B–Spline Based Monotone Multigrid Methods, with an Application to the Prizing of American Options

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Abstract

We propose a monotone multigrid method based on a B–spline basis of arbitrary smoothness for the efficient numerical solution of elliptic variational inequalities on closed convex sets. In order to maintain monotonicity (upper bound) and quasi–optimality (lower bound) of the coarse grid corrections, we propose coarse grid approximations of the obstacle function which are based on B–spline expansion coefficients. To illustrate the potential of the scheme, the method is applied to the pricing of American options in the Black–Scholes framework.

Keywords: Monotone multigrid method, variational inequalities, American option, Greek letters, finite elements, B–splines.
AMS-Classification: 65M55, 35J85, 65N30, 65D07.

1 Introduction

For the efficient numerical solution of elliptic variational inequalities on closed convex sets, multigrid methods have been investigated over the past decades with great success, e.g., [BC, HM, Ho, M, K]. However, not all of them have assured consequently that the obstacle criterion is met. As a consequence often no convergence theory is available. This disadvantage could be resolved by the work of Ralf Kornhuber [K] with the introduction of *monotone* multigrid methods (MMG).
Essential for the success of these methods is the appropriate approximation of the obstacle function on coarser grids. Such approximations satisfy the (upper) bound imposed by the obstacle (monotonicity) as well as a lower bound which corresponds to a condition of quasi–optimality. The construction is derived in [K] for piecewise linear finite element functions. On the other hand, there are a number of problems which would profit from higher order approximations. Among these are the problem of prizing American options which is formulated as a parabolic boundary value problem involving Black–Scholes' equation with a free boundary, the optimal exercise prize of the option. Here, of particular importance are accurate approximations to the derivatives of the solution, the so–called Greek letters. In [CP, O, RW] multigrid techniques have been already successfully used for the solution of the linear complementary problem which arises from a finite difference discretization of the problem to price American options with stochastic volatility.
In this paper, we first recall the main results from [HK, Hz2] where the monotone multigrid method from [K] is generalized to discretizations in terms of B–splines of arbitrary order and applied to the problem to prize American options. Then we provide details about the construction of *truncated* B–spline based monotone multigrid methods developed in [Hz1] which are known to converge faster than the standard version.
Using arguments from [K], global convergence and optimal complexity of the B–spline based monotone multigrid method can be proved. Monotone and quasi–optimal coarse grid approximations to the obstacle function are constructed for B–spline basis functions of arbitrary degree in optimal

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complexity of the number of unknowns. By construction, one can expect special robustness of the scheme and full multigrid efficiency in the asymptotic range. Moreover, due to the higher order discretization, the derivatives of the solution can be stably and accurately determined via direct differentiation of the basis functions. This is confirmed by computations for an American option pricing problem, where we additionally compare our results to the ones obtained in [LR].

2 B–Spline Based Finite Element Methods

In this section we introduce B–splines as finite element ansatz functions for the solution of variational inequalities. An introduction into finite element methods which are based on B–splines for the solution of variational equalities can be found in [Hg].

2.1 Elliptic Variational Inequalities

Let Ω be a domain in $\mathbb{R}^d$ and $\mathcal{J}(v) := \frac{1}{2}a(v,v) - f(v)$ a quadratic functional induced by a continuous, symmetric and H_0^1– elliptic bilinear form $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ and a linear functional $f : H_0^1(\Omega) \to \mathbb{R}$. As usual, $H_0^1(\Omega)$ is the subspace of functions belonging to the Sobolev space $H^1(\Omega)$ with zero trace on the boundary. We consider the constrained minimization problem

$$\text{find } u \in \mathcal{K} : \quad \mathcal{J}(u) \le \mathcal{J}(v) \quad \text{for all } v \in \mathcal{K} \tag{1}$$

on the closed and convex set

$$\mathcal{K} := \{v \in H_0^1(\Omega) : \ v(x) \le g(x) \ \text{ for all } x \in \Omega\} \subset H_0^1(\Omega).$$

The function $g \in H_0^1(\Omega)$ represents an upper obstacle for the solution $u \in H_0^1(\Omega)$. Lower obstacles can be treated analogously. If g satisfies $g(x) \ge 0$ for all $x \in \partial\Omega$, problem (1) admits a unique solution $u \in \mathcal{K}$ by the Lax–Milgram theorem. It can also be written as an elliptic variational inequality or as a linear complementary problem.
Discretizing in a finite dimensional spline space S_h of piecewise polynomials on a grid with uniform spacing h leads to the discrete formulation of (1),

$$\text{find } u_h \in \mathcal{K}_h : \quad \mathcal{J}(u_h) \le \mathcal{J}(v_h) \quad \text{for all } v_h \in \mathcal{K}_h \tag{2}$$

on the closed and convex set

$$\mathcal{K}_h := \{v_h \in S_h : \ v_h(x) \le g_h(x) \ \text{ for all } x \in \Omega\} \subset S_h.$$

In [BHR] regularity $u \in H^{5/2-\epsilon}(\Omega)$ of the solution to (1) is shown for arbitrary $\epsilon > 0$. Moreover, error estimates $\|u - u_h\|_{H^1(\Omega)} = O(h)$ and $\|u - u_h\|_{H^1(\Omega)} = O(h^{3/2-\epsilon})$ are proved in the case of piecewise linear, respectively piecewise quadratic, functions, provided the functions f, g are sufficiently regular.

