Error Measurement in Multiresolution Digital Elevation Models

Thomas Gerstner and Marc Hannappel
Division for Scientific Computing and Numerical Simulation
Department for Applied Mathematics
University of Bonn
Wegelerstr. 6, 53115 Bonn, Germany
e-mail: {gerstner,hannappe}@iam.uni-bonn.de

Keywords: DEM, generalization, resolution, scale

Abstract
Multiresolution models are getting more and more importance for a scale-sensitive modelling of spatial data. In principle, they allow the representation of the data at a variety of scales. Of special importance are adaptive methods, where the resolution does not need to be uniform but may be variable. Then, error indicators are used to control the local resolution of the model. In this paper, we show the construction and computation of such error indicators for multiresolution digital elevation models based on recursive triangle bisection for different applications.

1. Introduction
The amount of digital elevation data e.g. available from satellite measurements is increasing constantly. Consequently, many applications such as digital cartography, geographic informations systems, process modelling and simulation, or data visualization encounter their own specific problems related to scale when processing the data. On one hand, these problems may result from the sheer amount of data available requiring efficient data representation and compression. On the other hand, many important quantities, such as slope and curvature, are clearly scale-dependent and need to be handled carefully. Multiresolution models can address such problems.

We will here consider the construction of multiresolution digital elevation models for regular gridded data. The main ingredient of our method are triangulations generated by recursive bisection. These triangulations may be adaptive, if triangles are refined non-uniformly. Then, so-called error indicators control the local refinement since there will be an approximation error which occurs when a triangle is not refined. The focus of this paper will be on the selection and computation of such error indicators. Thereby, we will concentrate on a few specific application problems requiring geometric error control, inclusion of constraints, and preservation of topography. We will not consider measurement and positional uncertainties but assume that the input data is free of error.

The remainder of this paper is organized as follows. Section 2 shortly reviews the generation of multiresolution digital elevation models based on adaptive triangulations generated by recursive bisection. Section 3 will show the construction and usage of error indicators based on the wavelet expansion of the DEM. Different application problems and examples are then addressed in Section 4. Section 5 concludes with further remarks and applications of the used methodology.

2. Multiresolution Digital Elevation Models

A multiresolution digital elevation model (DEM) is a representation of a given input DEM from which is it possible to extract approximate DEMs with different levels-of-detail. These approximate models can be seen as representations of the input DEM on a coarser scale. Therefore, besides the resolution (e.g. mesh width), a second parameter ε is obtained which indicates the approximation error, e.g. the norm of the difference between the input and the approximation. Clearly, the norm will depend on the type of application. The parameter ε defines a scale dimension besides the coordinate directions *x* and *y*. Therefore, elevation can be seen as a function of three parameters, x, y, and ε. An overview of multiresolution digital elevation models and applications can be found in *(De Floriani et al. 1999, Dutton 1999).*

We will now shortly explain the construction of multiresolution models based on recursive bisection triangulations. These triangulations have been extensively used for adaptive grid refinement during the numerical solution of partial differential equations *(Rivara 1984)*. Basically the same approach has recently also been applied to the representation of terrain data *(Lindstrom et. al. 1996, Gerstner 1999)*. They are closely related to triangulations of the leaves of a restricted quadtree. However, recursive bisection triangulations are more flexible than quadtrees since they use twice the number of grid levels. Also, hierarchical triangulations have a greater potential for the multiresolution approximation of irregular distributed data points, although we will focus here on regular gridded data sets.

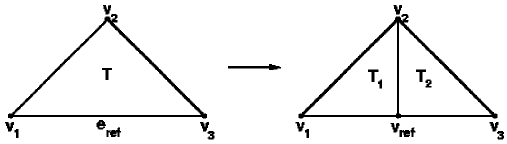

Figure 1: Bisection of a triangle.

The main idea is to start with a initial triangulation S^0 of level 0 and then to construct finer triangulations S^{l+1} recursively by splitting each triangle $T \in S^l$ in two. In the case of regular gridded data one can use isosceles triangles $T = (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3)$ with a right angle at $\mathbf{v}_2$. Then, by the selection of the midpoint of the longest edge $e_{\mathrm{ref}}(T) = (\mathbf{v}_1\mathbf{v}_3)$ as the refinement vertex $\mathbf{v}_{\mathrm{ref}}(T)$, two new triangles $T_1 = (\mathbf{v}_2\mathbf{v}_{\mathrm{ref}}\mathbf{v}_1)$ and $T_2 = (\mathbf{v}_3\mathbf{v}_{\mathrm{ref}}\mathbf{v}_2)$ are generated (Figure 1).

