Valuation of performance-dependent options in a Black-Scholes framework

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Abstract

In this paper, we introduce performance-dependent options as the appropriate financial instrument for a company to hedge the risk arising from the obligation to purchase shares as part of a bonus scheme for their executives. We determine the fair price of such options in a multidimensional Black-Scholes model which results in the computation of a multidimensional integral whose dimension equals the dimension of the underlying Brownian motion. The integrand is typically discontinuous, though, which makes accurate solutions difficult to achieve by numerical approaches. As a remedy, we derive a pricing formula which only involves the evaluation of smooth multivariate normal distributions. This way, performance-dependent options can efficiently be priced as it is shown by numerical results.

Keywords: option pricing, multivariate integration, Black-Scholes model

1 Introduction

Today, often long term incentive and bonus schemes form a major part of the wages of the executives of companies. One widespread form of such schemes consists in giving the participants a conditional award of shares. More precisely, if the participant stays with the company for at least a prescribed time period, he will receive a certain number of shares of the company at the end of the period. The exact amount of shares is usually linked to the success of the company measured via a performance criterion such as the company's gain over the period or its ranking among comparable firms.

It is now a huge risk for a company to leave the resulting positions unhedged. As the purchase of vanilla call options on the maximum number of possibly needed shares binds to much capital, the appropriate financial instruments in this situation are so-called performance-dependent options. These options are financial derivatives whose payoff depends on the performance of one asset in comparison to a set of benchmark assets. Thereby, we assume that the performance of an asset is determined by the relative increase of the asset price over the considered period of time. The performance of the asset is then compared to the performances of the benchmark assets. For each possible outcome of this comparison, a different payoff of the derivative can be realized.

We use a multidimensional Black-Scholes model, see, e.g., Karatzas [1] or Korn and Korn [2] for the dynamics of all asset prices required for the performance ranking. The martingale approach then yields a fair price of the performance-dependent option as a multidimensional integral whose dimension equals the dimension of the underlying Brownian motion. The integrand is typically discontinuous, however, which makes accurate numerical solutions difficult to achieve.

The main aim of this paper is to demonstrate that the combination of a closed-form solution to the pricing problem for performance-dependent options with suitable numerical integration methods clearly outperforms standard numerical approaches. The derived formula only involves the evaluation of smooth multivariate normal distributions which can be computed quickly and robustly by numerical integration. In various numerical results we illustrate the efficiency of this approach and its possibility to evaluate high-dimensional normal distributions in a superior way.

2 Performance-dependent options

Bonus schemes whose payoff depends on certain success criteria are a way to provide additional incentives for the executives of a company. Often, the executives obtain a conditional amount of shares of the company. The exact number depends on the ranking of the company's stock price increase in comparison to other (benchmark) companies. Such schemes induce uncertain future costs for the company, though. The appropriate way to hedge these risks are options which include the performance criteria in the definition of their payoff function, so-called performance-dependent options. In the following, we aim to derive pricing formulas for the fair price of these options.

We assume that there are n assets involved in total. The asset of the considered company gets assigned label 1 and the $n-1$ benchmark assets are labeled from 2 to n. The price of the i-th asset varying with time t is denoted by $S_i(t), 1 \leq i \leq n$. All stock prices at time t are collected in the vector $\mathbf{S}(t) = (S_1(t), \ldots, S_n(t))$.

2.1 Payoff profile

First, we need to define the payoff of a performance-dependent option at time T. To this end, we denote the relative price increase of stock i over the time interval $[0, T]$ by

$$\Delta S_i = S_i(T)/S_i(0).$$

We save the performance of the first asset in comparison to a given strike price K (typically, $K = S_1(0)$) and in comparison to the benchmark assets at time T in a ranking vector **Rank**$(\mathbf{S}(T)) \in \{+, -\}^n$ which is defined by

$$\text{Rank}_1(\mathbf{S}(T)) = \begin{cases} + & \text{if } S_1(T) \geq K, \\ - & \text{else} \end{cases} \quad \text{and } \text{Rank}_i(\mathbf{S}(T)) = \begin{cases} + & \text{if } \Delta S_1 \geq \Delta S_i, \\ - & \text{else} \end{cases}$$

for $i = 2, \ldots, n$. For each possible ranking $\mathbf{R} \in \{+, -\}^n$, a bonus factor $a_{\mathbf{R}} \in \mathbb{R}^+$ defines the payoff of the performance-dependent option. For explicit examples of such bonus factors see Section 3. In all cases we define $a_{\mathbf{R}} = 0$ if $\mathbf{R}_1 = -$.