2.2 B–Splines

In the following we use a B–spline basis $\Sigma_h := \{N_{1,k,h}, \ldots, N_{n,k,h}\}$ of order k to span the discrete space S_h. Let $\theta_1 = \ldots = \theta_k = a < \theta_{k+1} < \ldots < \theta_n < b = \theta_{n+1} = \ldots = \theta_{n+k}$ be an expanded knot sequence with uniform grid spacing h in the interior of the interval $I := [a,b]$. Then the B–spline basis functions $N_{i,k,h}$ of order k are recursively defined by

$$\begin{aligned} N_{i,1,h}(x) &= \begin{cases} 1, & \text{if } x \in [\theta_i, \theta_{i+1}) \\ 0, & \text{else} \end{cases} , \\ N_{i,k,h}(x) &= \frac{x - \theta_i}{(k-1)h} N_{i,k-1,h}(x) + \frac{\theta_{i+k} - x}{(k-1)h} N_{i+1,k-1,h}(x) \end{aligned} \tag{3}$$

for $x \in I$. In the case $k = 2$, $N_{i,k,h}$ equals the usual piecewise linear hat function which is one at θ_{i+1} and zero at all other grid points. In the following we will exploit that B–spline basis functions satisfy $\operatorname{supp} N_{i,k,h} \subseteq [\theta_i,\, \theta_{i+k}]$ (local support), $N_{i,k,h}(x) \geq 0$ for all $x \in I$ (nonnegativity) and $N_{i,k,h} \in C^{k-2}(I)$ (differentiability). Furthermore, we use the two–scale refinement relation

$$N_{i,k,2h} = \sum_{j=0}^{k} a_j \, N_{2i-k+j,k,h} \tag{4}$$

with the subdivision coefficients

$$a_j := 2^{1-k} \binom{k}{j} \qquad \text{for } j = 0, \ldots, k \tag{5}$$

as a natural prolongation operator.
In the multivariate case $\Omega := \prod_{\ell=1}^{d} [a_\ell, b_\ell] \subset \mathbb{R}^d$, the i-th d-dimensional tensor product B–spline of order k is defined on a tensorized grid by

$$N^{(d)}_{i,k,h}(x) := \prod_{\ell=1}^{d} N_{i_\ell,k,h}(x_\ell), \qquad x \in \Omega, \tag{6}$$

where $i := (i_1, \ldots, i_d)$ now denotes a multi-index. The same properties hold as in the univariate case. The multivariate variant of the refinement relation (4) is given by

$$N^{(d)}_{i,k,2h} = \sum_{j \in J} a^{(d)}_j \, N^{(d)}_{2i-k+j,k,h} \tag{7}$$

with the index set $J := \{ j \in \mathbb{N}^d : 0 \leq j_i \leq k \text{ for all } i = 1, \ldots, d\}$ and the subdivision coefficients

$$a^{(d)}_j := 2^{(1-k)d} \prod_{\nu=1}^{d} \binom{k}{j_\nu} \qquad \text{for } j \in J. \tag{8}$$

3 B–Spline Based Projected Gauss–Seidel Relaxation

In this section we propose a projected Gauss–Seidel scheme based on B–spline basis functions for the numerical solution of (2). In Section 4 the method is then incorporated as smoothing component into a multigrid cycle. The crucial point is the definition of a projection operator $\mathcal{P} : S_h \to \mathcal{K}_h$ in the case of higher–order basis functions.
In the case $k = 2$ S_h consists of hat functions so that the projection into the convex set $\mathcal{K}_h$ can be defined for given grid points $\{\theta_i\}_i$ as

$$\mathcal{P}\, v_h(\theta_i) := \min\{v_h(\theta_i),\, g_h(\theta_i)\}. \tag{9}$$

For higher–order functions v_h, the difficulty arises that the function v_h may violate the obstacle g_h even though $v_h(\theta_i) \leq g_h(\theta_i)$ holds on all grid points. Thus, controlling function values on grid points is not a sufficient criterion in this case. We propose here instead a construction using higher order B–splines, which compares B–spline *expansion coefficients* instead of function values. It is based on the following lemma which follows from the nonnegativity of B–splines.

