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Euler Characteristic of Epi-Lipschitz Subsets of Riemannian Manifolds

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In this paper we present some properties of tangent and normal cones of epi-Lipschitz subsets of complete Riemannian manifolds. The fact that epi-Lipschitz subsets of complete Riemannian manifolds are absolute neighborhood retracts is proved. A notion of Euler characteristic of epi-Lipschitz subsets of complete Riemannian manifolds is introduced. Moreover, we provide a sufficient condition which ensures that the Euler characteristic of this class of sets is equal to one. Then, these results are applied to equilibrium theory on complete parallelizable Riemannian manifolds.

Keywords: Clarke subdifferential, Epi-Lipschitz sets, Euler characteristic, Riemannian manifolds

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1. Introduction

In 1978, Rockafellar [35, 36] introduced the concept of epi-Lipschitz subsets of finite dimensional Euclidean spaces. In [13, 15] Cornet and Czarnecki proved that every nonempty epi-Lipschitz subset of $\mathbb{R}^n$ can be defined by a nondegenerate Lipschitz inequality, and provided necessary and sufficient conditions for the existence of equilibria, generalized equilibria and fixed points for set valued mappings defined on epi-Lipschitz subsets of $\mathbb{R}^n$. Other applications to the marginal cost pricing are given in [6]. Some other works and applications dealing with the class of epi-Lipschitz subsets of linear spaces include those by Borwein, Lucet and Mordukhovich [8], Cornet [13], Jourani [24], Czarnecki and Rifford [17], Lorenz [32], Gudovich, Kamenskii and Quincampoix [20].

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In various aspects of mathematics such as equilibrium theory, optimization and matrix analysis, smooth and nonsmooth functions arise naturally on smooth manifolds; see, e.g. [27, 29, 31]. Unlike an Euclidean space a manifold in general does not have a linear structure and therefore new techniques are needed for dealing with functions defined on manifolds.

This paper deals with the class of epi-Lipschitz subsets of Riemannian manifolds which is of particular importance, since it includes closed convex sets with nonempty interior and sets defined by finite smooth inequality constraints satisfying a nondegeneracy assumption. In [22] the authors studied epi-Lipschitz subsets of complete Riemannian manifolds and obtained a characterization of this class of sets as follows.

Let M be a complete finite dimensional Riemannian manifold and S be a closed subset of M. Then the following assertions are equivalent.

(a) S is epi-Lipschitz.
(b) There is a locally Lipschitz function $\varphi : M \to \mathbb{R}$ such that
  (i) $S = \{x \in M : \varphi(x) \leq 0\}$,
  (ii) If $\varphi(x) = 0$, then $0 \notin \partial\varphi(x)$,
  (iii) $\partial S = \partial(\operatorname{int} S) = \{x \in M : \varphi(x) = 0\}$.

In this paper, we present some properties of tangent and normal cones of epi-Lipschitz subsets of complete Riemannian manifolds. Then, we prove that if $f : M \to \mathbb{R}$ is a locally Lipschitz function defined on a complete Riemannian manifold M and $[a, b]$ is a closed interval of real line with compact inverse image $f^{-1}[a, b]$, then $M_a = \{x \in M : f(x) \leq a\}$ is a neighborhood retract of $M_b = \{x \in M : f(x) \leq b\}$ provided that $[a, b]$ does not contain any critical value of f, that is, $a \leq f(x) \leq b \Rightarrow 0 \notin \partial f(x)$. It provides a nonsmooth generalization of the "noncritical neck principle" of Morse's theory on Riemannian manifolds. Moreover, it has a fundamental role to prove our results in Section 6.

