# The Computation of American Option Price Sensitivities using a Monotone Multigrid Method for Higher Order B–Spline Discretizations

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**Abstract**

In this paper a fast solver for discrete free boundary value problems which is based on hierarchical higher order discretizations is presented. The numerical method consists of a finite element discretization with B–spline ansatz functions of arbitrary degree combined with a monotone multigrid method for the efficient solution of the resulting discrete system. In particular, the potential of the scheme in the fast and accurate computation of American style option prices in the Black–Scholes framework and of their derivatives with respect to the underlying is investigated. Due to the higher order discretization, the derivatives, also called Greek letters, can be stably and accurately determined via direct differentiation of the basis functions. Considering the valuation of plain vanilla American stock options, we show that our solution method is competitive to the best schemes proposed in the literature when accurate approximations to the derivatives are required. It provides the first multigrid approach based on higher order basis functions which is directly applicable to American option pricing.

**Keywords:** American options, Greek letters, free boundary value problems, linear complementary problems, finite elements, cardinal higher order B–splines, monotone multigrid methods
**AMS-Classification:** 65M55, 65N30, 35J85 65D07, 65D99, 90C33

## 1 Introduction

The overwhelming majority of all traded options are of American–style. However, no general closed–form solution for their valuation is known. Therefore, one has to resort to analytical approximations or to numerical pricing methods. Here, one has to keep in mind that often not only fair option prices are required, but also accurate approximations to the derivatives of the option price, e.g., with respect to time or to the underlying. The derivatives, also called *Greek letters* or *Greeks*, play a crucial role as hedge parameters in the analysis of market risks. They can – in contrast to option prices – not directly be observed in the market. This fact is further increasing the demand for numerical schemes for their approximation. Usually, the Greek letters are computed by numerical

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differentiation of option values. Here, the option values must be approximated up to high accuracy in order to obtain stable and reliable results.

In the Black–Scholes framework [BS], the valuation of American options requires the solution of a particular free boundary value problem. A finite difference or finite element discretization leads to a discrete linear complementary problem which can also be regarded as an obstacle problem. Since the work of [BC], it is known that the most efficient solvers for this kind of problems are multigrid techniques. However, due to inconsistent approximations of the free boundary on coarser grids, the scheme from [BC], called PFAS is sometimes lacking stability. Moreover, no convergence proof is known.

This disadvantage could be resolved by the work of Kornhuber [Ko1] with the introduction of *monotone multigrid methods (MMG)*. In the last decade, they have been applied in [Ko1, Ko2, Kr] with great success to problems from continuum mechanics. Their key ingredients are sophisticated restriction operators for obstacle functions which arise from a consistent handling of the free boundary, and a special truncation of the basis functions. Unfortunately, the method is restricted to continuous piecewise linear functions, also called hat functions. To our knowledge, this also applies to all other multigrid approaches (e.g., [BC, HM, Ho, Ko1, Ko2, Kr, Ma, Oo, Tai]) for linear complementary problems proposed in the literature. Taking derivatives of approximations obtained by hat functions, however, will lead to unstable and misleading results.

The main difficulty in generalizing monotone multigrid methods to higher order basis functions is a suitable handling of the obstacle condition of the problem. Via the use of a B–spline basis, this difficulty could be recently resolved in a companion paper [HK], which led to the first generalization of the monotone multigrid method to arbitrary smooth basis functions. Using arguments of [Ko1], global convergence and optimal complexity of the *B–spline–based monotone multigrid method* could be proved. By construction, one can expect special robustness of the scheme and full multigrid efficiency in the asymptotic range. Moreover, the use of smooth basis functions admits a direct determination of the derivatives of the solution.

In this paper, we apply the B–spline–based monotone multigrid method in order to compute American option prices as well as their derivatives with respect to the underlying up to high accuracy and investigate the performance of the scheme by numerical examples.

The paper is structured as follows. In Section 2, we apply a finite difference discretization in time and a higher order B–spline-based finite element discretization in space to the free boundary problem which describes the fair price of an American option in the Black–Scholes framework. In Section 3, we generalize the projective Gauss–Seidel scheme to higher order basis functions and illustrate the monotone multigrid method from [Ko1] as a fast solver for the discrete form. Finally, in Section 4, the potential of the new scheme for the approximation of plain vanilla American option price sensitivities is demonstrated by numerical experiments. We show that our scheme is competitive to the best schemes proposed the literature when accurate approximations to the derivatives of the option value are required. In Section 5, we indicate some possible extensions of our scheme to higher dimensional option pricing problems, to adaptive grid refinements and to higher order time discretizations.

# 2 The Computation of American Price Sensitivities via a Higher Order Finite Elements Method

In this section, the free boundary value problem which describes the fair price of an American option is introduced. A weak form is derived and discretized with finite differences in time and finite elements in space. This leads to a discrete linear complementary problem which is solved by a projective variant of the Gauss–Seidel relaxation. By using sufficiently smooth basis functions, the derivatives of the option value can be determined by direct differentiation of the finite element ansatz functions.

## 2.1 The Pricing of American Options

Options are financial contracts that give its owner the right, but not the obligation, to buy (call option) respectively to sell (put option) an underlying security (e.g. an asset) at specified times in the future for an agreed strike price $K$. *European* options can only be exercised at the maturity date $T$, whereas *American* options can be exercised at any time at or before the maturity date. For a plain vanilla put option we have at time $t = T$ a payoff of

$$\mathcal{H}(S,T) = \max\{0, K - S(T)\}. \tag{1}$$

The enormous success and systematic trade of financial derivatives in the recent years led to the need for fast and accurate pricing techniques. As options are getting more and more complex and their valuation more and more complicated, sophisticated computational methods have been developed and continue to be an active field of research. For traders not only the value $V = V(S,t)$ of an option at time $t < T$ is important, but also the derivatives, e.g., with respect to time, to the underlying or to volatility, as they play a crucial role in the analysis of market risks. In this paper we restrict ourself to the space derivatives

$$\text{Delta} := \frac{\partial V}{\partial S} \qquad \text{and} \qquad \text{Gamma} := \frac{\partial^2 V}{\partial S^2}.$$

In the famous and well–established Black–Scholes model [BS] the stochastic process generating the price $S(t)$ of the underlying asset is modeled as a geometric Brownian motion

$$dS(t) = \mu\, S(t)\, dt + \sigma\, S(t)\, dB(t).$$

As usual, $B(t)$ denotes a one–dimensional standard Brownian motion. The parameters $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ are the drift and the volatility of the stochastic process $S(t)$. Under additional assumptions on the financial market and the central assumption of absence of arbitrage, it is well–known that the fair price of a European option satisfies the linear, instationary, backward partial differential equation

$$\mathcal{L}\,V := \frac{\partial}{\partial t}V + \frac{\sigma^2}{2}S^2\frac{\partial^2}{\partial S^2}V + rS\frac{\partial}{\partial S}V - rV = 0, \tag{2}$$

where $r \in \mathbb{R}^+$ denotes the interest rate for a riskless investment. Taking suitable boundary conditions into account, an analytic solution of the *Black–Scholes equation* (2) can be derived and is given by the famous *Black–Scholes formula* [BS].

As American options can – in contrast to European ones – be exercised at all times $t \leq T$, their valuation is more involved. For all $t \leq T$ not only the option value must be determined but also whether or not the option should be exercised. In the Black–Scholes framework, the fair value of an American option can be determined by the solution of a *free boundary value problem* of the Black–Scholes equation (2) in the domain $\mathbb{R}^+ \times [0, T)$ (cf. [WHD]). We formulate it for the case of an American put option, i.e., for the payoff $\mathcal{H}(S, t)$ as in (1).