Lemma 3.1 *If the B–spline coefficients of $v_h, g_h \in S_h$ satisfy $v_i \leq g_i$ for all $i = 1, \ldots, n$, then $v_h(x) \leq g_h(x)$ holds for all $x \in I$.*

In view of Lemma 3.1 the projection can now be defined for B–spline functions of general order k similar to (9), but now involving expansion coefficients, by setting

$$\mathcal{P}\, v_i := \min\{v_i,\, g_i\}. \tag{10}$$

Using (10) we can define a projected Gauss–Seidel scheme for the solution of (2) with B–spline basis functions of arbitrary order. For a given iterate $u_h^\nu \in S_h$ a standard Gauss–Seidel sweep $\mathcal{S}$ is supplemented by the projection $\mathcal{P}$ into the convex set $\mathcal{K}_h$, i.e.,

$$u_h^{\nu+1} = \mathcal{P} \circ \mathcal{S}\,(u_h^\nu). \tag{11}$$

Since the discrete solution set $\{\mathbf{v} \in I\!R^n : v_i \le g_i \text{ for } i = 1, \ldots, n\}$ describes a cuboid in $I\!R^n$ the same arguments as in [C] can be used to prove the convergence of the resulting projected Gauss–Seidel scheme for symmetric and positive definite problems.
Employing the tensor-product approach from Section 2.2, the corresponding extensions of Lemma 3.1, the definition of the projection operator and of the projected Gauss–Seidel scheme immediately carry over to the d–dimensional setting.

4 B–Spline Based Monotone Multigrid Methods

The asymptotic convergence rate of the projected Gauss–Seidel relaxation is of order $1 - O(h^2)$ as it is well–known for the Gauss–Seidel scheme in the unconstrained case. Consequently, this implies unsatisfactorily slow convergence for small grid spacings h. This disadvantage can be overcome by multigrid techniques. A by now popular method for the solution of the variational inequality (2) is the monotone multigrid method from [K]. If we introduce a nested sequence of finite–dimensional spaces

$$S_1 \subset S_2 \subset, \ldots, \subset S_L \subset H_0^1(\Omega),$$

the monotone multigrid method can be implemented as a variant of a standard multigrid scheme by adding a projection step as in (9) or (10) and employing special restriction operators $\tilde{r}$ for the inter–grid transfer of the obstacle function. We choose the different components as follows: The spaces S_ℓ, $\ell = 1, \ldots, L$, are defined as the spaces of B–splines of order k with equidistant grid spacings $h_\ell := h_{\ell-1}/2$. The a–priori and the a–posteriori–smoothing steps $\mathcal{P} \circ \mathcal{S}$ are realized by the projective Gauss–Seidel scheme (11). The prolongation operator p is defined by the B–spline refinement relations (4) or (7). Following [H], we choose the restriction operator r which is used for the transport of defect and stiffness matrix to coarser grids as the adjoint of the prolongation p. In order to obtain suitable coarse grid approximations to the obstacle function, a special *restriction operator* $\tilde{r}$ should be used which provides monotone approximations of the obstacle. This ensures that coarse grid corrections do not violate the fine grid obstacle function which is one of the key ideas of monotone multigrid methods and which leads to special robustness of the scheme. Such operators are constructed in [K] for piecewise linear functions. In the next section the construction is generalised to B–spline functions of higher order. Using arguments of [K], global convergence and optimal complexity of the B–spline based monotone multigrid method can then be proved.

4.1 Construction of Monotone and Quasi–optimal Obstacle Approximations

In this section we provide the construction of monotone and quasi–optimal coarse grid approximations of the obstacle function which lead to suitable restriction operators $\tilde{r}$. For more transparency, we consider here only two grids, as the generalization to several grids is obvious.
In the following we denote the obstacle function by $\tilde{S}$ and its approximation by S. The approximation is defined on a coarse grid $T := \{\theta_i\}_{i=1,\ldots,n+k} \subset I$ with equidistant grid spacing $2h$.
The obstacle function is defined on the fine grid $\Delta := \{\tilde{\theta}_i\}_{i=1,\ldots,\tilde{n}+k}$. The fine grid is defined with a grid spacing h such that $\theta_i = \tilde{\theta}_{2i-k}$ for $i = k, \ldots, n+1$ and $\frac{1}{2}(\theta_{i-1} + \theta_i) = \tilde{\theta}_{2i-k-1}$ for $i = k+1, \ldots, n+1$. By construction, it follows that $T \subset \Delta$ and $\tilde{n} = 2n + 1 - k$. Then the obstacle function and the approximation have the representations

$$\tilde{S} = \sum_{i=1}^{\tilde{n}} \tilde{c}_i\, N_{i,k,h} =: \tilde{\mathbf{c}}^T\, \mathbf{N}_{k,h}, \qquad S = \sum_{i=1}^{n} c_i\, N_{i,k,2h} =: \mathbf{c}^T\, \mathbf{N}_{k,2h} \tag{12}$$

in the B–spline basis with coefficient vectors $\tilde{\mathbf{c}} \in \mathbb{R}^{\tilde{n}}$ and $\mathbf{c} \in \mathbb{R}^n$.
Given an obstacle function $\tilde{S}$ which is defined on the fine grid Δ, we present a construction which provides an coarse grid approximation S which satisfies