Furthermore, we introduce a notion of Euler characteristic of an epi-Lipschitz subset S of a complete Riemannian manifold M. When $S \subset \mathbb{R}^n$ is a compact C^1 submanifold with boundary, the Euler characteristic of S is defined by the topological degree of the Gauss mapping G_S with respect to zero, see [33]. In the absence of smoothness assumptions on the set S, Cornet followed the same approach, with the only difference that the Gauss mapping was set valued. He defined the Euler characteristic of S by using the Cellina-Lasota degree of upper semicontinuous mappings with compact convex values, see [14]. In [16, Remark 2.13] it was proved that this notion of Euler characteristic is equal to the classical Euler characteristic. Following [14] we define a notion of Euler characteristic of epi-Lipschitz subsets of complete Riemannian manifolds. Our definition of this notion is a generalization of Cornet's definition and coincides with the classical one in Euclidean spaces.

It is worthwhile to mention that the results regarding Euler characteristic χ and equilibrium theory which are obtained in this paper, are not local. Moreover, if S is an epi-Lipschitz subset of a complete Riemannian manifold M with $\chi(S) \neq 0$, then for every open neighborhood U of M which $U \cap S \neq \emptyset$, $\chi(S \cap U)$ is not necessarily nonzero.

The rest of the paper is organized as follows. In Section 2, we introduce some basic definitions and notations. Section 3 is concerned with the properties of tangent and normal cones of epi-Lipschitz subsets of complete Riemannian manifolds. In Section 4, the fact that epi-Lipschitz subsets of complete Riemannian manifolds are absolute neighborhood retracts, is proved. Section 5 is devoted to the definition of topological degree of set valued mappings defined from n-dimensional manifold M to $2n$-dimensional manifold cotangent bundle TM^* or tangent bundle TM. In Section 6, we define the Euler characteristic of epi-Lipschitz sets in complete Riemannian manifolds. Then some results regarding this notion which are useful in equilibrium theory are proved. Moreover, a sufficient condition on the function φ defined in [22, Theorem 5.5] which ensures that the Euler characteristic of this class of sets is equal to one, is obtained. Finally, in Section 7 we provide sufficient conditions for the existence of equilibria for a class of set valued mapping F defined on a compact epi-Lipschitz subset S of a complete parallelizable Riemannian manifold M with valued in tangent bundle. Our assumption on the epi-Lipschitz set S is a topological one, which involves the Euler characteristic of the set S.

2. Preliminaries

In this paper, we use the standard notations and known results of Riemannian manifolds, see, e.g. [18, 25, 28]. Throughout this paper, M is an n-dimensional complete manifold endowed with a Riemannian metric $\langle .,. \rangle_x$ on the tangent space T_xM. As usual we denote by $B(x,\delta)$ the open ball centered at x with radius δ, by $\operatorname{int} N (\operatorname{cl} N)$ the interior (closure) of the set N. Also, let S be a nonempty closed subset of a Riemannian manifold M, we define $d_S : M \longrightarrow \mathbb{R}$ by

$$d_S(x) := \inf\{d(x,s) : s \in S\},$$

where d is the Riemannian distance on M. Moreover, $B(S,\varepsilon) := \{x \in M : d_S(x) \leq \varepsilon\}$. Recall that the set S in a Riemannian manifold M is called convex if every two points $p_1, p_2 \in S$ can be joined by a unique geodesic whose image belongs to S. For the point $x \in M$, $\exp_x : U_x \to M$ will stand for the exponential function at x, where U_x is an open subset of T_xM. Recall that $\exp_x$ maps straight lines of the tangent space T_xM passing through $0_x \in T_xM$ into geodesics of M passing through x. An n-dimensional parallelizable manifold M is a manifold of dimension n having vector fields $E_1, ..., E_n$ such that at any point p of M the tangent vectors $E_1(p), ..., E_n(p)$ provide a basis of the tangent space at p. An example with $n = 1$ is the circle: we can take E_1 to be the unit tangent vector field, say pointing in the anti-clockwise direction. More generally, any Lie group G is parallelizable. It is worthwhile to mention that if M is an n-dimensional parallelizable manifold, there are forms $w_1, ..., w_n$ such that at any point p of M the cotangent vectors $w_1(p), ..., w_n(p)$ provide a basis of the cotangent space at p.