The payoff of the performance-dependent option at time T is then defined by

$$V(\mathbf{S}(T), T) = a_{\textbf{Rank}(\mathbf{S}(T))} \left(S_1(T) - K \right). \tag{1}$$

In the following, we aim to determine the fair price $V(\mathbf{S}(0), 0)$ of such an option at the current time $t = 0$.

2.2 Multivariate Black-Scholes model

We assume that the stock price dynamics are given by

$$dS_i(t) = S_i(t) \left(\mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t) \right) \tag{2}$$

for $i = 1, \ldots, n$, where μ_i denotes the drift of the i-th stock, σ the $n \times n$ volatility matrix of the stock price movements and $W_j(t), 1 \leq j \leq n$, an n-dimensional Brownian motion. The matrix $\sigma\sigma^T$ is assumed to be positive definite.

The explicit solution of the stochastic differential equation (2) is then given by

$$S_i(T) = S_i(\mathbf{X}) = S_i(0) \exp\left(\mu_i T - \bar{\sigma}_i + X_i \right) \tag{3}$$

for $i = 1, \ldots, n$ with

$$\bar{\sigma}_i := \frac{1}{2} \sum_{j=1}^{n} \sigma_{ij}^2 T$$

and

$$X_i := \sum_{j=1}^{n} \sigma_{ij} W_j(T).$$

Hence, $\mathbf{X} = (X_1, \ldots, X_n)$ is a $N(\mathbf{0}, \mathbf{\Sigma})$-normally distributed random vector with $\mathbf{\Sigma} = \sigma\sigma^T T$.

2.3 Martingale approach

In the above multi-dimensional Black-Scholes setting, the option price $V(\mathbf{S}(0), 0)$ is given by the discounted expectation

$$V(\mathbf{S}(0), 0) = e^{-rT} E[V(\mathbf{S}(T), T)] \tag{4}$$

of the payoff under the unique equivalent martingale measure, i.e. the drift μ_i in (3) is replaced by the riskless interest rate r for each stock i. Plugging in the density function $\varphi_{\mathbf{0},\mathbf{\Sigma}}$ of the random vector $\mathbf{X}$, we get that the fair price of a performance-dependent option with payoff (1) is given by the n-dimensional integral

$$V(\mathbf{S}(0),0) = e^{-rT} \int_{R^n} \sum_{\mathbf{R}\in\{+,-\}^n} a_{\mathbf{R}}(S_1(T)-K)\,\chi_{\mathbf{R}}(\mathbf{S}(T))\varphi_{\mathbf{0},\mathbf{\Sigma}}(\mathbf{x})\,d\mathbf{x}. \quad (5)$$

Thereby, the expectation runs over all possible rankings $\mathbf{R}$ and the characteristic function $\chi_{\mathbf{R}}(\mathbf{S}(T))$ is defined by

$$\chi_{\mathbf{R}}(\mathbf{S}(T)) = \begin{cases} 1 & \text{if } \mathbf{Rank}(\mathbf{S}(T)) = \mathbf{R} \\ 0 & \text{else} \end{cases}.$$