$$L(x) \leq S(x) \leq \tilde{S}(x) \quad \text{for all } x \in I$$

in the optimal complexity of $O(n)$ arithmetic operations of the coarse grid. The lower barrier $L(x)$ is provided below in (15). If the upper inequality is satisfied, we call S a *monotone lower coarse grid approximation* to the obstacle $\tilde{S}$. This condition leads to special robustness of the multigrid scheme. The lower inequality corresponds to the condition of *quasi–optimality* in [K] and ensures an asymptotic reduction of the method to a linear relaxation.
For hat functions such approximations are constructed in [M, K]. A corresponding construction for higher–order functions has to our knowledge not been provided previous to [HK]. In view of Lemma 3.1, our method controls B–spline expansion coefficients. In the following, we use the notation that all terms which involve c_i or q_i with $i < 1$ or $i > n$ or $\tilde{c}_i$ with $i < 1$ or $i > \tilde{n}$ have to be omitted.

Theorem 4.1 (Monotone Coarse Grid Approximation) *The spline $S = \mathbf{c}^T \mathbf{N}_{k,2h}$ satisfies*

$$S(x) \leq \tilde{S}(x) \quad \textit{for all } x \in I$$

if the expansion coefficients of S and $\tilde{S}$ satisfy the linear inequality system

$$A_k \, \mathbf{c} \leq \tilde{\mathbf{c}} \tag{13}$$

where the matrix $A_k \in \mathbb{R}^{\tilde{n} \times n}$ is defined by $(A_k)_{ij} := a_{i+k-2j}$ with the subdivision coefficients a_j from (5) and the additional convention that $a_j := 0$ if $j < 0$ or $j > k$.

Proof: Substituting (4) into (12) and sorting according to the basis functions $N_{i,k,h}$ leads to

$$\tilde{S}(x) - S(x) = \sum_{i=1}^{\tilde{n}} \left[\tilde{\mathbf{c}} - (A_k \, \mathbf{c})\right]_i \, N_{i,k,h}(x). \tag{14}$$

By Lemma 3.1 we obtain $\tilde{S}(x) - S(x) \geq 0$ for all $x \in I$ if $\left[\tilde{\mathbf{c}} - (A_k \, \mathbf{c})\right]_i \geq 0$ holds for all $i = 1, \ldots, \tilde{n}$. □

In the multivariate case $d > 1$, Theorem 4.1 holds if the matrix A_k is replaced by the tensor product matrix $A_k^{(d)} := A_k \otimes \ldots \otimes A_k \in \mathbb{R}^{(\tilde{n} \times n)^d}$.
As all row sums of A_k are equal to one, the vector $\mathbf{q} := (q_1, \ldots, q_n)^T$ defined below in (15) satisfies the inequality system $A_k \, \mathbf{q} \leq \tilde{\mathbf{c}}$. By Theorem 4.1, we have thus derived a monotone lower coarse grid approximation to $\tilde{S}$.

Proposition 4.2 *The spline $L := \mathbf{q}^T \, \mathbf{N}_{k,2h}$ with coefficients*

$$q_i := \min\left\{\tilde{c}_{2i-k}, \, \ldots, \, \tilde{c}_{2i}\right\} \quad \textit{for } i = 1, \ldots, n \tag{15}$$

satisfies

$$L(x) \leq \tilde{S}(x) \quad \textit{for all } x \in I.$$

In the multivariate case $d > 1$ the definition (15) is replaced by

$$q_i := \min\{\tilde{c}_j : \; 2i_m - k \leq j_m \leq 2i_m, \; m = 1, \ldots, d\} \text{ for } i \in \mathbb{I}_c$$

with indices from $\mathbb{I}_c := \{i \in \mathbb{N}^d : \, 1 \leq i_m \leq n, \; m = 1, \ldots, d\}$.
The approximations from Proposition 4.2 correspond to the quasi–optimality condition in [K]. It can be proved that multigrid methods with such obstacle approximations asymptotically reduce to a linear relaxation.
As it is illustrated in Figure 1 for the cases $d = 1$, $k = 2$ and $d = 2$, $k = 3$, the approximation L can be further improved in many cases. Such approximations are obtained next by the following Theorem. Within a monotone multigrid scheme, we expect that better approximations of the obstacle function on coarse grids lead to more efficient coarse grid corrections and, thus, to faster convergence.