We will also use the parallel transport of vectors along geodesics. Recall that for a given curve $\gamma : I \to M$, number $t_0 \in I$, and a vector $V_0 \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field $V(t)$ along $\gamma(t)$ such that $V(t_0) = V_0$. Moreover, the mapping defined by $V_0 \mapsto V(t_1)$ is a linear isometry between the tangent spaces $T_{\gamma(t_0)}M$ and $T_{\gamma(t_1)}M$, for each $t_1 \in I$. In the case when γ is a minimizing geodesic and $\gamma(t_0) = x, \gamma(t_1) = y$, we will denote this mapping by L_{xy}, and we will call it the parallel transport from T_xM to T_yM along the curve γ. Note that, L_{xy} is well defined when the minimizing geodesic which connects x to y, is unique. For example, the parallel transport L_{xy} is well defined when x and y are contained in a convex neighborhood. In what follows L_{xy} will be used wherever it is well defined. The isometry L_{yx} induces another linear isometry L^*_{yx} between T_xM^* and T_yM^*, such that for every $\sigma \in T_xM^*$ and $v \in T_yM$, we have $\langle L^*_{yx}(\sigma), v\rangle = \langle \sigma, L_{yx}(v)\rangle$. We will still denote this isometry by $L_{xy} : T_xM^* \to T_yM^*$.

Recall that a real valued function f defined on a Riemannian manifold M is said to satisfy a Lipschitz condition of rank k on a given subset S of M if $\mid f(x) - f(y) \mid \leq kd(x, y)$ for every $x, y \in S$, where d is the Riemannian distance on M. A function f is said to be Lipschitz near $x \in M$ if it satisfies the Lipschitz condition of some rank on an open neighborhood of x. A function f is said to be locally Lipschitz on M if f is Lipschitz near x, for every $x \in M$. Also, a set valued mapping $F : X \rightrightarrows Y$, where X, Y are topological spaces is said to be upper semicontinuous at x if for every open neighborhood U of $F(x)$, there exists an open neighborhood V of x, such that

$$y \in V \Longrightarrow F(y) \subseteq U.$$

Furthermore, a set valued mapping $F : X \rightrightarrows Y$, where X, Y are topological spaces, is said to be lower semicontinuous at x if for every open neighborhood U with $U \cap F(x) \neq \emptyset$, there exists an open neighborhood V of x, such that

$$y \in V \Longrightarrow F(y) \cap U \neq \emptyset.$$

It is worth pointing out that a set valued mapping $F : X \rightrightarrows Y$, where X, Y are topological spaces, is said to be lower semicontinuous (upper semicontinuous) if F is lower semicontinuous (upper semicontinuous) at every point $x \in X$.

3. Tangent and normal cones

We start with the definition of the generalized gradient of locally Lipschitz functions on Riemannian manifolds; for more details see [22].

Definition 3.1. Let M be a Riemannian manifold, $x \in M$ and let $f : M \to \mathbb{R}$ be a locally Lipschitz function. The generalized gradient of f at x, denoted by $\partial f(x)$ is defined as follows:

$$\partial f(x) = \partial(f \circ \exp_x)(0_x),$$

where $\partial(f \circ \exp_x)(0_x)$ is the generalized gradient of $f \circ \exp_x$ at 0_x as a locally Lipschitz function defined on a subset of $T_xM \cong \mathbb{R}^n$.

Remark 3.2. Let M be a finite dimensional Riemannian manifold and let $f : M \to \mathbb{R}$ be a locally Lipschitz function. Suppose that $(y, v) \in TM$ and $B_r(y)$ is a geodesic ball around y. Then the function $x \mapsto \sigma(L_{xy}(\partial f(x)), v)$ is upper semicontinuous on $B_r(y)$, where σ is the support function of the set $L_{xy}(\partial f(x))$. Hence for every $\gamma \in \mathbb{R}$ and $(y,v) \in TM$,

$$\{x \in B_r(y) : \ \gamma < \inf_{\xi \in \partial f(x)} \langle L_{yx}(v), \xi \rangle\}, \tag{1}$$

is an open subset of M, see [22].

Let us present some definitions and properties of normal and tangent cones.