**Problem 2.1 (Free Boundary Problem)** *Find $V = V(S, t)$ and $S_f = S_f(t)$, such that*

$$\begin{aligned} \mathcal{L}V(S,t) &= 0 \qquad && \text{for } \; S > S_f \;\; \text{and} \;\; 0 \leq t < T, \\ V(S,t) &= \mathcal{H}(S,t) \qquad && \text{for } \; S \leq S_f \;\; \text{and} \;\; 0 \leq t < T, \end{aligned}$$

*with the boundary data*

$$V(S,t) = 0 \qquad \text{for } \; S \to \infty \;\; \text{and} \;\; 0 \leq t < T,$$

*the final data*

$$V(S,T) = \mathcal{H}(S,T) \qquad \text{for} \quad S \geq 0$$

*and the conditions, that $V$ and $\partial V/\partial S$ are continuous on the free boundary $S_f$.*

Note that the boundary condition $V(S,t) = K$ for $S \to 0$ is already implied by the condition $V(S,t) = \mathcal{H}(S,t)$ for $S \leq S_f$. The free boundary $S_f$, which is part of the solution, separates the part of the domain where it is optimal to exercise the option immediately from the part where it is favorable to keep the option. It therefore describes the optimal exercise price of the option. Note that the right of early exercise leads to the condition $V(S,t) \geq \mathcal{H}(S,t)$ for all $S > 0$ and $0 \leq t \leq T$ and can therefore also be regarded as an lower obstacle condition to the American option price $V$.
Despite much effort, a closed–form solution of Problem 2.1 is not yet known except for the infinite horizon case [Mc]. Thus, the derivation of analytical approximations or numerical methods for the valuation of American options is still an active field of research.
Due to their flexibility and simplicity the most common approaches used by financial institutions are binomial methods introduced by [CRR]. There are many extensions and improvements on this approach (e.g., [Bn, By, FG, BG2, LR]). Analytical approximations include the works [Mi, BW, BjS, Ro, Ge, Wh, GJ]. Other approaches include Monte Carlo simulation (see [BG3] for a review), the method of lines [Me, CF], penalty methods [FV] and techniques from linear optimization theory [DH]. A discretization of the free boundary problem by finite difference methods was initially proposed by [BrS] and is also considered in [GS, HW, TR]. Finite element methods were used by [WHD, FVZ, ZS, ZFV]. Note that finite element or finite difference approaches provide an approximation to the whole surface, which is defined by all option values $V(S,t)$ for $S > 0$ and $0 \leq t \leq T$, whereas most of the other methods approximate just a certain single value $V(S_0, 0)$. The computation of derivatives of American option prices is explicitly considered in [C, BG1, PV, R, WW]. In [WW], an analytic approximation [BW], a finite difference method with hat functions [BrS] and a binomial tree in the variant of [LR], called *Leisen–Reimer trees*, are compared with regard to their performance in approximating American option price sensitivities. The author concludes that Leisen–Reimer trees are the superior method.

One can generalize the standard Black–Scholes model by the assumption of stochastic volatility (cf., e.g., [BR]). This way, a free boundary value problem of the above form is obtained, but in two space dimensions. Solutions via finite difference discretization combined with the PFAS multigrid scheme from [BC] are discussed in [CP, Oo]. Since the work shows the need for higher order Black–Scholes solvers, we also mention the asymptotic model [FPS] to price American options with stochastic volatility, which is based on accurate approximations to the third derivative $\partial^3 V/\partial S^3$ of the solution $V$ of an one–dimensional Black–Scholes model.

### 2.2 Transformation and Weak Formulation

The starting point of finite element methods is a weak formulation of Problem 2.1. The first step in the derivation, which can be found in more detail in [Hz, WHD], is a reformulation into a linear complementary problem such that the free boundary does not show up explicitly anymore. Then, the nonlinear transformations

$$S = Ke^x, \quad t = T - \frac{2\,\tau}{\sigma^2}, \quad V(S,t) = K\, e^{-\frac{x}{2}(q-1)-\left(\frac{1}{4}(q-1)^2+q\right)\tau}\, y(x,\tau), \tag{3}$$

with $q := 2r/\sigma^2$, are used to transform the Black–Scholes equation (2) into the parabolic heat equation

$$\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} = 0 \tag{4}$$

in the new variable $y(x,\tau)$ in order to avoid numerical complications with the treatment of the convective term and to simplify the implementation. The transformation of the payoff function $\mathcal{H}(S,t)$ gives the transformed payoff function

$$g(x,\tau) := e^{\frac{1}{4}(q+1)^2\tau} \max\{e^{\frac{x}{2}(q-1)} - e^{\frac{x}{2}(q+1)}, 0\}. \tag{5}$$

The transformation of the boundary data leads to

$$\lim_{x\to\pm\infty} y(x,\tau) = \lim_{x\to\pm\infty} g(x,\tau), \quad y(x,0) = g(x,0). \tag{6}$$

To simplify the treatment of the boundary conditions, we reduce the problem (in contrast to [WHD]) to homogeneous boundary data by substituting

$$u(x,\tau) := y(x,\tau) - g(x,\tau). \tag{7}$$

In the next step, the unbounded transformed domain $\mathbb{R} \times (0, \sigma^2 T/2]$ is substituted by the computational domain

$$\Omega := I_x \times I_\tau := [x_{\min}, x_{\max}] \times (0, \sigma^2 T/2] \subset \mathbb{R}^2$$

with fixed values $x_{\min} < 0 < x_{\max}$. In [JLL], it is proved that the resulting localization error decreases uniformly for increasing $\Omega$.
Finally, a weak formulation can be derived by multiplying the transformed problem by a test function and integrating by parts. The obstacle condition as well as the homogeneous

boundary and initial conditions are integrated in the definition of the set of admissible test functions

$$\mathcal{K} := \{v \in H(\Omega) : \, v(x,\tau) \geq 0, \, v(x_{\min},\tau) = v(x_{\max},\tau) = 0, \, v(x,0) = 0 \text{ for all } (x,\tau) \in \Omega\},$$

where $H(\Omega)$ denotes the Sobolev space of all functions $v = v(x,\tau) \in L^2(\Omega)$, which are weakly differentiable with respect to $x$ and strongly differentiable with respect to $\tau$. The transformed solution $u(x,\tau)$ can now be determined by the solution of the following parabolic variational inequality, where $\langle \cdot, \cdot \rangle$ denotes the $L^2$– inner product in $I_x$.

**Problem 2.2 (Weak Formulation)** *Find $u \in \mathcal{K}$, such that*

$$\langle \frac{\partial u}{\partial \tau}, v - u \rangle + a(u, v-u) \geq f(v-u) \quad \textit{for all } v \in \mathcal{K},$$

*where $a : \, H(\Omega) \times H(\Omega) \to \mathbb{R}$ and $f : \, H(\Omega) \to \mathbb{R}$ are defined by*

$$a(u,v) := \int_{x_{min}}^{x_{max}} \frac{\partial}{\partial x} u \frac{\partial}{\partial x} v \, dx, \qquad f(v) := -\int_{x_{min}}^{x_{max}} \frac{\partial g}{\partial \tau} v + \frac{\partial g}{\partial x} \frac{\partial v}{\partial x} dx.$$

Since the bilinear form $a(\cdot,\cdot)$ is symmetric and positive definite, Problem 2.2 admits a unique solution $u \in \mathcal{K}$ (cf. [EO]). From Problem 2.2 the American option value $V$ can finally be derived by a back transformation of $u$ via (7) and (3).

## 2.3 Discretization

We discretize Problem 2.2 by finite differences in time and finite elements in space. Due to the demand of accurate approximations to the derivatives we use a B–spline basis of arbitrarily order $k$ for the space discretization. As we show in subsection 2.3.2, this basis also has advantages in the handling of the obstacle condition.

### 2.3.1 Time discretization

Let $\tau_m := m \, \Delta\tau$, $m = 0, \ldots, M$, $M \in \mathbb{N}$ be an equidistant discretization of time interval $I_\tau$ with step size $\Delta\tau := \frac{1}{2}\sigma^2 T/M$. Let further $H_0^1(I_x)$ denote the Sobolev space of functions with zero trace on the boundary and define $u^m := u(\cdot, \tau_m) \in H_0^1(I_x)$. A finite difference discretization of the time derivative in Problem 2.2 by a $\theta$–scheme, which interpolates between an explicit ($\theta = 0$) and an implicit ($\theta = 1$) representation, leads to an elliptic variational inequality in each time step.