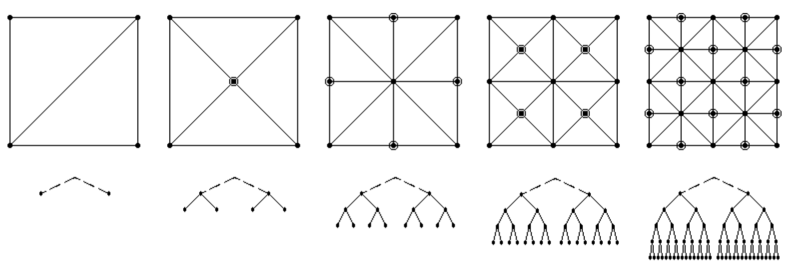

Figure 2: Hierarchical triangulation and corresponding binary trees.

Clearly, by this refinement procedure a binary tree hierarchy is inferred on the triangles. Note that in the interior of the domain, all refinement vertices will be shared by two triangles (Figure 2).

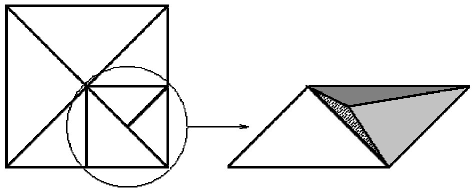

Figure 3: Hanging nodes will lead to cracks in the DEM.

Now, an adaptive triangulation can be defined by the selection of a subtree of the triangle binary tree. However, such triangulations can contain hanging nodes which occur if two triangles sharing a refinement vertex are not refined conformingly (Figure 3). Hanging nodes are undesirable because they can lead to cracks in the DEM since the surface defined by the triangulation is no more continuous. Cracks are undesirable, since they will result in holes when drawing the DEM, infer discontinuities in isolines, and generally cause problems in all algorithms assuming continuity of the surface.

One possibility to avoid hanging node is to ensure that, whenever a triangle is refined, the triangle sharing its refinement vertex is refined as well. This can be achieved by the definition of error indicators η on the refinement vertices, i.e. $\eta(T) = \eta(\mathbf{v}_{\text{ref}}(T))$, and by selection of all triangles where $\eta(T) > \varepsilon$ for some prescribed threshold ε. If the error indicator values fulfill the condition

$$\eta(T) \geq \max\{\eta(T_1), \eta(T_2)\}$$

for all triangles $T \in S^l$ with level $l < l_{\max}$, no hanging nodes can occur for all possible values of ε. If an error indicator η does not fulfil this condition it can easily be adjusted in a precomputing step. In a level-wise bottom-up traversal of the hierarchy it is possible to construct the minimal error indicator $\bar{\eta}$ larger than or equal to η by setting

$$\bar{\eta}(T) := \max\left\{\eta(T), \bar{\eta}(T_1), \bar{\eta}(T_2)\right\}.$$

This framework allows a great freedom for the choice of an initial error indicator η. This is necessary, since different applications will typically require different types of error indicators. Since $\bar{\eta}(T) \geq \eta(T)$, error bounds for η will also hold for $\bar{\eta}$.

3. Wavelet Representation

In order to be able to give concrete constructions for possible error indicators η, it is first necessary to define a surface model. Here we will use a piecewise linear interpolation inside each triangle which is uniquely defined by the elevation values at its vertices. Together with the refinement rule, we have thereby defined a hierarchical basis which is in fact a biorthogonal wavelet basis *(Cohen, Daubechies, Feauveau 1992)*. The wavelet-transformed DEM can be written as a continuous elevation function f,

$$f(\mathbf{x}) = \sum_{l=0}^{l_{\max}} \sum_{i=1}^{n_l} c_{li} \cdot \psi_{li}(\mathbf{x})$$

with wavelet basis functions $\psi_{li}(\mathbf{x})$ and wavelet coefficients c_{li}. The loop for the indices i runs over the n_0 vertices of the initial triangulation and over all n_l refinement vertices for level $l > 0$.

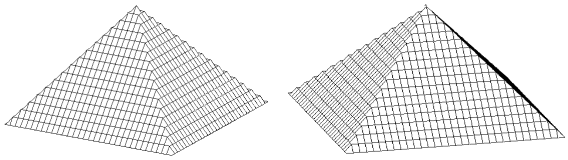

Figure 4: Pyramidal hat functions as wavelet basis.

If the DEM contains no hanging nodes, the wavelet functions are scaled and translated versions of two mother wavelets (Figure 4). The left basis function has the shape of a four-sided pyramid and is used in odd levels *l*. The right basis function is used in even levels and is a scaled by $\sqrt{2}$ and rotated by 45 degrees version of the left one.