Definition 3.3. Let S be a nonempty closed subset of a Riemannian manifold M, $x \in S$ and (φ, U) be a chart of M at x. Then the (Clarke) tangent cone to S at x, denoted by $T_S(x)$ is defined as follows:

$$T_S(x) := d\varphi(x)^{-1}[T_{\varphi(S\cap U)}(\varphi(x))],$$

where $T_{\varphi(S\cap U)}(\varphi(x))$ is tangent cone to $\varphi(S\cap U)$ as a subset of $\mathbb{R}^n$ at $\varphi(x)$.

Obviously, $0_x \in T_S(x)$ and $T_S(x)$ is closed and convex. In the case of submanifolds of $\mathbb{R}^n$, the tangent space and the normal space are orthogonal to one another. In an analogous manner, for a closed subset S of a Riemannian manifold M the normal cone to S at x, denoted by $N_S(x)$, is defined as the (negative) polar of the tangent cone $T_S(x)$, i.e.

$$N_S(x) := T_S(x)^\circ := \{\xi \in T_xM^* : \langle \xi, z\rangle \leq 0 \ \forall z \in T_S(x)\}.$$

An easy consequence of this definition is the following proposition.

Proposition 3.4.

(*a*) *$N_S(x)$ is a closed convex cone.*

(*b*) *$N_S(x) = d\varphi(x)^*(N_{\varphi(S\cap U)}(\varphi(x)))$, where $N_{\varphi(S\cap U)}(\varphi(x))$ is the normal cone to $\varphi(S\cap U)$ as a subset of $\mathbb{R}^n$ at $\varphi(x)$.*

For the definitions of tangent and normal cones to subsets of $\mathbb{R}^n$ see [12, 34].

The following theorem is a consequence of [26, Theorem 1.4.1].

Theorem 3.5. *Let S be a closed convex subset of a Riemannian manifold M, then*

(*a*) *$N_S(x) = \{\xi \in T_xM^* : \ \langle \xi, \exp_x^{-1}(y)\rangle \leq 0$ for every $y \in S\}$.*

(*b*) *$T_S(x) = \mathrm{cl}\{\lambda \exp_x^{-1}(y) : \ \lambda \geq 0, \ y \in S\}$.*

Let us define epi-Lipschitz sets in Riemannian manifolds and present some properties of tangent and normal cones to epi-Lipschitz sets in complete Riemannian manifolds.

Definition 3.6. Let M be a Riemannian manifold. The subset $S \subset M$ is said to be epi-Lipschitz if at every point $x \in S$, $N_S(x) \cap (-N_S(x)) = \{0\}$.

Example 3.7. Every closed convex set S with nonempty interior in a manifold M is epi-Lipschitz.

Theorem 3.8. *Let M be a complete Riemannian manifold and S be an epi-Lipschitz closed set in M. Then,*

(a) *the set valued mapping $x \mapsto N_S(x)$ is upper semicontinuous on S.*
(b) *the set valued mappings $x \mapsto \operatorname{int} T_S(x)$ and $x \mapsto T_S(x)$ are lower semicontinuous on S.*

To prove Theorem 3.8, we need the following lemma from [22].

Lemma 3.9. *Let M be an n-dimensional manifold. Consider the set valued mapping $G : M \rightrightarrows TM^*$ such that $G(x) \subseteq T_xM^*$ for every $x \in M$. Suppose that in a chart (ψ, W) at $x \in M, G$ is represented by*

$$G(y) = \left\{ \sum_{i=1}^{n} g_i(y) dx_i|_y : \ \{dx_i|_y\} \text{ is a local basis of } T_yM^* \text{ in the chart } (\psi, W) \right\}.$$

Then, G is an upper semicontinuous (lower semicontinuous) mapping at x if and only if $g : W \rightrightarrows \mathbb{R}^n$ defined by $g(y) := \{(g_1(y), g_2(y), \ldots, g_n(y))\}$ is upper semicontinuous (lower semicontinuous) at x.

Analogously, this lemma can be proved when the set valued mapping $G : M \rightrightarrows TM$ is defined such that $G(x) \subseteq T_xM$ for every $x \in M$.