**Problem 2.3 (Semi–discrete Form)** *Find $u^{m+1} \in \mathcal{K}$, such that*

$$a_{\Delta\tau}(u^{m+1}, v - u^{m+1}) \geq f^m(v - u^{m+1}) \quad \textit{for all } v \in \mathcal{K}$$

*where*

$$\mathcal{K} := \left\{ v \in H_0^1(I_x) : \, v(x) \geq 0 \textit{ for all } x \in I_x \right\}$$

*and $a_{\Delta_\tau} : \, H_0^1(I_x) \times H_0^1(I_x) \to \mathbb{R}$ and $f^m : \, H_0^1(I_x) \to \mathbb{R}$ are given by*

$$a_{\Delta\tau}(u,v) := \int_{x_{min}}^{x_{max}} u \, v + \theta \Delta\tau \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx, \quad f^m(v) := \Delta\tau f(v) + \int_{x_{min}}^{x_{max}} u^m \, v + \left[ (\theta - 1) \Delta\tau \frac{\partial u^m}{\partial x} \right] \frac{\partial v}{\partial x} \, dx.$$

Note that $f^m$ depends on the solution $u^m \in H_0^1(I_x)$ of the previous time step. Specific for $\theta = \frac{1}{2}$ the Crank–Nicholson scheme is obtained. In [BHR] the regularity $u^{m+1} \in H^{5/2-\epsilon}$ is shown for arbitrary $\epsilon > 0$.

### 2.3.2 A B–spline–based Finite Element Discretization in Space

Now, a higher order finite element discretization is applied to Problem 2.3, which is based on cardinal B–spline functions. Let $x_i := x_{\min} + i\,h,\ i = 0, \dots, N-1,\ N \in \mathbb{N}$, be an equidistant discretization of the space interval $I_x := [x_{\min}, x_{\max}]$ with step size $h := (x_{\max} - x_{\min})/(N-1)$ and let $S_h \subset H_0^1(I_x)$ be a finite dimensional space of piecewise polynomials. If piecewise linear functions $v_h \in S_h$ are used for the space discretization, the side condition

$$v_h(x) \geq 0 \quad \text{for all } \ x \in I_x \tag{8}$$

from Problem 2.3 is obviously satisfied if the set of pointwise inequalities

$$v_h(x_i) \geq 0 \quad \text{for all } \ i = 0, \dots, N-1 \tag{9}$$

hold. Trying to generalize this idea to piecewise functions $v_h$ of higher degree, one is confronted with the problem that for given $x \in [x_i, x_{i+1}]$ the estimate

$$\min\{v_h(x_i), v_h(x_{i+1})\} \leq v_h(x) \leq \max\{v_h(x_i), v_h(x_{i+1})\} \tag{10}$$

is not valid anymore. This shows that controlling function values on grid points does not suffice to ensure a side condition of the form (8) in the higher order case and explains why higher order nodal Lagrange basis functions are not suited for the problem at issue.
Instead, we propose here a construction using B–splines as higher order basis functions, which compares B–spline expansion coefficients instead of function values, and heavily profits from the fact that B–splines are nonnegative. More information on finite element methods for boundary value problems with B–splines can, e.g., be found in [Hg]. There, a modification of the B–spline basis, which leads to so called web–splines, is used to handle general domains and general boundary conditions. Due to the homogeneous boundary conditions (the solution is even zero in a neighborhood of the boundary as shown in [Hz]) and due to the rectangular domain, such a modification, however, is not necessary for the problem under consideration. For readers' convenience, we recall the relevant facts about B–splines from [Bo].

**Definition 2.4 (B–splines)** *For $k \in \mathbb{N}$ and $n = N + k - 1$ let $\Delta_h := \{\theta_i\}_{i=1,\dots,n+k}$ be an equidistant* expanded knot sequence *in the interval $I_x$ with grid spacing $h$ of the form*

$$\theta_1 = \dots = \theta_k = x_{min} < \theta_{k+1} < \dots < \theta_n < x_{max} = \theta_{n+1} = \dots = \theta_{n+k} \tag{11}$$

*with $\theta_{k+i} = x_i$. Then the* B–splines *$N_{i,k,\Delta_h}$ of order $k$ are recursively defined by*

$$\begin{aligned} N_{i,1,\Delta_h}(x) &= \begin{cases} 1, & \text{if } x \in [\theta_i, \theta_{i+1}) \\ 0, & \text{else} \end{cases} \\ N_{i,k,\Delta_h}(x) &= \frac{x - \theta_i}{(k-1)\,h} N_{i,k-1,\Delta_h}(x) + \frac{\theta_{i+k} - x}{(k-1)\,h} N_{i+1,k-1,\Delta_h}(x). \end{aligned} \tag{12}$$

*for $i = 1, \dots, n$.*

We also use the abbreviation $N_{i,k} = N_{i,k,\Delta_h}$, when only one grid is considered. It is known that B–splines have the properties $\operatorname{supp} N_{i,k} \subseteq [\theta_i, \theta_{i+k}]$ (local support), $N_{i,k}(x) \geq 0$ for all $x \in I_x$ (nonnegativity) and $N_{i,k} \in C^{k-2}(I_x)$ (differentiability). Moreover, the set $\Sigma_h := \{N_{1,k}, \ldots, N_{n,k}\}$ constitutes a locally independent and unconditionally stable basis with respect to to $\|\cdot\|_{L_p}$, $1 \leq p \leq \infty$, for the finite dimensional space $\mathcal{N}_{k,\Delta_h} := \operatorname{span} \Sigma_h$ of the splines of order $k$.

**Lemma 2.5** *If the B–spline coefficients $v_i, g_i$ of two B–spline functions $v_h, g_h \in \mathcal{N}_{k,\Delta_h}$ satisfy $v_i \geq g_i$ for all $i = 1, \ldots, n$, then $v_h(x) \geq g_h(x)$ holds for all $x \in I_x$.*

*Proof:* Using the representation

$$v_h = \sum_{i=1}^{n} v_i \, N_{i,k}, \qquad g_h = \sum_{i=1}^{n} g_i \, N_{i,k}$$

and the nonnegativity $N_{i,k}(x) \geq 0$ for all $x \in I_x$, we deduce

$$v_h(x) - g_h(x) = \sum_{i=1}^{n} (v_i - g_i) \, N_{i,k}(x) \geq 0 \qquad \text{for all } x \in I_x.$$

□

Here and below, we use the subscript $i$ in $v_i = (v_h)_i$ to denote B–spline expansion coefficients and boldface letters $\mathbf{v}$ to denote B–spline coefficient vectors $\mathbf{v} := (v_1, \ldots, v_n)^T \in \mathbb{R}^n$. By applying Lemma 2.5 with $g_h = 0$, the side condition (8) can now be ensured for B–spline functions of general order $k$ similar to (9) by the conditions

$$v_i \geq 0 \quad \text{for all } \; i = 1, \ldots, n. \tag{13}$$

Choosing $S_h := \mathcal{N}_{k,\Delta_h}$ the space of B–splines of order $k$ and approximating the exact solution $u^{m+1} \in H_0^1(I_x)$ of Problem 2.3 by a discrete function $u_h^{m+1} \in S_h$ with the B–spline coefficient vector $\mathbf{u}^{m+1} \in \mathbb{R}^n$, the obstacle condition $u^{m+1} \geq 0$ in $I_x$ of Problem 2.3 can now be replaced by the pointwise side condition $\mathbf{u}^{m+1} \geq 0$. Following the computations of [Hz, WHD], one therefore obtains the following discrete form of Problem 2.2, which has to be solved in each time step.