2.4 Pricing formula

We will now derive an analytical expression for the solution of (5) in terms of smooth functions. We denote the Gauss kernel by

$$\varphi_{\mu,\mathbf{\Sigma}}(\mathbf{x}) := \frac{1}{(2\pi)^{d/2}(\det \mathbf{\Sigma})^{1/2}}\; e^{-\frac{1}{2}(\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}$$

and denote the multivariate normal distribution corresponding to $\varphi_{\mathbf{0},\mathbf{\Sigma}}$ with mean zero and covariance matrix $\mathbf{\Sigma}$ and the integral limits

$$c_i = \begin{cases} b_i & \text{if } R_i = + \\ -\infty & \text{else} \end{cases} \quad \text{and} \quad d_i = \begin{cases} \infty & \text{if } R_i = + \\ b_i & \text{else} \end{cases} \quad \text{for } i = 1,\ldots,n$$

which are depending on the ranking $\mathbf{R} \in \{+,-\}^n$ by

$$\Phi_{\mathbf{R}}(\mathbf{\Sigma},\mathbf{b}) := \int_{c_1}^{d_1} \ldots \int_{c_n}^{d_n} \varphi_{0,\mathbf{\Sigma}}(\mathbf{x})d\mathbf{x}.$$

Furthermore, we define the comparison relation $\mathbf{x} \geq_{\mathbf{R}} \mathbf{y}$ for two vectors $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ with respect to the ranking $\mathbf{R}$ by

$$\mathbf{x} \geq_{\mathbf{R}} \mathbf{y} \;:\Leftrightarrow\; R_i(x_i - y_i) \geq 0 \;\text{ for } 1 \leq i \leq n.$$

To proof our main theorem we need the following two lemmas.

Lemma 2.1 *Let* $\mathbf{b},\,\mathbf{q} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n\times n}$ *with full rank and* $\mathbf{\Sigma} \in \mathbb{R}^{n\times n}$ *symmetric and positive definite. Then*

$$\int_{\mathbf{A}\mathbf{x}\geq_{\mathbf{R}}\mathbf{b}} e^{\mathbf{q}^T\mathbf{x}}\varphi_{0,\mathbf{\Sigma}}(\mathbf{x})d\mathbf{x} = e^{\frac{1}{2}\mathbf{q}^T\mathbf{\Sigma}\mathbf{q}}\Phi_{\mathbf{R}}(\mathbf{A}\mathbf{\Sigma}\mathbf{A}^T, \mathbf{b} - \mathbf{A}\mathbf{\Sigma}\mathbf{q}).$$

Proof: A simple computation shows

$$e^{\mathbf{q}^T\mathbf{x}}\varphi_{0,\mathbf{\Sigma}}(\mathbf{x}) = e^{\frac{1}{2}\mathbf{q}^T\mathbf{\Sigma}\mathbf{q}}\varphi_{\mathbf{\Sigma}\mathbf{q},\mathbf{\Sigma}}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$. Using the substitution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} + \mathbf{\Sigma}\mathbf{q}$ we obtain

$$\begin{aligned} \int_{\mathbf{A}\mathbf{x}\geq_{\mathbf{R}}\mathbf{b}} e^{\mathbf{q}^T\mathbf{x}}\varphi_{0,\mathbf{\Sigma}}(\mathbf{x})d\mathbf{x} &= e^{\frac{1}{2}\mathbf{q}^T\mathbf{\Sigma}\mathbf{q}} \int_{\mathbf{A}\mathbf{x}\geq_{\mathbf{R}}\mathbf{b}} \varphi_{\mathbf{\Sigma}\mathbf{q},\mathbf{\Sigma}}(\mathbf{x})d\mathbf{x} \\ &= e^{\frac{1}{2}\mathbf{q}^T\mathbf{\Sigma}\mathbf{q}} \int_{\mathbf{y}\geq_{\mathbf{R}}\mathbf{b}-\mathbf{A}\mathbf{\Sigma}\mathbf{q}} \varphi_{0,\mathbf{A}\mathbf{\Sigma}\mathbf{A}^T}(\mathbf{y})d\mathbf{y} \end{aligned} \quad (6)$$

and thus the assertion. □

Lemma 2.2 *We have* **Rank**$(\mathbf{S}(T)) = \mathbf{R}$ *exactly if* $\mathbf{AX} \geq_{\mathbf{R}} \mathbf{b}$ *with*

$$\mathbf{A} := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & -1 & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad \textit{and} \quad \mathbf{b} := \begin{pmatrix} \ln \frac{K}{S_1(0)} - rT + \bar{\sigma}_1 \\ \bar{\sigma}_1 - \bar{\sigma}_2 \\ \vdots \\ \bar{\sigma}_1 - \bar{\sigma}_n \end{pmatrix}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{R}^n$.

