 |1-t||1 - P(\xi)|\mathrm{d}t \\
&= C|y(\xi)| \int_0^1 1 - t\mathrm{d}t \\
&= \frac{C}{2}|y(\xi)| \in O(e^{-a\xi})
\end{aligned}$$

. Boundedness of the functions A_i follows from continuity of F and boundedness of arguments for F in (4.71). All that remains to be shown for Proposition 4.2.6 to be applicable are the lower bounds for $A_2, \cdots, A_N$. Fix $i \in \{2, \cdots, N\}$. We already know that $A_i(\xi) \to A_{i,+} > 0$ for $\xi \to \infty$. Hence $\exists \tau > 0 : \forall \xi \geq \tau : A_i(\xi) \geq \frac{A_{i,+}}{2}$. Moreover the continuous function A_i takes a minimum $\alpha_{i,0} > 0$ on the compact set $[0, \tau]$. Thus we have $\forall \xi \geq 0 : A_i(\xi) \geq \min\{\frac{A_{i,+}}{2}, \alpha_{i,0}\} > 0$.

Proposition 4.2.6 is now applicable for the case $\xi \to \infty$. As $P(\xi) < 1$ implies $y = 1 - P > 0$ and hence y does not vanish identically on any $[\tau, \infty)$ with $\tau \in \mathbb{R}$, Proposition 4.2.6 implies that $\lambda^s = \lambda_+^- > -\infty$ and $\exists C_+ > 0, \exists \epsilon > 0$ such that

$$1 - P(\xi) = y(\xi) = C_+ e^{\lambda^s \xi} + O(e^{(\lambda^s - \epsilon)\xi}) \qquad (4.75)$$

for $\xi \to \infty$. Moreover the formula obtained by formally differentiating 4.75 holds. Subtraction of 1 and multiplication with -1 in 4.75 finishes the proof. □

We know prove strict monotonicity of solutions joining the equilibria ± 1.

Proposition 4.3.4

Let $c \neq 0$, $\rho \in W$, and $P \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$ be a solution to (4.1) satisfying the boundary conditions (4.2).

Then $\forall \xi \in \mathbb{R} : P^{(1)}(\xi) > 0$.

Proof. By Theorem 4.3.3, i.e. from formally differentiating (4.69), we have $\tau_0 \in \mathbb{R}$: $\forall|\xi| \geq \tau_0 : P^{(1)}(\xi) > 0$. P takes a minimum $1 > P_- > -1$ and a maximum $1 > P_+ < 1$ in the compact interval $[-\tau_0, \tau_0]$. As P satisfies the boundary conditions there exists some $\tau_1 > \tau_0, \tau_2 > \tau_0$ such that $\forall \xi \geq \tau_1 : P(\xi) > P_+$ and $\forall \xi \leq -\tau_2 : P(\xi) < P_-$. Taking $\tau = \max\{\tau_0, \tau_1, \tau_2\}$ we obtain

$$\begin{aligned} \forall|\xi| \geq \tau : &\quad P^{(1)}(\xi) > 0 \\ \forall|\xi| < \tau : &\quad P(-\tau) < P(\xi) < P(\tau) \end{aligned} \tag{4.76}$$

. Hence we have $\forall k \geq 2\tau : \forall \xi \in \mathbb{R} : P(\xi + k) > P(\xi)$. Suppose now that $P^{(1)}(\xi) < 0$ for some $\xi \in \mathbb{R}$ and set

$$k_0 \coloneqq \inf\{k > 0 | \forall \xi \in \mathbb{R} : P(\xi + k) > P(\xi)\} \tag{4.77}$$

. As $P^{(1)}(\xi) < 0$ for some ξ we certainly have $k_0 > 0$. Also, $k_0 \leq 2\tau$ and $\forall \xi \in \mathbb{R}$: $P(\xi + k_0) \geq P(\xi)$. By definition, if $0 < k < k_0$ then $P(\xi_0 + k) \leq P(\xi_0)$ for some ξ, where necessarily $|\xi_0| \leq \tau$ and $|\xi_0 + k| \leq \tau$. We consider the continuous function $k \mapsto P(\xi_0 + k) - P(\xi_0)$ and hence $P(\xi_0 + k_0) - P(\xi_0) = \lim_{k \nearrow k_0} P(\xi_0 + k) - P(\xi_0) \leq 0$. Therefore $P(\xi_0 + k_0) = P(\xi_0)$. Both P and $\tau_{-k_0} P$ are solutions of (4.1) so Lemma 4.1.1 implies that $P(\xi + k_0) = P(\xi)$ either $\forall \xi \leq \xi_0$ or $\forall \xi \geq \xi_0$ which are both impossible. We conclude that $P^{(1)}(\xi) < 0$ is impossible.