**Problem 2.6 (Discrete Variational Inequality)** *Find $0 \leq \mathbf{u}^{m+1} \in \mathbb{R}^n$, such that*

$$(\mathbf{v} - \mathbf{u}^{m+1})^T \left( \mathbf{C} \, \mathbf{u}^{m+1} - \mathbf{b}^m \right) \geq 0 \tag{14}$$

*hold for all $0 \leq \mathbf{v} \in \mathbb{R}^n$. Here the matrix $\mathbf{C} = (C_{i,j}) \in \mathbb{R}^{n \times n}$ is defined by*

$$C_{i,j} := \int_{x_{min}}^{x_{max}} N_{i,k} N_{j,k} dx + \theta \Delta\tau \int_{x_{min}}^{x_{max}} N'_{i,k} N'_{j,k} dx =: B_{i,j} + \theta \Delta\tau A_{i,j} \tag{15}$$

*where $N'_{i,k} := \partial N_{i,k} / \partial x$. The right hand side $\mathbf{b}^m$ is given by*

$$\mathbf{b}^m := \Delta\tau \, \mathbf{r}^m + (\mathbf{B} + (\theta - 1)\Delta\tau \mathbf{A}) \; \mathbf{u}^m$$

*where*

$$r_i^m := -\int_{x_{min}}^{x_{max}} \frac{\partial g}{\partial \tau}(x, \tau_m) \, N_{i,k}(x) + \frac{\partial g}{\partial x}(x, \tau_m) \, N'_{i,k}(x) dx. \tag{16}$$

For the solution of Problem 2.6 the error estimates $\|u^{m+1} - u_h^{m+1}\|_1 = O(h)$ and $\|u^{m+1} - u_h^{m+1}\|_1 = O(h^{3/2-\epsilon})$ in the $H^1$– Sobolev norm are proved in [BHR], provided $S_h$ is the space of continuous piecewise linear resp. piecewise quadratic functions. Note that Problem 2.6 can be equivalently written in the form of the linear complementary problem

$$
\begin{aligned}
\mathbf{C}\,\mathbf{u}^{m+1} &\geq \mathbf{b} \\
\mathbf{u}^{m+1} &\geq 0 \\
(\mathbf{u}^{m+1})^T \left(\mathbf{C}\,\mathbf{u}^{m+1} - \mathbf{b}\right) &= 0
\end{aligned} \tag{17}
$$

and be regarded as an obstacle problem with the zero function as obstacle (cf. [EO]) .

**Remark 2.7** Note that this solution approach can immediately be adapted to the pricing of European and Bermudan options (cf. [WHD]). In each time step, where prior exercise of the option is not possible, the variational inequality (14) must just be replaced by the corresponding variational equality.

In the case of equidistant grids, simple explicit formulas for the entries $A_{i,j}$ and $B_{i,j}$ of the stiffness matrix $\mathbf{A}$ and the mass matrix $\mathbf{B}$ from (15) can be found in [Hg].
To compute the vector $\mathbf{r}^m \in \mathbb{R}^n$ at time step $m$ from (16), the transformed payoff function $g(x, \tau_m) \in H^1(I_x)$ at time step $\tau_m$ is approximated by a function $g_h^m \in S_h$ and expanded as

$$
g_h^m(x) = \sum_{i=1}^{n} g_i^m N_{i,k}(x). \tag{18}
$$

By substituting this representation into (16), one derives the discrete right hand side

$$
\mathbf{r}^m = \left(\mathbf{C}\,\mathbf{g}^{m+1} - (\mathbf{B} + (\theta - 1)\Delta\tau\mathbf{A})\ \mathbf{g}^m\right)/\Delta\tau. \tag{19}
$$

By the approximation properties of B–splines (see [Bo]) the resulting approximation error is of same order as the discretization error. Alternatively, an explicit formula for the computation of $\mathbf{r}^m$ in the special case of equidistant grids and the function $g$ from (5) is given in [Hz].

## 2.4 Approximations to the Greek Letters

The use of smooth basis functions in the finite element approach is motivated by the possibility to determine the space derivatives of the solution by direct differentiation of the ansatz functions. This way, numerical differentiation can be avoided and a much higher accuracy can be expected. This will indeed be confirmed in Section 4 by numerical examples.

**Lemma 2.8** *For fixed $\tau$, let $y(x, \tau)$ be the solution of the heat equation (4) with boundary data (6). Then the Greeks letters Delta and Gamma are given by the identities*

$$
\begin{aligned}
\frac{\partial V(S,t)}{\partial S} &= e^{-\frac{x}{2}(q+1)-(\frac{1}{4}(q-1)^2+q)\tau} \left(y'(x,\tau) - \frac{1}{2}(q-1)\,y(x,\tau)\right), \\
\frac{\partial^2 V(S,t)}{\partial S^2} &= e^{-\frac{x}{2}(q+3)-(\frac{1}{4}(q-1)^2+q)\tau} \left(y''(x,\tau) - q\,y'(x,\tau) + \frac{1}{4}(q^2-1)\,y(x,\tau)\right)/K.
\end{aligned}
$$

*Proof:* Using the transformations (3) and the identity $\partial S/\partial x = Ke^x$, the assertion can easily be verified by the product rule of differentiation. □

For $\tau = \tau_m$, $y(x, \tau_m)$ and the partial derivatives $y^{(j)}(x, \tau_m)$ can be obtained from the finite element solution $u_h^m(x)$ of Problem 2.6 using (7) and (18). The function values $u_h^m(x)$ and the derivatives $(u_h^m)^{(j)}(x)$ can efficiently and stably be determined by the well–known recursion formulas for the valuation of B–splines and their derivatives (cf. [Bo]), provided B–splines of order $k \geq j + 2$ are used.

# 3 Fast Solution of the discrete form

In this section we present a fast solver for Problem 2.6. In the first subsection, the projective Gauss–Seidel scheme from [Cr] is generalized to a B–spline basis. This scheme is then used as smoothing component within a multigrid scheme to obtain convergence rates which are independent of the grid spacing $h$.

## 3.1 B–spline–based Projective Gauss–Seidel Schemes

In order to emphasize the $h$–dependency, we adapt our notation and write the discrete linear complementary problem (17) in the operator form

$$\begin{aligned} \mathcal{L}_h u_h &\geq f_h, \\ u_h &\geq g_h, \\ (u_h - g_h)(\mathcal{L}_h u_h - f_h) &= 0 \end{aligned} \tag{20}$$

with $f_h$, $g_h$, $u_h \in S_h$ and the linear operator $\mathcal{L}_h$. Note that the zero obstacle function in (17) is replaced by a general discrete obstacle $g_h \in S_h$, and that the time index $m$ is omitted for the sake of clarity. Since the operator $\mathcal{L}_h$, which corresponds to the matrix $\mathbf{C}$ from (15), is symmetric and positive definite, the linear complementary problem (20) can be solved by the projective Gauss–Seidel scheme, provided $S_h$ denotes the space of continuous piecewise linear functions (cf. [Cr]). Given an iterate $u_h^\nu$ in the $\nu$–th iteration, a standard Gauss–Seidel sweep

$$\bar{u}_h^\nu := \mathcal{S}\,(u_h^\nu)$$

is supplemented by a projection

$$u_h^{\nu+1} := \mathcal{P}\,\bar{u}_h^\nu$$

in order to satisfy the side condition $u_h^{\nu+1} \geq g_h$. For each grid point $x_i \in I_x$, the projection of a piecewise linear function $v_h$ is usually performed by

$$\mathcal{P}\,v_h(x_i) := \max\{v_h(x_i),\, g_h(x_i)\}. \tag{21}$$

In order to generalize the projection to higher order functions $v_h$, one is, however, again confronted by the problem that the estimate (10) is not valid anymore. Once more, the difficulty can be resolved by the use of a B–spline basis. In that case, the projection can be realized by Lemma 2.5 similar to (21) but involving B–spline coefficients $v_i$ by setting

$$\mathcal{P}\,v_i := \max\{v_i,\, g_i\} \tag{22}$$

for $i = 1, \ldots, n$. Because of the tensor–product structure of the discrete solution set $\{\mathbf{v} \in \mathbb{R}^n : v_i \geq g_i \text{ for } i = 1, \ldots, n\} \subset \mathbb{R}^n$, the convergence of the resulting projective Gauss–Seidel scheme follows using the same arguments as in [Cr]. Furthermore, under the assumption of no degeneracy (cf. [Cr, EO]), which is satisfied for Problem 2.6, it follows that the contact set of the solution $u_h$, defined by all coefficients $u_i$ for which equality holds, is identified after a finite number of iterations. This implies that the asymptotic convergence rate of the projective Gauss–Seidel scheme is of the order $1 - O(h^2)$, as it is well–known for the Gauss–Seidel scheme in the unconstrained case. Therefore, one suffers under unsatisfactorily slow convergence rates for small grid spacings $h$.