The wavelet coefficients c_{li} can be computed by linear combination of the elevation values e on the refinement edge corresponding to the index i,

$$c_{li} = -\frac{1}{2}e(\mathbf{v}_1) + e(\mathbf{v}_{\mathrm{ref}}) - \frac{1}{2}e(\mathbf{v}_3).$$

An approximate DEM f_ε corresponding to an adaptive triangulation constructed as shown in the previous section is defined by setting all coefficients c_{li} where $\bar{\eta}(\mathbf{v}_{\mathrm{ref}}) < \varepsilon$ to zero, that is

$$f_\varepsilon(\mathbf{x}) = \sum_{l,i:\,\bar{\eta}(\mathbf{v}_{\mathrm{ref}}) \geq \varepsilon} c_{li} \cdot \psi_{li}(\mathbf{x}).$$

4. Error Measurement and Examples

With the (piecewise linear) surface model of the previous section in mind, we can now take a closer look at possible methods of error measurement. Of course, there is a great variety of methods that can be used. We will here restrict ourselves to geometric error measurement but also consider inclusion of constraints and preservation of topography.

4.1. Geometric Error Measurement

Let us at first assume, that there is a norm $\|\cdot\|$ measuring the geometric approximation error. There is a large variety of norms that are commonly used for the computation of errors. Popular examples are integral norms such as the Lebesgue norms L_p. For $p = \infty$ this norm corresponds to the maximum vertical distance of the appoximation and the original, for $p = 1$ to the integral of the absolute difference DEM. Other geometric norms can involve derivatives (such as Sobolev norms), discrete curvature computations, or Haussdorff measures.

Let us recall, that the difference between the original DEM and the approximation is defined by $f_\varepsilon(\mathbf{x}) - f(\mathbf{x})$. Therefore, the global norm of the approximation error can be computed, respectively estimated by

$$\| f_\varepsilon(\mathbf{x}) - f(\mathbf{x}) \| = \left\| \sum_{l,i:\eta(\mathbf{v}_{\mathrm{ref}})<\varepsilon} c_{li} \cdot \psi_{li}(\mathbf{x}) \right\| \leq \sum_{l,i:\eta(\mathbf{v}_{\mathrm{ref}})<\varepsilon} |c_{li}| \cdot \| \psi_{li}(\mathbf{x}) \|.$$

The error indicator η reflects the local application of this norm, i.e. the restriction of the norm to the quadrilateral formed by a pair of triangles having a common refinement vertex. Remember that error indicator values are defined on the vertices of the hierarchical triangulation. Therefore, η should measure the error on both triangles sharing the refinement vertex. For an overview of error estimation techniques see *(Verführt 1996).*

As an example we consider a part of the global *gtopo30* data set of the US Geological Survey showing the North Sea and its surroundings. We show approximations based on the L_1-norm (Figure 6). It is clearly visible that in rough, mountainous areas such as in Norway and Scotland more triangles have to be used to keep the global error bounded while in relatively flat areas such as in northern Germany and the Netherlands, larger triangles can be used.

4.2. Constraints

Some measures of importance are not easily defined using a geometric norm such as in the previous example. For instance, one may want to focus on a few areas of greater interest. Such measures of importance are typically modelled using constraints. It is very easy to include constraints into our multiresolution digital elevation model while preserving the continuity of the surface. This is simply done by increasing error indicator values in selected areas.

Let us consider the North Sea example again. Using only the geometric norm, coastlines are somehow not resolved satisfactory since a coarse approximation of flat coasts does not lead to a large error. Depending on the application, however, coastlines may be important. Since ocean areas are marked in the data set, constraints on coastlines can be imposed easily by the multiplication of corresponding error indicator values with a constant factor (Figure 7).

4.3. Preservation of Topography

Especially in hydrological modelling and simulation, changes in the topography of the DEM may yield surprising and unwanted effects. Small changes in elevation values can lead to large changes in catchment size and structure.

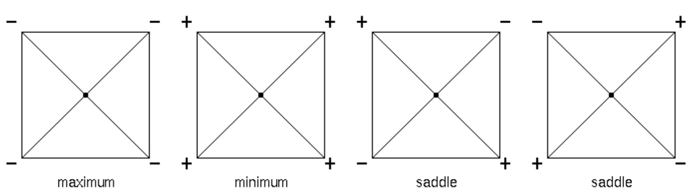

Figure 5: The four topological cases where critical points can arise.