Proof: Using (3) we see that $\text{Rank}_1 = +$ is equivalent to

$$S_1(T) \geq K \quad \Longleftrightarrow \quad X_1 \geq \ln \frac{K}{S_1(0)} - rT + \bar{\sigma}_1$$

which yields the first row of the system $\mathbf{AX} \geq_{\mathbf{R}} \mathbf{b}$. Moreover, for $i = 2, \dots, n$ the outperformance criterion $\text{Rank}_i = +$ can be written as

$$\frac{S_1(T)}{S_1(0)} \geq \frac{S_i(T)}{S_i(0)} \quad \Longleftrightarrow \quad X_1 - X_i \geq \bar{\sigma}_1 - \bar{\sigma}_i$$

which yields rows 2 to n of the system. □

Now we can state the following pricing formula which, in a slightly more special setting, can be found in Korn [3].

Theorem 2.3 *In our market setting determined by the price model (2), the price of a performance-dependent option with payoff (1) is given by*

$$V(\mathbf{S}(0), 0) = \sum_{\mathbf{R}\in\{+,-\}^n} a_{\mathbf{R}} \left(S_1(0)\, \Phi_{\mathbf{R}}(\mathbf{C}, \mathbf{d}) - e^{-rT}\, K\Phi_{\mathbf{R}}(\mathbf{C}, \mathbf{b})\right)$$

where $\mathbf{C} := \mathbf{A}\mathbf{\Sigma}\mathbf{A}^T$ *and* $\mathbf{d} := \mathbf{b} - \mathbf{A}\mathbf{\Sigma}\mathbf{e}_1$ *with* $\mathbf{A}$ *and* $\mathbf{b}$ *defined as in Lemma 2.2 and with* $\mathbf{e}_1$ *being the first unit vector.*

Proof: The characteristic function $\chi_{\mathbf{R}}(\mathbf{S}(T))$ in the integral (5) can be eliminated using Lemma 2.2 and we get

$$V(\mathbf{S}(0),0) = e^{-rT} \sum_{\mathbf{R}\in\{+,-\}^n} a_{\mathbf{R}} \int_{\mathbf{A}\mathbf{x}\geq_{\mathbf{R}}\mathbf{b}} (S_1(T)-K)\varphi_{\mathbf{0},\mathbf{\Sigma}}(\mathbf{x})d\mathbf{x}. \qquad (7)$$

By (3), the integral term can be written as

$$S_1(0)e^{rT-\bar{\sigma}_1} \int_{\mathbf{A}\mathbf{x}\geq_{\mathbf{R}}\mathbf{b}} e^{x_1}\,\varphi_{\mathbf{0},\mathbf{\Sigma}}(\mathbf{x})d\mathbf{x} - K \int_{\mathbf{A}\mathbf{x}\geq_{\mathbf{R}}\mathbf{b}} \varphi_{\mathbf{0},\mathbf{\Sigma}}(\mathbf{x})d\mathbf{x}.$$

Application of Lemma 2.1 with $\mathbf{q} = \mathbf{e}_1$ shows that the first integral equals

$$e^{\frac{1}{2}\mathbf{e}_1^T\mathbf{\Sigma}\mathbf{e}_1}\Phi_{\mathbf{R}}(\mathbf{A}\mathbf{\Sigma}\mathbf{A}^T, \mathbf{b}-\mathbf{A}\mathbf{\Sigma}\mathbf{e}_1) = e^{\bar{\sigma}_1}\Phi_{\mathbf{R}}(\mathbf{C},\mathbf{d}).$$