### 3.2 The Monotone Multigrid Algorithm (MMG)

For boundary value problems, it is well-known that the disadvantage of the $h$–dependency of the Gauss–Seidel relaxation can be overcome by multigrid techniques. This also applies to variational inequalities and linear complementary problems as it is demonstrated by a wide range of literature (e.g. [BC, HM, Ho, Ko1, Ko2, Kr, Ma, Oo, CP, Tai]). However, all cited works are restricted to discretizations with piecewise linear functions. This gap was closed in [HK], where the monotone multigrid method (MMG) from [Ko1] could be generalized to smooth basis functions by using a B–spline basis. If we introduce a nested sequence of finite–dimensional spaces

$$S_1 \subset S_2 \subset, \ldots, \subset S_L \subset H_0^1(I_x)$$

with equidistant grids $\Delta_\ell := \Delta_{h_\ell}$, $\ell = 1, \ldots, L$, $L \in \mathbb{N}$, and grid spacings $h_{\ell-1} := 2\,h_\ell$, the monotone multigrid method can be implemented as a variant of a standard multigrid scheme by adding a projection step as in (22) and employing special restriction operators $r$, $\tilde{r}$ for the inter–grid transfer.

**Algorithm 3.1 MMG$_\ell$** *Let $u_\ell^\nu := u_{h_\ell}^\nu \in S_\ell$ be a given approximation in the $\nu$–th cycle on level $\ell \geq 1$. Then, the MMG algorithm consists of the following steps:*

1. *A–priori–smoothing:* $u_\ell^{\nu,1} := (\mathcal{P} \circ \mathcal{S})^{\eta_1}(u_\ell^\nu)$.
2. *Coarse grid correction:*
$$\begin{aligned} d_\ell &:= f_\ell - \mathcal{L}_\ell u_\ell^{\nu,1}, \\ f_{\ell-1} &:= r d_\ell, \\ g_{\ell-1} &:= \tilde{r}(g_\ell - u_\ell^{\nu,1}), \\ \mathcal{L}_{\ell-1} &:= r \mathcal{L}_\ell p. \end{aligned}$$
*If $\ell = 1$, exactly solve the linear complementary problem*
$$\begin{aligned} \mathcal{L}_{h_{\ell-1}} v_{\ell-1} &\geq f_{\ell-1}, \\ v_{\ell-1} &\geq g_{\ell-1}, \\ (v_{\ell-1} - g_{\ell-1})(\mathcal{L}_{\ell-1} v_{\ell-1} - f_{\ell-1}) &= 0. \end{aligned}$$
*If $\ell > 1$, do $\gamma$ steps of* **MMG**$_{\ell-1}$ *with start value $u_{\ell-1}^0 := 0$ and solution $v_{\ell-1}$.*
*Set $u_\ell^{\nu,2} := u_\ell^{\nu,1} + p\, v_{\ell-1}$.*
3. *A–posteriori–smoothing:* $u_\ell^{\nu+1} := (\mathcal{P} \circ \mathcal{S})^{\eta_2}(u_\ell^{\nu,2})$.

The number of a–priori and a–posteriori–smoothing steps is denoted by $\eta_1$ and $\eta_2$, respectively. For $\gamma = 1$ one obtains a V–cycle, for $\gamma = 2$ a W–cycle iteration. To apply Algorithm

3.1 to the valuation of American options (i.e., to Problem 2.6), we choose the different components as follows: The spaces $S_\ell := \mathcal{N}_{k,\Delta_\ell}$ are defined as the spaces of B–splines of order $k$ as motivated before. The a–priori and the a–posteriori–smoothing steps $\mathcal{P} \circ \mathcal{S}$ are realized as in (22) by the projective Gauss–Seidel scheme. The prolongation operator $p = p^\ell_{\ell+1} : S_\ell \to S_{\ell+1}$ is defined by the B–spline refinement relation (cf. [Bo])

$$N_{i,k,\Delta_\ell} = \sum_{j=0}^{k} 2^{1-k} \binom{k}{j} N_{2i-1+j,k,\Delta_{\ell+1}}.$$

Following [Ha2], we choose the restriction $r = r^{\ell+1}_\ell : S_{\ell+1} \to S_\ell$, which is used to transfer the defect $d_\ell$ to coarser grids, as the adjoint of the prolongation $p$. In our case, one obtains the weighted restriction $r = \frac{1}{2}\, p$. In order to obtain the coarse grid obstacle function $g_{\ell-1} := \tilde{r}(g_\ell - u_\ell^{\nu,1})$, a special restriction operator $\tilde{r} = \tilde{r}^{\ell+1}_\ell : S_{\ell+1} \to S_\ell$ should be used, which differs from $r$ in general and leads to monotone approximations of the obstacle. As we show in the next section, the monotonicity of restriction operators leads to admissible new iterates $u_\ell^{\nu,2}$ in the sense that the side condition

$$u_\ell^{\nu,2} \geq g_\ell \tag{23}$$

is satisfied. This is one of the underlying ideas of monotone multigrid methods and leads to special robustness of the scheme.

**Remark 3.2** The convergence of the scheme can be significantly further accelerated if the coarse grid basis functions are adapted in each iteration step to the actual position of the free boundary by a suitable truncation operator. This leads to the *truncated version (TrMMG)* of the monotone multigrid method (cf. [Ko1, HK]).

### 3.2.1 The Construction of Monotone Obstacle Approximations

In this section we summarize the main results from [HK] with regard to the construction of monotone restriction operators $\tilde{r}$. The construction is not obvious for B–splines of general order $k$. It is based on the nonnegativity and on the refinement properties of B–splines. In the following, we fix two levels $\ell$ and $\ell + 1$ and expand the lower obstacle function $\tilde{S} := g_\ell - u_\ell^{\nu,1} \in S_\ell$ and its approximation $S := g_{\ell-1} \in S_{\ell-1}$ with B–spline coefficient vectors $\tilde{\mathbf{c}} \in \mathbb{R}^{n_\ell}$ and $\mathbf{c} \in \mathbb{R}^{n_{\ell-1}}$ as

$$\tilde{S} = \sum_{i=1}^{n_\ell} \tilde{c}_i\, N_{i,k,\Delta_\ell} =: \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta_\ell}, \qquad S = \sum_{i=1}^{n_{\ell-1}} c_i\, N_{i,k,\Delta_{\ell-1}} =: \mathbf{c}^T\, \mathbf{N}_{k,\Delta_{\ell-1}},$$

where $n_\ell$ depends on the number of grid points in $\Delta_\ell$ and $n_{\ell-1} = (n_\ell - 1 + k)/2$.