Theorem 4.3 *Let $\tilde{S} = \tilde{\mathbf{c}}^T \mathbf{N}_{k,h}$ be a given obstacle and let $L = \mathbf{q}^T \mathbf{N}_{k,2h}$ be defined as in (15). Let further the vector $\mathbf{c}$ be recursively defined by*

$$c_i = \min\{2\,\tilde{c}_{2i-2} - c_{i-1},\ \tilde{c}_{2i-1},\ 2\,\tilde{c}_{2i} - q_{i+1}\} \tag{16}$$

in the case $k = 2$,

$$c_i = \min\left\{4\,\tilde{c}_{2i-3} - 3\,c_{i-1}, \tfrac{4}{3}\tilde{c}_{2i-2} - \tfrac{1}{3}c_{i-1}, \tfrac{4}{3}\tilde{c}_{2i-1} - \tfrac{1}{3}q_{i+1}, 4\,\tilde{c}_{2i} - 3\,q_{i+1}\right\} \tag{17}$$

in the case $k = 3$ and

$$\begin{aligned} c_i \quad := \quad \min\big\{&8\tilde{c}_{2i-4} - c_{i-2} - 6c_{i-1}, 2\tilde{c}_{2i-3} - c_{i-1}, \tfrac{4}{3}\tilde{c}_{2i-2} - \tfrac{1}{6}(c_{i-1} + q_{i+1}), \\ &2\tilde{c}_{2i-1} - q_{i+1},\ 8\tilde{c}_{2i} - 6q_{i+1} - q_{i+2}\big\} \end{aligned} \tag{18}$$

in the case $k = 4$. Then it holds

$$L(x) \le S(x) \le \tilde{S}(x) \qquad \text{for all } x \in I$$

and the construction requires only optimal $O(n)$ arithmetic operations.

The proof of Theorem 4.3 which exploits the special structure of the matrices A_k, the (involved) formula for a general order $k \in \mathbb{N}$ and an extension to dimension $d > 1$ can be found in [HK].
The approximations from Theorem 4.3, which we will refer to as *OCGC (optimized coarse grid correction) approximations*, are visualised in Figure 1 for the cases $d = 1$, $k = 2$ and $d = 2$, $k = 3$. Additionally, we display the quasi–optimal coarse grid approximations L according to Proposition 4.2.

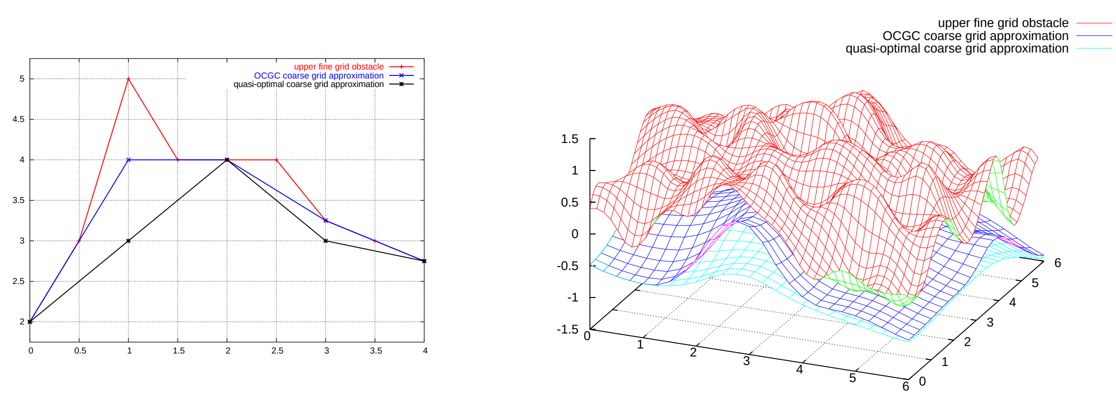

Figure 1: Left: Univariate continuous piecewise linear upper obstacle function on the fine grid $[0, 4] \cap \mathbb{Z}/2$ and coarse grid approximations from Proposition 4.2 and Theorem 4.3, respectively, on the coarse grid $[0, 4] \cap \mathbb{Z}$. Right: Two–dimensional C^1–smooth, quadratic, upper obstacle function defined on fine grid $[0, 6]^2 \cap (\mathbb{Z}/2)^2$ and coarse grid approximations from Proposition 4.2 and Theorem 4.3, respectively, on the coarse grid $[0, 6]^2 \cap \mathbb{Z}^2$.

The improvement of the OCGC approximation over the quasi–optimal approximation L is clearly visible. In the special case $k = 2$ the quasi–optimal coarse grid approximations from Proposition 4.2 coincide with the restrictions from [M] whereas the OCGC approximations recover the restriction operator from [K].

4.2 Truncated B–Spline Based Monotone Multigrid Methods

Within a time discretization, when solving instationary problems, we can expect that the asymptotic phase dominates the convergence behavior of the multigrid scheme since the solutions from the previous time steps can be used as good initial guesses.

However, standard monotone multigrid methods often suffer from unsatisfying convergence rates in their asymptotic convergence range. This is due to the fact that the demand for monotonicity excludes the use of many coarse grid functions near the free boundary in the correction step of the multigrid scheme which leads to a less efficient coarse grid transport.

As shown in [K], similar convergence rates as the unconstrained case can be achieved if the coarse grid basis functions are adapted in each iteration step to the actual position of the free boundary by a suitable truncation operator. This leads to the *truncated version (TrMMG)* of the monotone multigrid method. As described next, the B–spline based method can be modified in a similar way to obtain a truncated variant.