By a further application of Lemma 2.1 with $\mathbf{q} = \mathbf{0}$, we obtain that the second integral equals $K\Phi_{\mathbf{R}}(\mathbf{C},\mathbf{b})$ and thus the assertion holds. □

Note that the price of a performance-dependent option does not depend on the stock prices $S_2(0),\ldots,S_n(0)$ of the benchmark companies but only on the joint volatility matrix Σ. The pricing formula of Theorem 2.3 allows an efficient valuation of performance-dependent options in the case of moderate-sized benchmarks. It requires the computation of up to 2^n many n-dimensional normal distributions. The actual number of integrals equals twice the number of nonzero bonus factors $a_{\mathbf{R}}$. In the case of large benchmarks, the complexity and dimensionality of the pricing formula can prevent its efficient application, though. These problems can be circumvented by using a reduced Black-Scholes model and suitable tools from computational geometry, for details see [4].

3 Numerical Results

In this Section, we present numerical examples to illustrate the use of the pricing formula of Theorem 2.3. In particular, we compare the efficiency of our algorithm to the standard pricing approach (denoted by STD) of quasi-Monte Carlo simulation of the expected payoff (4) based on Sobol point sets, see, e.g., Glasserman [5]. Monte Carlo instead of quasi-Monte Carlo simulation led to significantly less accurate results in all our experiments. We systematically compare the use of our pricing formula with

- Quasi-Monte Carlo integration based on Sobol point sets (QMC),
- Product integration based on the Clenshew Curtis rule (P),
- Sparse Grid integration based on the Clenshew Curtis rule (SG)

for the evaluation of the multivariate cumulative normal distributions (see Genz [6]). The Sparse Grid approach is based on [7]. All computations were performed on a dual Intel(R) Xeon(TM) CPU 3.06GHz processor.

We consider a Black-Scholes market with $n = 5$ assets. Thereby, we investigate the following three choices of bonus factors $a_{\mathbf{R}}$ in the payoff function (1):

Example 3.1 *Linear ranking-dependent option:*

$$a_{\mathbf{R}} = \begin{cases} m/(n-1) & \textit{if } R_1 = + \\ 0 & else. \end{cases}$$

Here, m denotes the number of outperformed benchmark assets. The payoff depends on the rank of our company among the benchmark assets. If the company ranks first, there is a full payoff $(S_1(T) - K)^+$. If it ranks last, the payoff is zero. In between, the payoff increases linearly with the number of outperformed benchmark assets.

Example 3.2 *Outperformance option:*

$$a_{\mathbf{R}} = \begin{cases} 1 & \textit{if } \mathbf{R} = (+, \ldots, +) \\ 0 & else. \end{cases}$$

A payoff only occurs if $S_1(T) \geq K$ and if all benchmark assets are outperformed.

Example 3.3 *Linear ranking-dependent option combined with an outperformance condition:*

$$a_{\mathbf{R}} = \begin{cases} m/(n-1) & \textit{if } R_1 = + \textit{ and } R_2 = + \\ 0 & else. \end{cases}$$

The bonus depends linearly on the number m of outperformed benchmark companies like in Example 3.1. However, the bonus is only paid if company two is outperformed. Company two could, e.g., be the main competitor of our company.

In all cases, we use the model parameters $K = 100$, $S_1(0) = 100$, $T = 1$, $r = 5\%$ and as volatility matrix

$$\sigma = \begin{pmatrix} 0.1515 & 0.0581 & 0.0373 & 0.0389 & 0.0278 \\ 0.0581 & 0.2079 & 0.0376 & 0.0454 & 0.0393 \\ 0.0373 & 0.0376 & 0.1637 & 0.0597 & 0.0635 \\ 0.0389 & 0.0454 & 0.0597 & 0.1929 & 0.0540 \\ 0.0278 & 0.0393 & 0.0635 & 0.0540 & 0.2007 \end{pmatrix}.$$

The computed option prices and discounts compared to the price of the corresponding plain vanilla option given by 9.4499 are displayed in the second and third column of Table 1. The number of normal distributions (# Int) which have to be computed is shown in the last column.