**Definition 3.3 (Monotone Coarse Grid Approximation)** *A function $S \in S_{\ell-1}$ is called an upper* monotone coarse grid approximation *to $\tilde{S} \in S_\ell$ if $S(x) \geq \tilde{S}(x)$ holds for all $x \in I_x$.*

**Remark 3.4** The condition (23) is satisfied if $g_{\ell-1}$ is an upper monotone coarse grid approximation to $g_\ell - u_\ell^{\nu,1}$, since

$$u_\ell^{\nu,2} := u_\ell^{\nu,1} + p\, v_{\ell-1} \geq u_\ell^{\nu,1} + p\, g_{\ell-1} \geq u_\ell^{\nu,1} + g_\ell - u_\ell^{\nu,1} \geq g_\ell.$$

For hat functions such approximations are constructed in [Ma] and [Ko1]. For B–splines of general order $k$ monotone coarse grid approximations can be obtained by the following proposition, which we cite from [HK].

**Proposition 3.5** *The B–spline* $L_k := \mathbf{q}_k^T \mathbf{N}_{k,\Delta_{\ell-1}}$ *with expansion coefficients*

$$q_{k,i} := \max\{\tilde{c}_{2i-k}, \ldots, \tilde{c}_{2i}\} \quad \text{for } i = 1, \ldots, n_{\ell-1} \tag{24}$$

*is a monotone upper coarse grid approximation to the B–spline* $\tilde{S} = \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta_\ell}$*.*

Within a monotone multigrid scheme, it can be expected that better approximations of the obstacle function on coarse grids lead to more efficient coarse grid corrections and thus to a faster convergence. As shown in [HK], the approximations $L_k$ can further be improved via a linear optimization formulation. By Fourier–Motzkin Elimination in the case $k = 2$ and a new optimization algorithm, called OCGC, in the case $k > 2$ (approximate) solutions can be obtained in optimal $O(n_\ell)$ operations. These lead to the following coarse grid approximations $S$ which we cite only for the cases $k = 2, 3, 4$ for the sake of simplicity. The general formula and all proofs can be found in [HK].

**Proposition 3.6** *The B–spline* $S = \mathbf{c}^T \mathbf{N}_{k,\Delta_{\ell-1}}$ *with recursively defined coefficients* $c_1 := q_{2,1}$ *and*

$$c_i := \max\{2\,\tilde{c}_{2i} - q_{2,i+1},\, \tilde{c}_{2i-1},\, 2\,\tilde{c}_{2i-2} - c_{i-1}\} \quad \text{for } i = 2, \ldots, n$$

*in the case* $k = 2$*,* $c_1 = q_{3,1}$ *and*

$$c_i := \max\left\{4\,\tilde{c}_{2i-3} - 3\,c_{i-1}, \tfrac{4}{3}\tilde{c}_{2i-2} - \tfrac{1}{3}c_{i-1}, \tfrac{4}{3}\tilde{c}_{2i-1} - \tfrac{1}{3}q_{3,i+1}, 4\,\tilde{c}_{2i} - 3\,q_{3,i+1}\right\}$$

*in the case* $k = 3$*, and* $c_1 = q_{4,1}$ *and*

$$\begin{aligned} c_i \;:=\; \max\Big\{ & 8\,\tilde{c}_{2i-4} - 6\,c_{i-1} - c_{i-2},\, 2\,\tilde{c}_{2i-3} - c_{i-1},\, \tfrac{4}{3}\tilde{c}_{2i-2} - \tfrac{1}{6}c_{i-1} - \tfrac{1}{6}q_{i+1}, \\ & 2\,\tilde{c}_{2i} - q_{i+1},\, 8\,\tilde{c}_{2i} - 6\,q_{i+1} - q_{i+2}\Big\} \end{aligned}$$

*in the case* $k = 4$*, where* $q_{k,i}$ *is defined as in (24), is an upper monotone coarse grid approximation to the lower obstacle* $\tilde{S} = \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta_\ell}$*. It is an improvement of the approximation* $L_k$ *from Proposition 3.5 in the sense that* $\mathbf{c} \le \mathbf{q}_k$ *holds.*

**Remark 3.7** In the special case $k = 2$ our B–spline–based monotone multigrid scheme corresponds to the multigrid method from [Ma] if the obstacle transfer is performed as described in Proposition 3.5. Moreover, if the coarse grid approximations are constructed according to Proposition 3.6, one recovers the monotone multigrid scheme from [Ko1].

For illustration, in Figure 1 lower obstacle functions $g_{\ell-1} := \tilde{r}(g_\ell - u_\ell^{\nu,1})$ are displayed for levels $\ell = 2, 3, 4$ as they typically arise if the MMG algorithm is applied to American option valuation with continuous piecewise linear (left) and $C^1$–smooth piecewise quadratic (right) basis functions. The obstacle functions restrict the size of the coarse grid correction $v_{\ell-1}$ such that the new iterate $u_\ell^{\nu,2} = u_\ell^{\nu,1} + p\,v_{\ell-1}$ is admissible, i.e., equal or above the transformed payoff function $g_\ell$. The restriction operator $\tilde{r}$ is chosen as derived in Proposition 3.6. Note that the value $x = 0$, where the coarse grid correction is least restricted, corresponds to the strike price $S = K$ by back transformation.

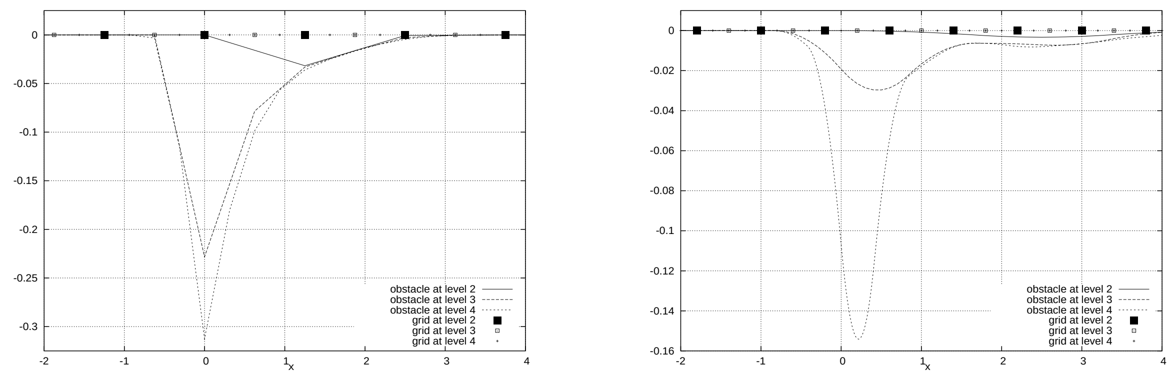


Figure 1: Lower coarse grid obstacle functions $g_\ell$ for the coarse grid correction and grids $\Delta_\ell$, $\ell = 2, 3, 4$ as they typically arise when the MMG–algorithm is applied to American option pricing with continuous, piecewise linear (left) and $C^1$–smooth, piecewise quadratic finite element ansatz functions (right).

The restriction operators $\tilde{r}$ from Proposition 3.5 as well as from Proposition 3.6 can immediately be modified such that they can be used in the truncated version TrMMG from Remark 3.2. In [HK] it is shown by numerical experiments that the truncated version combined with the obstacle approximations according to Proposition 3.6 is the fastest convergent variant. When applied to American option pricing, the scheme requires only one or two smoothing steps on each refinement level. It recovers asymptotic convergence rates which are independent of the grid size $h$. In the case $k = 3$ the convergence rates are bounded by about 0.27 for one smoothing step and 0.17 for two smoothing steps on each refinement level.

# 4 Numerical Results

In this section we investigate the performance of our scheme by numerical examples. In the first two subsections, we compute fair prices of short and long term plain vanilla American Put options and its derivatives with respect to the stock price. In particular, we analyze the influence of basis functions of different smoothness. In subsection 4.3, we then compare the performance of our scheme to the performance of Leisen–Reimer trees [LR] in the approximation of the second derivative Gamma.