We denote by u_i, $i = 1, \ldots, \tilde{n}$, the B–spline coefficients of the smoothed approximation $\bar{u}_h^\nu$ in the ν–th iteration step of the monotone multigrid scheme and define the contact set of $\bar{u}_h^\nu$ by

$$K^\bullet(\bar{u}_h^\nu) := \{i \in \{1, \ldots, \tilde{n}\} : u_i = g_i\}$$

where g_i, $i = 1, \ldots, \tilde{n}$, denote the B–spline coefficients of the obstacle function g_h.

Truncated coarse grid B–spline basis functions can now be obtained by a recursive application of the following two steps. First, coarse grid basis functions are transferred to the next finer grid via (4). Then all B–spline coefficients of the transferred function with index $i \in K^\bullet(\bar{u}_h^\nu)$ are set to zero. To achieve such a truncation, the implementation of the monotone multigrid method has to be modified in the following way:

First, when computing restrictions of the stiffness matrix and of the residual via (4), all entries in the prolongation and restriction matrix are set to zero which correspond to active components $i \in K^\bullet(\bar{u}_h^\nu)$. Second, when constructing monotone and quasi–optimal coarse grid approximations of the obstacle function via Proposition 4.2 or Theorem 4.3, all B–spline coefficients of the obstacle functions with index $i \in K^\bullet(\bar{u}_h^\nu)$ are set to infinity in order to be less restrictive for the coarse grid corrections. Third, corrections are only added to components of the smoothed iterate $\bar{u}_h^\nu$ with index $i \in K^\bullet(\bar{u}_h^\nu)$.

According to the results obtained from the numerical experiments in [K, HK], such modifications lead to the most efficient monotone multigrid variants. They exhibit full multigrid efficiency in the asymptotic convergence range.

The modification of the obstacle function in the case $k = 3$ is illustrated in Figure 2. There, a quadratic upper obstacle function $g_h \in S_h$ with $S_h = \operatorname{span} \Sigma_h$ and $\Sigma_h = \{N_{1,3,1}, \ldots, N_{1,3,26}\}$ defined on the fine grid $[0, 13] \cap \mathbb{Z}/2$ and the corresponding OCGC approximations are displayed. The modifications within the truncated version are illustrated in the right figure where the contact set is set to be $K^\bullet(\bar{u}_h^\nu) = \{6, 7, \ldots, 10, 23\}$. One can see that smooth parts of the obstacle are

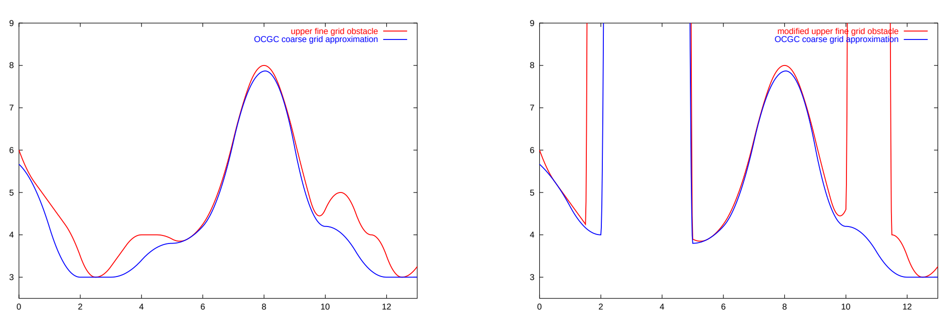

Figure 2: Left: C^1–smooth, quadratic upper obstacle function on the fine grid $[0, 13] \cap \mathbb{Z}/2$ with quadratic approximation according to Theorem 4.3 on grid $[0, 13] \cap \mathbb{Z}$. Right: Modified obstacle function and approximation where the contact set is set to be $K^\bullet(\bar{u}_h^\nu) = \{6, 7, \ldots, 10, 23\}$.

very well approximated, while variations of the obstacle of higher frequency can only be partly approximated as it is visible in the interval $[9, 13]$.

5 Numerical Example

To present a numerical example from Mathematical Finance, we choose the domain $\Omega := \mathbb{R}^+ \times [0, T)$, the function $\mathcal{H}(S) := \max\{K - S, 0\}$ and consider the free boundary value problem to find $V(S,t) \in H^1(\Omega)$ and the free boundary $S = S_f(t)$ such that for $0 \le t < T$,

$$
(19) \qquad \begin{aligned} \frac{\partial}{\partial t}V(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}V(S,t) + rS\frac{\partial}{\partial S}V(S,t) - rV(S,t) &= 0 \quad \text{for } S > S_f, \\ V(S,t) &= \mathcal{H}(S) \quad \text{for } S \le S_f, \end{aligned}
$$

with boundary data $V(S,t) = 0$ for $S \to \infty$, $V(S,t) = \mathcal{H}(S)$ for $S \to 0$, final data $V(S,T) = \mathcal{H}(S)$, and the conditions that V and $\partial V / \partial S$ are continuous on the free boundary S_f.