The convergence behaviour of the four different approaches (STD, QMC, P, SG) to price the performance-dependent options from the Examples 3.1 – 3.3 are displayed in Figure 1. There, the time is displayed which is needed to obtain a given accuracy. One can see that the standard approach (STD) and the product integration approach (P) perform worst for all accuracies. The convergence rates are clearly lower than one in all Examples. The integration scheme STD suffers

Example	$V(S_1, 0)$	Discount	# Int
3.1	6.2354	34.02%	30
3.2	3.0183	68.06%	2
3.3	4.5612	51.73%	16

Table 1: Option prices, discounts compared to the corresponding plain vanilla option and number of computed normal distributions.

under the irregularity of the integrand which is highly discontinuous and not of bounded variation. The product integration approach suffers under the curse of dimension. The use of the pricing formula from Thereom 2.3 combined with QMC or SG integration clearly outperforms the STD approach in terms of efficiency in all considered Examples. The QMC scheme exhibits a convergence rate of about one independent of the problem. The combination of Sparse Grid integration with our pricing formula (SG) leads to the best overall accuracies and convergence rates in all cases. Even very high accuracy demands can be fulfilled in less than a few seconds.

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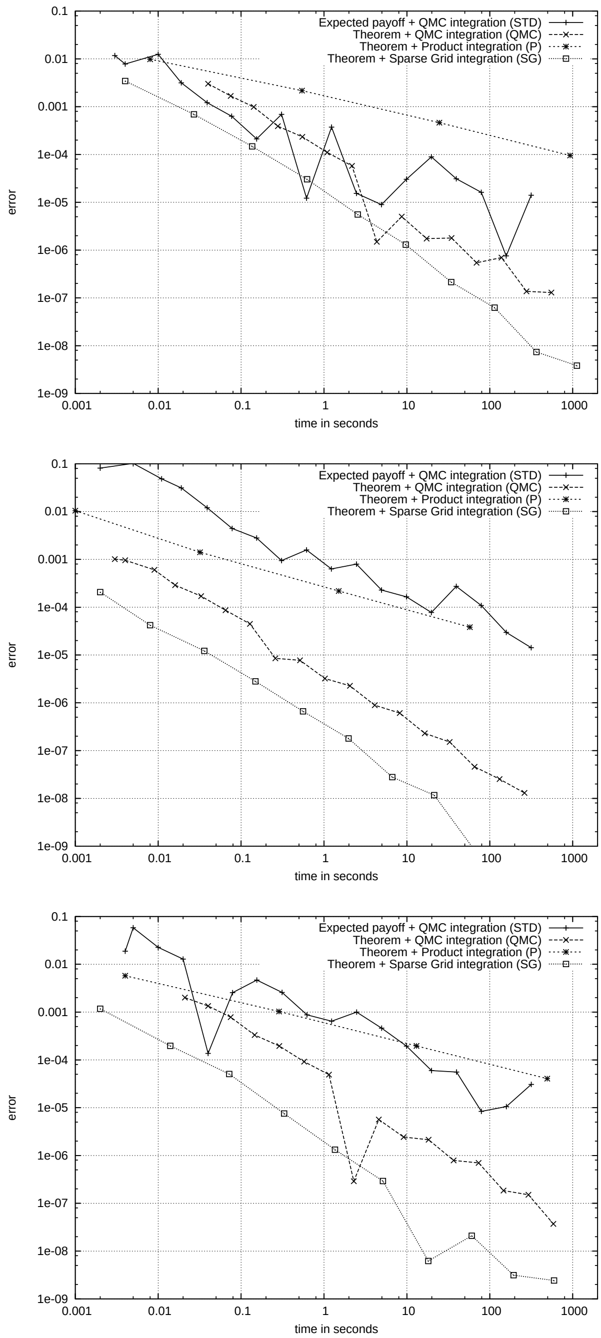

Figure 1: Errors and timings of the different numerical approaches to price the performance-dependent options of Examples 3.1 (top), 3.2 (middle) and 3.3 (bottom).