The strict inequality follows from yet another application of Lemma 4.1.1, this time to the linearization about P, where we take the two solutions $x_1 = P^{(1)}$ and $x_2 = 0$, knowing that $P^{(1)}(\xi) > 0$ for large $|\xi|$. □

We know see that if $P \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$ satisfies Equation (4.1) for some $\rho \in W$ and $c \neq 0$, then P is strictly increasing and the operator $\Lambda_{c,L}$ associated to the linearization about P satisfies all the conditions of Theorem 4.2.8, exponential approach of the limiting equations following from exponential approach of the equilibria (see 4.3.3 and its proof), boundedness of the matrix coefficients A_i following directly from the properties of F and the exponential approach of the limiting coefficients, $A_{\Sigma\pm} < 0$ also directly following from properties of F and taking $p = P^{(1)}$ for some solutions P as the nontrivial, nonnegative, bounded solution.

The strict monotonicity of solutions joining the equlibria ± 1 and their smoothness allows us to seek a uniquely determined (for each particular solution) translate satisfying $x(0) = 0$. We shall therefore seek solutions in the subspace

$$W_0^{1,\infty} \coloneqq \{x \in W^{1,\infty} | x(0) = 0\} \tag{4.78}$$

.

At last we arrive at our final theorem/proposition, Proposition 6.4 from [3], and again refer the interested readers to the original papers for more results and references.

Proposition 4.3.5

Let

$$\mathfrak{M} \coloneqq \{(c, P, \rho) \in \mathbb{R} \setminus \{0\} \times W_0^{1,\infty} \times W | P \text{ solves (4.1) with (4.2)}\} \tag{4.79}$$

,

$$\begin{aligned} &\mathfrak{G} : \mathbb{R} \times W_0^{1,\infty} \times W \to L^\infty \\ &\big(\mathfrak{G}(c,P,\rho)\big)(\xi) = -cP^{(1)}(\xi) - F(P(\xi+r_1),\cdots,P(\xi+r_N),\rho) \end{aligned} \tag{4.80}$$

and $(c_0, P_0, \rho_0) \in \mathfrak{M}$.

Then the derivative of $\mathfrak{G}$ at this point with respect to the first to arguments, is an isomorphism from $\mathbb{R} \times W_0^{1,\infty}$ onto L^∞, i.e.

$$\begin{aligned} &D_{1,2}\mathfrak{G} : \mathbb{R} \times W_0^{1,\infty} \times W \to (L^\infty)^{\mathbb{R} \times W_0^{1,\infty}} \\ &\bigg(\Big(\big(D_{1,2}\mathfrak{G}\big)(c,P,\rho)\Big)(w,y)\bigg)(\xi) = -wP^{(1)}(\xi) + (\Lambda_{c,L}y)(\xi) \end{aligned} \tag{4.81}$$

where $(w, y) \in \mathbb{R} \times W_0^{1,\infty}$ and

$$A_i(\xi) = \big(\frac{\partial}{\partial u_i}F\big)(P(\xi+r_1),\cdots,P(\xi+r_N),\rho) \tag{4.82}$$

is the linear operator associated with the linearization at P and

$$\big(D_{1,2}\mathfrak{G}\big)(c_0,P_0,\rho_0) : \mathbb{R} \times W_0^{1,\infty} \to L^\infty \tag{4.83}$$

is an isomorphism.

Thus by the implicit function theorem, there exist for each ρ near ρ_0 a unique point $(c(\rho), P(\rho)) \in (\mathbb{R} \setminus \{0\}) \times W_0^{1,\infty}$ near (c_0, ρ_0), depending C^1 smoothly on ρ for which

$$\mathfrak{G}(c(\rho), P(\rho), \rho) = 0 \tag{4.84}$$

For each such ρ the solution $P(\rho)$ to (4.1) satisfies the boundary conditions (4.2), hence $(c(\rho), P(\rho), \rho) \in \mathfrak{M}$.

Proof. We first take note that G really is Fréchet C^1 with

$$\Big(\big(D\mathfrak{G}(c,P,\rho)\big)(d,Q,\sigma)\Big)(\xi) = -dP^{(1)}(\xi) + (\Lambda_{c,L}Q)(\xi) + (\frac{\partial}{\partial\rho}F)(\kappa(P,\xi),\sigma)$$

for $(c, P, \rho), (d, Q, \sigma) \in \mathbb{R} \times W_0^{1,\infty} \times W$. To prove this we first note that