Topography can be defined by the set of critical points. A critical point is defined as a point in space, where an isoline changes its topology. Since our data model is piecewise linear, critical points can arise only at vertices of the triangulation. Multiresolution DEMs, however, require special care. A point in space which is not critical on the finest resolution may become critical on a coarser resolution. Therefore, critical points have to be defined hierarchically.

Whenever a pair of triangles is refined, the common refinement which is inserted in the DEM is a candidate for a critical point on the current level. The four possible cases where the refinement vertex is really critical are depicted in (Figure 5). A ´-´ indicates, that the elevation value at this vertex is smaller than the elevation value of the refinement vertex, a ´+´ that it is larger. By setting error indicator values η at critical points to ∞, any approximate DEM will have the same topographical structure as the input DEM for all values of ε.

As an example, we consider a DEM (courtesy of LVerA Rheinland-Pfalz) in western Germany in the vicinity of a lake (Laacher See). In (Figure 8) we show isolines and the corresponding adaptive triangulations for the L_∞-norm without topography preservation. We see that for the coarse triangulation the lake in the foreground will get an opening at its upper border. With topography preservation, all isolines will retain their structure and such unwanted effects are eliminated (Figure 9).

5. Concluding Remarks

In this paper we have considered the construction of continuous multiresolution digital elevation models based on adaptive hierarchical triangulations. We have adressed the computation of error indicators using the corresponding wavelet expansion of the DEM. We have also shown how the construction can be extended to handle constraints as well as preservation of topography.

Hierarchical triangulations have already been successfully used for the handling of large-scale data in spatial data bases, data compression and interactive visualization. Further applications of the methodology are the computation of scale-dependent quantities such as slope and curvature, as well as analysis and fractal classification of landform. In the future, multiresolution DEMs will also be used for scale-sensitive process modelling and simulation.

Acknowledgements

This research has been supported by SFB 350 ´Interactions between and modelling of continental geosystems´ and graduate college ´Landform – a structured and variable boundary layer´ of the German Research Foundation (DFG).

References

- *Cohen, A., Daubechies, I., Feauveau, J., 1992,* Bi-orthogonal bases of compactly supported wavelets, Comm. Pure Appl. Math 45.
- *De Floriani, L., Puppo, E., Magillo, P., 1999,* Geometric Structures and Algorithms for Geographic Information Systems, Chapter 7 in Handbook of Computational Geometry, J.R. Sack, J. Urrutia (eds.), Elsevier Science.
- *Dutton, G.H., 1999,* A Hierarchical Coordinate System for Geoprocessing and Cartography, Lecture Notes in Earth Science 79, Springer Verlag.
- *Gerstner, T., 1999,* Adaptive Hierarchical Methods for Landscape Representation and Analysis, in: Process Modelling and Landform Evolution, S. Hergarten, H.-J. Neugebauer (eds.), Lecture Notes in Earth Science 78, Springer Verlag.
- *Lindstrom, P., Koller, D., Ribarsky, W., Hodges, L.F., Faust, N., Turner, G., 1996,* Real-Time, Continuous Level of Detail Rendering of Height Fields, Computer Graphics (Proc. SIGGRAPH ´96).
- *Rivara, M.C., 1984,* Algorithms for Refining Triangular Grids suitable for Adaptive and Multigrid Techniques, International Journal for Numerical Methods in Engineering 20.
- *Verführt, R., 1996,* A Review of a Posteriori Error Estimation and Adaptive Mesh-refinement Techniques, Teubner Verlag.

Figure 6: Adaptive triangulations and corresponding illumination shaded DEMs of the North Sea area based on the L_1-norm without focus on coastlines.

Figure 7: Adaptive triangulatiosn and corresponding illumination shaded DEMs of the North Sea area based on the L_1-norm with focus on coastlines.