## 4.1 Plain Vanilla American Put Option Prices

In our first experiment we consider an American put option with strike price $K = 100$, maturity $T = 0.5$ and an underlying stock with volatility $\sigma = 0.4$ which pays no dividends. The interest rate is assumed to be $r = 6\%$. Following [AC], the value $V(S, 0)$ obtained from the average of a 1000 step and a 1001 step binomial method is regarded as 'exact'. Against this benchmark we compute the pointwise errors of our B–spline–based finite element scheme (B-FEM) with

| Stock price | $S = 80$ | $S = 90$ | $S = 100$ | $S = 110$ | $S = 120$ | Comp. time |
|---|---|---|---|---|---|---|
| "exact" $V(S, 0)$ | 21.6059 | 14.9187 | 9.9458 | 6.4352 | 4.0611 | 3.45 Sec. |
| B-FEM, $k = 2$ | 0.0073 | 0.0154 | 0.0471 | 0.0410 | 0.0341 | 0.006 Sec. |
| B-FEM, $k = 3$ | 0.0194 | 0.0299 | 0.0223 | 0.0177 | 0.0200 | 0.008 Sec. |
| B-FEM, $k = 4$ | 0.0399 | 0.0074 | 0.0412 | 0.0332 | 0.0324 | 0.01 Sec. |

Table 1: Short term American Put value ($K = 100$, $\sigma = 0.4$, $d = 0.0$, $r = 0.06$, $T = 0.5$), and pointwise errors of B-FEM scheme with $N = 128$ and $M = 16$ space and time steps.

| Stock price | $S = 80$ | $S = 90$ | $S = 100$ | $S = 110$ | $S = 120$ | Comp. time |
|---|---|---|---|---|---|---|
| "exact" $V(S, 0)$ | 29.2601 | 24.8023 | 21.1294 | 18.0849 | 15.5428 | 3.45 Sec. |
| B-FEM, $k = 2$ | 0.0363 | 0.0511 | 0.0237 | 0.0287 | 0.0334 | 0.028 Sec. |
| B-FEM, $k = 3$ | 0.0341 | 0.0504 | 0.0241 | 0.0300 | 0.0351 | 0.02 Sec. |
| B-FEM, $k = 4$ | 0.0216 | 0.0411 | 0.0169 | 0.0243 | 0.0304 | 0.036 Sec. |

Table 2: Long term American Put value ($K = 100$, $\sigma = 0.4$, $d = 0.02$, $r = 0.06$, $T = 3$), and pointwise errors of B-FEM scheme with $N = 128$ and $M = 64$ space and time steps.

- $C^0$– smooth, piecewise linear B–splines ($k = 2$),
- $C^1$– smooth, piecewise quadratic B–splines ($k = 3$),
- $C^2$– smooth, piecewise cubic B–splines ($k = 4$).

We used an uniform grid, $\theta = 1$ and the space interval $I_x = [-5, 5]$. Since the scheme is much more sensible to the the number of space steps $N$ compared to the number of times steps $M$, we chose $N = 128$ and $M = 16$. This leads to a overall number of $2,048$ unknowns. The results are listed in Table 1. The speed of the scheme is measured in computational time in seconds and displayed in the last column. The computations were performed on a dual Intel(R) Xeon(TM) CPU 3.06GHz workstation. No calculation took longer then 0.01 seconds. The computation of the five option values by the average of a 1000 step and a 1001 step binomial method took 3.45 seconds. Note, that finite element or finite difference approaches provide an approximation to the whole surface $V(S, t)$ with $S \geq 0$ and $0 \leq t \leq T$, whereas tree methods have to be restarted for every single value $V(S_0, 0)$. In Table 2 we list the errors, which result from the valuation of the same American put, but with much longer maturity $T = 3$ and a dividend payment of $d = 2\%$ of the underlying. Due to the longer maturity, we chose a larger number of time steps $M = 64$.

As the errors are of similar size for different orders $k$ we conclude that for the computation of the option value the use of piecewise linear functions ($k = 2$) in the finite element scheme suffice. This, however, is no longer true if also derivatives of the option price with respect to the underlying have to be computed as we show in our next numerical examples.

### 4.2 American Option Price Sensitivities

We now consider an plain vanilla American call option with parameters

$$K = 10, \ \sigma = 0.6, \ r = 2.5\%, \ d = 0.0, \ T = 1. \tag{25}$$

In this scenario, the value of an American and European Call are identical, such that the Black–Scholes formula [BS] can be used to obtain benchmark values for the option price and for the Greek letters. For the numerical computations, we used an uniform grid, $\theta = 1/2$, the space interval $I_x = [-5, 5]$ and B–spline ansatz functions of orders $k = 2, 3, 4$ as in the last subsection. If the basis functions are sufficiently smooth, the derivatives of the solution $V$ are determined by direct differentiation via Lemma 2.8. In the other case numerical differentiation is used.

On the left hand side of Figure 2, 3 and 4, we displayed the $L^2$–errors at time $t = 0$ which arise if we compute the option value, Delta and Gamma with our scheme, respectively, for different numbers $N = M$ of unknowns. Due to the Crank–Nicholson time discretization quadratic convergence is the best that can be expected. One can see that the option value is computed in quadratic convergence for all orders $k$, but not the derivatives. The $j$–th derivative $\partial^{(j)}V/\partial S^{(j)}$ is only determined in quadratic convergence if basis functions of order $k \geq j + 2$ are used.

For the same choice of basis functions, we displayed on the right hand side of Figures 2, 3 and 4 the corresponding distributions of the pointwise error at time $t = 0$ in the case $N = M = 275$. The much more accurate approximation of Delta and Gamma obtained by the use of higher order basis functions and direct differentiation is clearly visible.

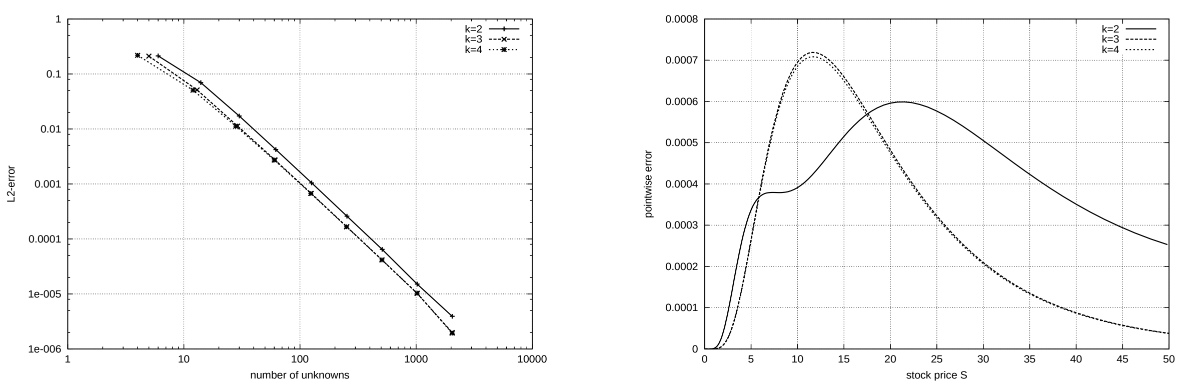


Figure 2: Mean square errors for $M = N$ (left) and distribution of the pointwise errors for $N = M = 275$ (right) which arise in the computation of the option value at time $t = 0$ with parameters from (25).