As it is shown in [WHD], the solution V describes the fair value of an *American put option* with strike price $K \in \mathbb{R}^+$, maturity $T \in \mathbb{R}^+$ and an underlying stock with value $S \in \mathbb{R}^+$ and volatility $\sigma \in \mathbb{R}^+$. The interest rate for risk free investments is denoted by $r \in \mathbb{R}^+$. If the obstacle condition is neglected the solution V describes the fair value of an *European put option.* In that case an analytical solution is given by the well-known Black–Scholes formula. Employing a time discretization, a weak form as in (1) of problem (19) can be derived [WHD, Hz2] which we solve by the finite element based scheme from Sections 2, 3 and 4.

In the numerical experiments we used the parameters $K = 10$, $T = 1$, $\sigma = 0.6$ and $r = 0.025$. Employing a Crank–Nicholson finite difference scheme for the time discretization and at least continuous piecewise finite elements for the space discretization, the method is known to converge quadratically.

Employing higher–order finite element functions, the derivatives of the solution V which provide important hedge parameters in the option pricing context can be determined by *direct differentiation* of the basis functions. To illustrate the difference a variable order k may offer, the pointwise errors of the second derivative $\Gamma := \frac{\partial^2 V}{\partial S^2}$ at time $t = 0$ are shown in Figure 3 for orders $k = 2, 3, 4$ in the case of 275×275 unknowns in time and space.

On the right hand side of Figure 3, the pointwise convergence rates to $\frac{\partial^2}{\partial S^2}V(K, 0)$ are displayed for our scheme with piecewise linear ($k = 2$) and piecewise cubic ($k = 4$) B–spline ansatz functions. Additionally, the convergence rate of a lattice scheme is plotted which was proposed by Leisen and Reimer [LR] and which was considered to be the superior method for the approximation of American option price sensitivities in a comparison of various pricing methods in [WW]. The costs (plotted on the x–axis) of the Leisen–Reimer scheme with m steps are of order $O(m^2)$, whereas the costs of our B–spline based finite element scheme with M time and N space steps have complexity $O(N\,M)$. One can see that the Leisen–Reimer scheme as well as the finite element scheme with hat functions exhibit a pointwise convergence rate of about $\rho = 1/2$. In contrast, the finite element scheme with piecewise cubic functions attains a much better convergence rate of nearly $\rho = 1$.

In the next experiments only one time step of problem (19) with a random initial guess is considered to analyze the performance of the multigrid scheme. In Figure 4, the iteration errors of the projective Gauss–Seidel scheme are displayed for different orders k and $M = 256$ unknowns in space. The impact of the order k is clearly visible.

Next we compare for the case of quadratic basis functions ($k = 3$) the convergence behavior of the following methods:

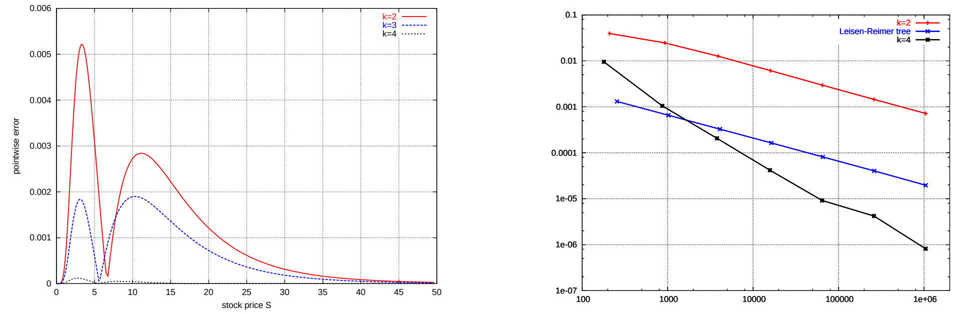

Figure 3: Left: Comparison of pointwise error for Gamma $\partial^2 V/\partial S^2(S,0)$ for orders $k = 2, 3, 4$ and 275×275 unknowns in time and space. Right: Pointwise convergence to Gamma $\partial^2 V/\partial S^2(K,0)$.

PSOR	projective Gauss–Seidel scheme
MMG	monotone multigrid method with OCGC approximation of the obstacle according to Theorem 4.3
TrMMG	truncated version of the monotone multigrid method with OCGC approximation of the obstacle according to Theorem 4.3
MMG (q–opt)	monotone multigrid method with quasi–optimal approximation of the obstacle according to Proposition 4.2
TrMMG (q–opt)	truncated version of the monotone multigrid method with quasi–optimal approximation of the obstacle according to Proposition 4.2
MG	linear multigrid method applied to the unrestricted problem

In the experiments we used the finest level $L = 7$ and only one smoothing step on each refinement level. The convergence behaviour is displayed in Figure 4. One can see that the truncated version