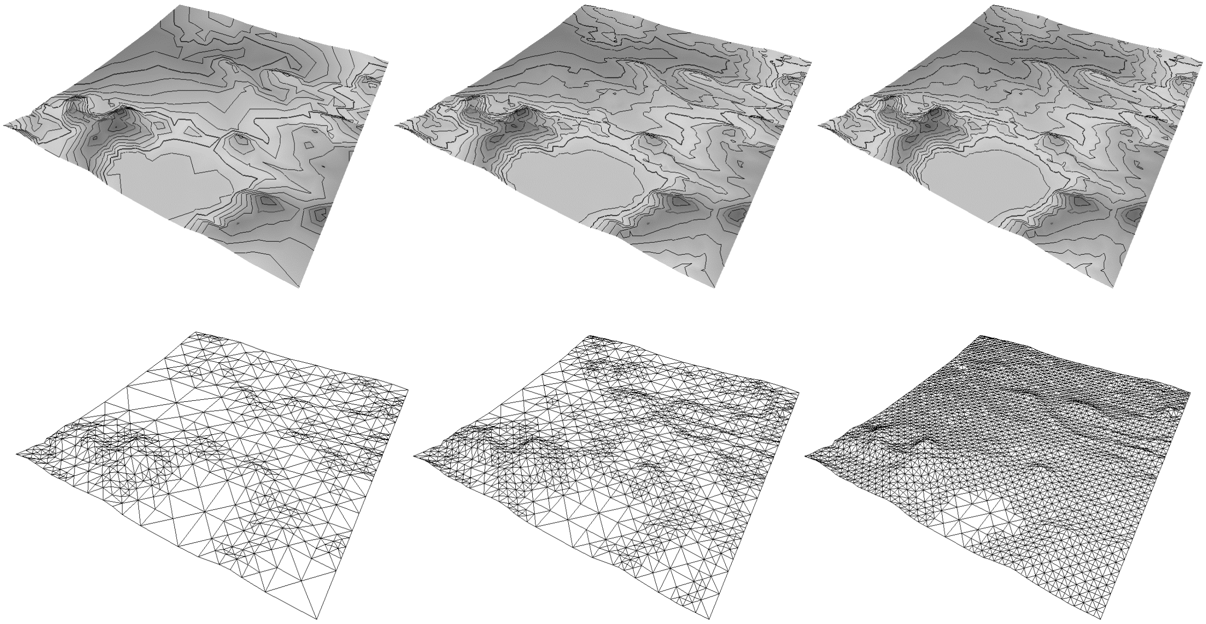

Figure 8: Adaptive triangulations and correspoonding hypsoshaded DEMs with isolines of the Laacher See area based on the L_∞-norm without topography preservation.

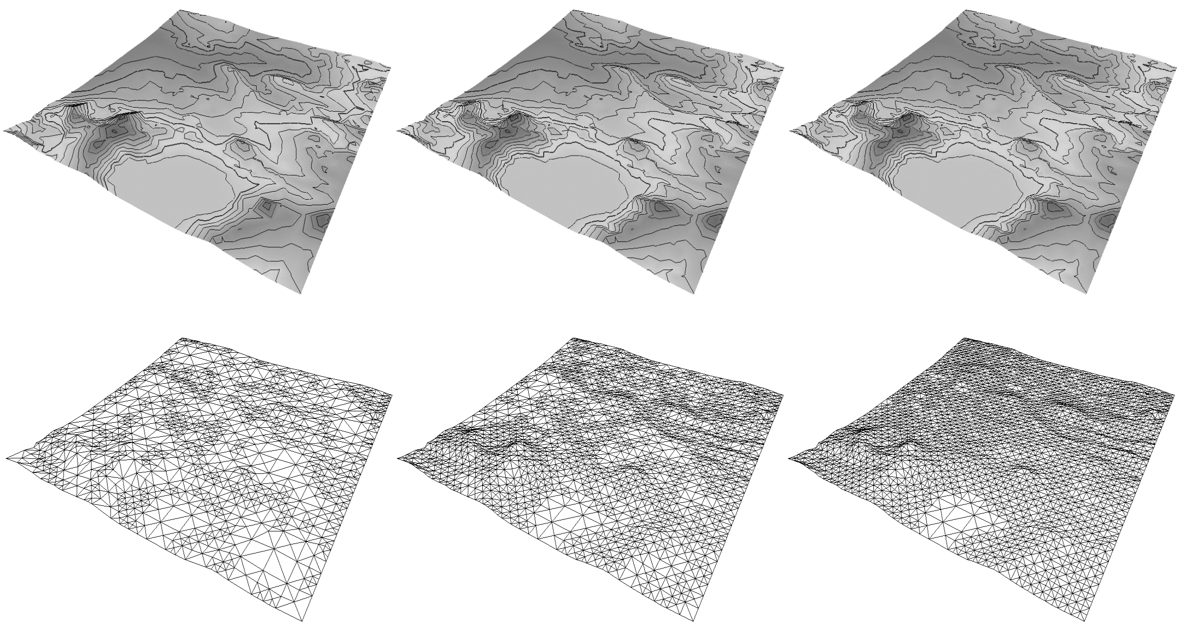

Figure 9: : Adaptive triangulations and corresponding hypsoshaded DEMs with isolines of the Laacher See area based on the L_∞-norm with topography preservation.