Now we come to the proof of Theorem 3.8.

Proof. We just prove part (a). The proof of part (b) is similar. Let M be an n-dimensional manifold and (φ, U) be a chart of an arbitrary point x. For each $z \in U \cap S$, we define

$$T(z) := N_{\varphi(U\cap S)}(\varphi(z)).$$

Note that $\varphi(U \cap S)$ is an epi-Lipschitz subset of $\mathbb{R}^n$, so the mapping

$$F : \varphi(U \cap S) \rightrightarrows \mathbb{R}^n,$$

defined by $F(y) := N_{\varphi(U\cap S)}(y)$ is upper semicontinuous on $\varphi(U \cap S)$ and $T(z) = F \circ \varphi(z)$. Thus, T is upper semicontinuous on $U \cap S$. On the other hand

$$N_S(x) = d\varphi(x)^*[N_{\varphi(U\cap S)}(\varphi(x))],$$

and

$$d\varphi(x)^* \left[\sum_{i=1}^{n} \pi_i(N_{\varphi(U\cap S)}(\varphi(x)))e_i \right] = \sum_{i=1}^{n} \pi_i(N_{\varphi(U\cap S)}(\varphi(x)))d\varphi(x)^*(e_i),$$

where $\{e_i\}$ is a basis of $\mathbb{R}^n$ and π_i is the projection function on the ith coordinate. Since $\{d\varphi(y)^*(e_i)\}$ is a local basis of T_yM^* in the chart (φ, U) and T is upper semicontinuous at x, it follows from Lemma 3.9 that N_S is upper semicontinuous at x. $\square$

4. Main results

In this section, we prove that if $f : M \to \mathbb{R}$ is a locally Lipschitz function defined on a complete Riemannian manifold M and $[a, b]$ is a closed interval of real line with compact inverse image $f^{-1}[a, b]$, then $M_a = \{x \in M : f(x) \leq a\}$ is a neighborhood retract of $M_b = \{x \in M : f(x) \leq b\}$ provided that $[a, b]$ does not contain any critical value of f, that is, $a \leq f(x) \leq b \Rightarrow 0 \notin \partial f(x)$. Then we provide a nonsmooth generalization of the "noncritical neck principle" of Morse's theory.

Let us recall the notion of convexity radius of a Riemannian manifold which plays an important role in the rest of the paper.

Theorem 4.1 (Whitehead). *Let M be a Riemannian manifold. For every $x \in M$, there exists $c > 0$ such that for all $r > 0$ with $0 < r < c$, the open ball $B(x, r) = \exp_x B(0_x, r)$ is convex.*

The convexity radius of a Riemannian manifold M at a point $x \in M$ is the supremum of the numbers $r > 0$ such that ball $B(x, r)$ is convex. We denote this supremum by $c(x, M)$. The global convexity radius of M is defined by $c(M) = \inf\{c(x, M) : x \in M\}$. By Whitehead's theorem $c(x, M) > 0$, for every $x \in M$. On the other hand, it is well known that the function $x \mapsto c(x, M)$ is continuous on M [25, Corollary 1.9.10]. Therefore, if $K \subseteq M$ is compact, then $c(K) > 0$.

The following remark contains some properties of parallel transport which will be used in the rest of the paper.

Remark 4.2. Let M be a Riemannian manifold.

(a) An easy consequence of the definition of the parallel translation along a curve as a solution to an ordinary linear differential equation, implies that the mapping

$$(x, \xi) \in T_xM \mapsto L_{xx_0}(\xi),$$

where x is in a neighborhood of x_0 is well defined and continuous at (x_0, ξ_0), that is, if $(x_n, \xi_n) \to (x_0, \xi_0)$ in TM, then $L_{x_nx_0}(\xi_n) \to L_{x_0x_0}(\xi_0) = \xi_0$, for every $(x_0, \xi_0) \in TM$; see [2, Remark 6.11].