### 4.3 Comparison to other Schemes

In [WW], a comparison of various pricing methods leads to the conclusion that Leisen–Reimer trees [LR] are the superior method for the approximation of American option price sensitivities. Thus, we chose this scheme as a benchmark for our finite element scheme. We compare the pointwise error in the computation of Gamma $\partial^2V/\partial S^2(K, 0)$ of an American option with the parameters from (25). The results are displayed on the left hand side of Figure 5 for an American call option and on the right hand side for an American put option, respectively. In the first case, an exact benchmark value can be obtained by partial

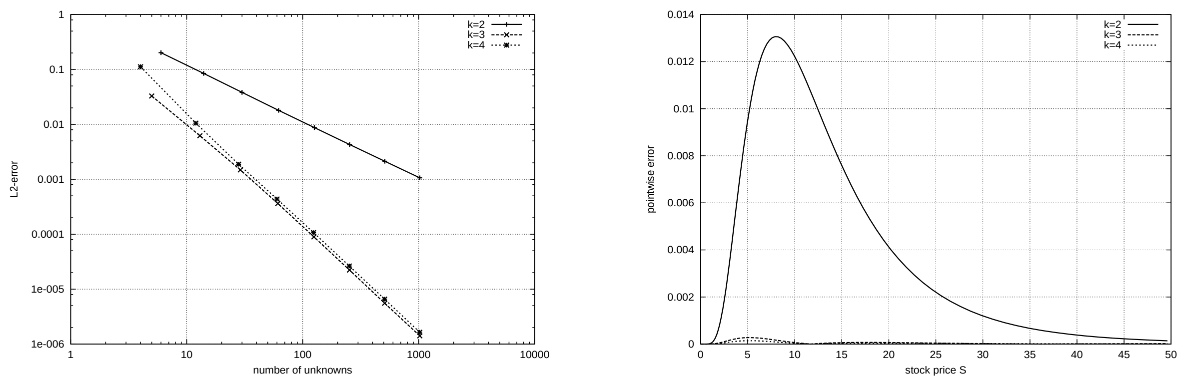


Figure 3: Mean square errors for $M = N$ (left) and distribution of the pointwise errors for $N = M = 275$ (right) which arise in the computation of Delta at time $t = 0$ with parameters from (25).

differentiation of the Black–Scholes formula. In the other case, we approximated Gamma by numerical differentiation of the average of a 20,000 step and a 20,001 step binomial method [CRR]. Using the parameter set (25), we obtain $\partial^2 V/\partial S^2(K, 0) \approx 0.064572055$. Against this benchmark value we computed the pointwise errors of our scheme (B-FEM) with basis functions of order $k = 2$ and $k = 4$ and the pointwise errors of Leisen–Reimer trees. Recall that the costs, which are plotted on the $x$–axis, of a binomial scheme with $n$ steps are of order $O(n^2)$, whereas the costs of our finite element scheme with $M$ time and $N$ space steps add up to $O(N \cdot M)$.

One can see in Figure 5 that Leisen–Reimer trees outperform the finite element scheme with hat functions ($k = 2$). Both schemes exhibit a pointwise convergence rate of about $\rho = 1/2$. In contrast, the finite element scheme with piecewise cubic functions ($k = 4$) attains a much better convergence rate of nearly $\rho = 1$. While this could be expected for American call options without dividends, where the solution is known to be smooth for all $t < T$, it is remarkable for American put options, where the solution is known to have a jump in the second derivative on the free boundary.

# 5 Concluding Remarks

In this paper we presented a finite element method which is based on higher order B–spline discretizations for the approximation of American option prices and their space derivatives Delta and Gamma. The method was supplemented by an monotone multigrid scheme for the efficient solution of the discrete form in order to achieve convergence rates which are independent of the grid spacing $h$.

Applying the scheme on uniform grids to the pricing of plain vanilla American options, our numerical experiments lead to the conclusion that just for the computation of the option value an increase of the polynomial degree of the ansatz functions is not advantageous. This, however, is no longer true if also the derivatives of the option value with respect to the underlying are required. Then, the correct choice of the order $k$ of the ansatz functions

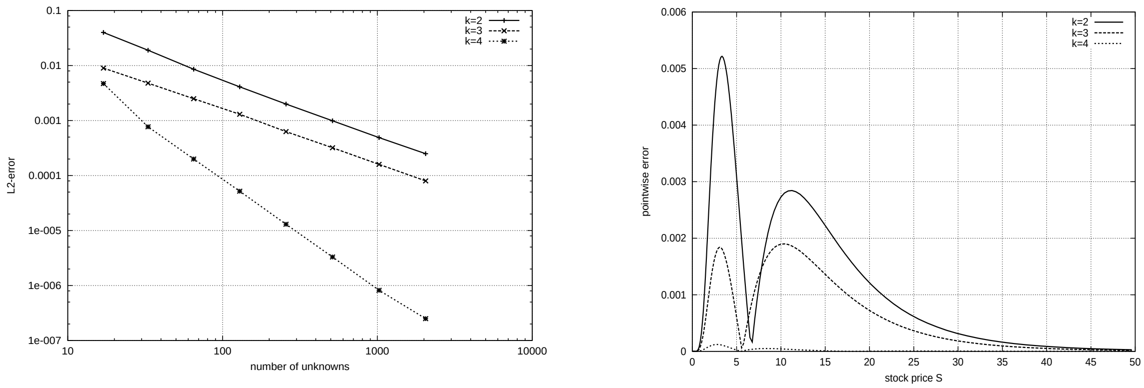


Figure 4: Mean square errors for $M = N$ (left) and distribution of the pointwise errors for $N = M = 275$ (right) which arise in the computation of Gamma at time $t = 0$ with parameters from (25).

(i.e., $k = j + 2$ for the $j$–th derivative) leads to a much higher accuracy and a much faster convergence. This way, our scheme, which provides the first multigrid approach which is applicable to higher order discretizations of free boundary value problems, is competitive to the best schemes proposed in the literature.

We wish to point out that the solution approach can, in principle, be generalized to higher spatial dimensions and thus also to higher dimensional option pricing problem, as long as partial differential equations can be derived for the option value, like, e.g, in the case of American options with stochastic volatility (cf. [BR, CP, Oo]) or in the case of convertible bonds with stochastic interest rate (cf. [BN, WHD]). Moreover, though the refinement relations of B–splines become more complicated (cf. [Bo]), all results can be generalized to non–uniform grids. It can be expected that a finer grid in the most interesting region near the strike price and a further refined grid near the free boundary would enhance the performance of the scheme significantly (cf. [PH, CP]). Using a suitable error estimator, as e.g. in [Ko2], it would then also be possible to perform the grid refinement adaptively. Due to the Crank–Nicholson time discretization at most quadratic convergence of our scheme can be expected. This limitation could be overcome by a higher order discretization of the time, e.g. by a Runge–Kutta scheme or by higher order finite elements. Then, also Theta, the derivative of the option value with respect to time, could be approximated with much higher accuracy.


**Acknowledgments.**

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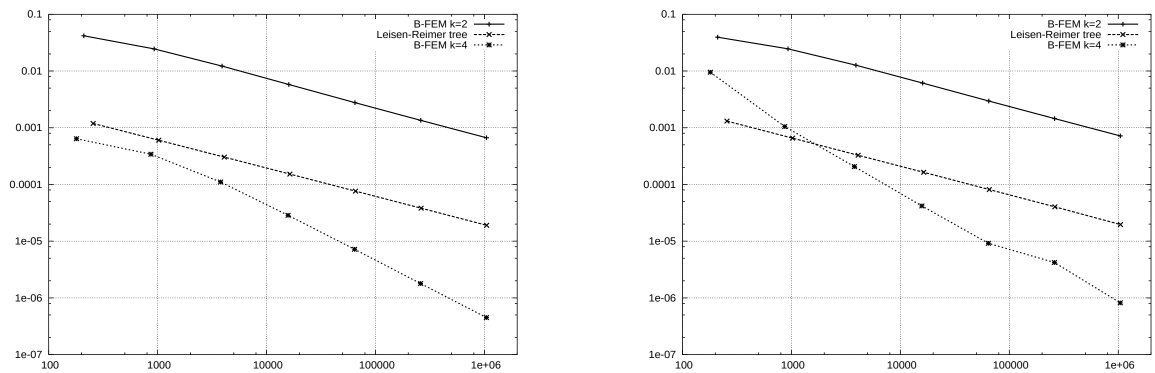


Figure 5: Pointwise convergence to Gamma $\partial^2 V/\partial S^2(K,0)$ of an American Call (left) and American put (right) with the parameter set (25).

implementation of the Leisen–Reimer trees.

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