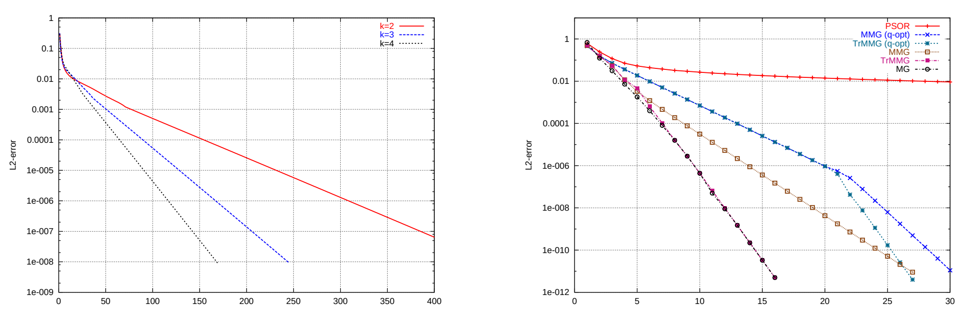

Figure 4: Iteration history of the projected Gauss–Seidel scheme (left) and of different MMG–variant based on piecewise quadratic, C^1–smooth B–spline basis functions (right).

TrMMG combined with the OCGC approximations is the fastest convergent variant. It exhibits, apart from a very slight variation in the first iterations which is caused by the search for the free boundary, almost the same convergence behavior as a linear multigrid method applied to the unrestricted problem; the lines overlap. For hat functions, this corresponds to the results in [K]. For the higher–order case, this indicates the quality of the obstacle approximations from Section 4.1 together with its enhancement by including truncated basis functions from Section 4.2. Estimated bounds for the asymptotic convergence rates for the truncated version TrMMG are listed in Table 1 for different numbers of smoothing steps on each refinement level.

η	1	2	3	4	5	6
ρ_∞	0.27	0.16	0.13	0.1	0.05	0.04

Table 1: Estimated asymptotic convergence rates for different numbers of smoothing steps η on each refinement level for the truncated version TrMMG with piecewise quadratic basis functions.

In summary, we recover the favorable convergence rates of standard multigrid schemes which are bounded for the truncated version TrMMG with $k = 3$ by $\rho_\infty \approx 0.27$ in the case of only one smoothing step on each refinement level.

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References

[BC] A. Brandt, C.W. Cryer, *Multigrid algorithms for the solution of linear complementary problems arising from free boundary problems*, SIAM J. Sci. Stat. Comput. 4, 655–684, 1983.

[BHR] F. Brezzi, W. Hager, P.A. Raviart, *Error estimates for the finite element solution of variational inequalities*, Numer. Math. 28, 431–443, 1977.

[C] C.W. Cryer, *The solution of a quadratic programming problem using systematic overrelaxation*, SIAM J. Control 9, 385–392, 1971.

[CP] N. Clarke, K. Parrott, *Multigrid for American option pricing with stochastic volatility*, Appl. Math. Finance 6, 177-195, 1999.

[H] W. Hackbusch, *Multi–Grid Methods and Applications*, Springer, 1985.

[Hg] K. Höllig, *Finite Element Methods with B-Splines*, SIAM, Frontiers in Applied Mathematics 26, 2003.

[HK] M. Holtz, A. Kunoth, *B–spline based monotone multigrid methods*, Manuscript, September 2004, submitted for publication, revised, October 2005.

[HM] W. Hackbusch, H.D. Mittelmann, *On multigrid methods for variational inequalities*, Numer. Math. 42, 65–76, 1983.

[Ho] R.H.W. Hoppe, *Multigrid algorithms for variational inequalities*, SIAM J. Numer. Anal. 24, 1046–1065, 1987.

[Hz1] M. Holtz, *Konstruktion B–Spline–basierter monotoner Mehrgitterverfahren zur Bewertung Amerikanischer Optionen* (in German), Diploma Thesis, Universität Bonn, May 2004.

[Hz2] M. Holtz, *The computation of American option price sensitivities using a monotone multigrid method for higher order B–spline discretizations*, Manuscript, October 2004, submitted for publication.

[K] R. Kornhuber, *Monotone multigrid methods for elliptic variational inequalities I*, Numer. Math. 69, 167–184, 1994.

[LR] D. Leisen, R. Reimer, *Binomial models for option valuation – examining and improving convergence*, Appl. Math. Finance 3, 319-346, 1996.

[M] J. Mandel, *A multilevel iterative method for symmetric, positive definite linear complementarity problems*, Appl. Math. Optimization 11, 77–95, 1984.

[O] C.W. Oosterlee, *On multigrid for linear complementarity problems with application to American–style options*, Electron. Trans. Numer. Anal. 15, 165-185, 2003.

[RW] C. Reisinger, G. Wittum, *On multigrid for anisotropic equations and variational inequalities*, Computing and Visualization in Science 7, 189–197, 2004.

[WHD] P. Wilmott, S. Howison, J. Dewynne, *The Mathematics of Financial Derivatives*, Cambridge, University Press, 1995.

[WW] C. Wallner, U. Wystup, *Efficient computation of option price sensitivities for options of American style*, Working paper, 2004.