$$\begin{aligned} &\|\Big(\mathfrak{G}(c+d,P+Q,\rho+\sigma) - \mathfrak{G}(c,P,\rho) - \big(D\mathfrak{G}(c,P,\rho)\big)(d,Q,\sigma)\Big)(\xi)\| \\ &= \|F(\kappa(P,\xi)+\kappa(Q,\xi),\rho+\sigma) - F(\kappa(P,\xi),\rho) - \sum_{i=1}^{N}\big(\frac{\partial}{\partial u_i}F\big)(\kappa(P,\xi),\rho)h(\xi+r_i) - \big(\frac{\partial}{\partial\rho}F\big)(\kappa(P,\xi),\sigma)\| \\ &\le \|F(\kappa(P,\xi)+\kappa(Q,\xi),\rho+\sigma) - F(\kappa(P,\xi),\rho+\sigma) - \big((D_1F)(\kappa(P,\xi),\rho+\sigma)\big)(\kappa(Q,\xi))\| \\ &+ \|F(\kappa(P,\xi),\rho+\sigma) - F(\kappa(P,\xi),\rho) - \big(\frac{\partial}{\partial\rho}F\big)(\kappa(P,\xi),\sigma)\| \end{aligned}$$

where

$$\kappa(P,\xi) := (P(\xi_1 + r_1), \cdots, P(\xi + r_N))$$

. Hence, by C^1ness of F, we have that

$$\lim_{(d,Q,\sigma)\to 0} \frac{\|\mathfrak{G}(c+d, P+Q, \rho+\sigma) - \mathfrak{G}(c,P,\rho) - \big(D\mathfrak{G}(c,P,\rho)\big)(d,Q,\sigma)\|_{L^\infty}}{\|(d,Q,\sigma)\|_{\mathbb{R}\times W_0^{1,\infty}\times W}} = 0$$

so $\mathfrak{G}$ is C^1 as $D\mathfrak{G}$ is clearly continuous in (c, P, ρ). Therefore we can apply the implicit function theorem as soon as we know that $D_{1,2}\mathfrak{G}$ is an isomorphism.

Consider the linearization of (4.1) about P_0, i.e. the linear equation (4.11) with A_i as in (4.82). Let $\Lambda_{c_0,L}$ denote the associated linear operator from $W^{1,\infty} \to L^\infty$. We see that this operator satisfies all the conditions of Theorem 4.2.8 as already mentioned in the paragraph before this theorem. In particular, $P_0^{(1)}$ solves 4.11 so Proposition 4.3.4 gives the nonnegative element $p = P_0^{(1)}$. Thus by theorem 4.2.8 the kernel $\mathfrak{K}_{c_0,L}$ of $\Lambda_{c_0,L}$ is precisely the one dimensional span of $P_0^{(1)}$.

The strict positivity $P_0^{(1)} > 0$ and particularily $P_0^{(1)}(0) > 0$ implies $P \notin W_0^{1,\infty}$. Hence the restriction of $\Lambda_{c_0,L}$ to $W^{1,\infty}$ is injective. Hence it is naturally an isomorphism from $W_0^{1,\infty}$ to its range $\mathfrak{R}_{c_0,L} \subseteq L^\infty$, which has codimension one by 4.2.8. 4.2.8 further provides existance of $0 < p^* \in \mathfrak{K}_{c_0,L^*}$, so

$$\int_{\mathbb{R}} p^*(\xi) P_0^{(1)}(\xi) \mathrm{d}\xi > 0 \tag{4.85}$$

Theorem 4.2.8 now implies $P_0^{(1)} \notin \mathfrak{R}_{c_0,L}$ by a Fredholm Alternative. We conclude from this, the formula of $D_{1,2}\mathfrak{G}$ and the fact that $\mathfrak{R}_{c_0,L}$ is complemented by the one dimensional span of $P_0^{(1)}$ that $D_{1,2}\mathfrak{G} : \mathbb{R}\times W_0^{1,\infty} \to L^\infty$ is an isomorphism. With this the implicit function theorem yields $c(\rho)$ and $P(\rho)$ satisfying (4.84).

It remains to be shown that this $P(\rho)$ satisfies the boundary conditions (4.2). As $P(\rho)$ varies continuously with ρ, so do $\mu_\pm(\rho) := \mu_\pm(P(\rho))$ and $\mu_{\pm,\pm\infty}(\rho) := \mu_{\pm,\pm\infty}(P(\rho))$. P_0 satisfying the boundary conditions (4.2) implies that $\mu_{\pm,\infty}(\rho_0) = 1$ and $\mu_{\pm,-\infty}(\rho_0) = -1$. By continuity and Lemma 4.3.1, i.e. Equation (4.62), we conclude that $\mu_{\pm,\infty}(\rho) = 1$ and $\mu_{\pm,-\infty}(\rho) = -1$ for any ρ near ρ_0 and hence have finished the proof. □

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