(b) By the continuity properties of the parallel transport and the geodesics, see [3, Theorem 35], for fixed point z and for each $\varepsilon > 0$, there exists a number $\delta > 0$ such that:

$$\|L_{xy}L_{zx} - L_{zy}\| \leq \varepsilon \text{ provided that } d(x, y) < \delta.$$

(c) Utilizing the properties of the exponential map on M, for fixed point $x \in M$ and for each $\varepsilon > 0$, we may find number $\delta_x > 0$ such that:

$$\|d(\exp_x^{-1})(y) - L_{yx}\| \leq \varepsilon \text{ provided that } d(x, y) < \delta_x.$$

(d) Recall that the exponential map is defined on an open subset U of TM. Define a new map $F : U \to M \times M$ by

$$F(q, v) = (q, \exp_q v).$$

Then by the inverse function theorem, F is diffeomorphism from some open neighborhood V of $(x, 0)$ to its image [30, Lemma 5.12]. Therefore, one can find a geodesic ball $B(x, r)$ around x such that for every $p \in B(x, r)$ the function $f_{p,x} : B(x, r) \to T_x M$ defined by

$$f_{p,x}(y) = L_{yx}(d(\exp_y^{-1})(p)),$$

is continuous at x.

Now, we define the notion of locally Lipschitz vector fields on Riemannian manifolds.

Definition 4.3. Let M be a Riemannian manifold. A mapping $X : M \to TM$ satisfying $X_y \in T_y M$ for all $y \in M$ is said to be Lipschitz vector field of rank k near a given point $x \in M$, if for some $\varepsilon > 0$, we have

$$\|L_{yz} X(y) - X(z)\| \leq k d(y, z) \quad \text{for all } z, y \in B(x; \varepsilon),$$

where $B(x; \varepsilon)$ is convex, and L_{yz} is parallel transport along the unique geodesic connecting z and y.

Note that if we consider the Riemannian metrics on M and TM, then the above definition is equivalent to the usual definition of locally Lipschitz functions on metric spaces, see [9, p. 241]. Any two Riemannian metrics being each bounded locally by a constant multiple of the other, give equivalent concepts of Lipschitz continuity though not the same local Lipschitz constant.

Remark 4.4. If X is a smooth vector field on a compact subset of a Riemannian manifold M. Then, k defined by

$$k = \sup \frac{\|L_\gamma(X(\gamma(0))) - X(\gamma(1))\|}{\operatorname{length} \gamma},$$

is finite, where $\gamma : [0, 1] \to M$ varies over all geodesics and L_γ is parallel transport along γ. For more details see [9, p. 241].

In all the following, suppose that a, b are real numbers, $a < b$, and let

$$M_a = \{x \in M : \ f(x) \leq a\}, \qquad M_b = \{x \in M : \ f(x) \leq b\},$$
$$M_{ab} = f^{-1}([a, b]).$$

Theorem 4.5. *Let $f : M \to \mathbb{R}$ be a locally Lipschitz function on a complete Riemannian manifold M. Assume that a, b are real numbers, $a < b$ such that the set M_{ab} is nonempty and compact and for all $x \in M_{ab}$, 0 does not belong to $\partial f(x)$. Then*

(*a*) *there exists a neighborhood $\widetilde{M}$ of M_b and a locally Lipschitz mapping r from $\widetilde{M}$ to M_a such that*

(*i*) *$r(x) = x$, for all $x \in M_a$,* (*ii*) $f(r(x)) = a$, *for all* $x \in \widetilde{M} \setminus M_a$.

(*b*) *there exists* $\varepsilon \in (0, b-a)$ *such that for all* x *in* $f^{-1}((a, a+\varepsilon])$ *and* y *in* $f^{-1}([a, a+\varepsilon])$, *with* $r(x) = r(y)$,

$$\langle \exp_x^{-1} r(x), L_{yx}(\xi) \rangle < 0, \quad \textit{for all } \xi \in \partial f(y).$$

We prepare the proof of Theorem 4.5 by the following lemmas; see [5].

Lemma 4.6. *Under the assumptions of Theorem* 4.5, *there exists a bounded open neighborhood* Ω *of* M_{ab}, *two positive real numbers* β, θ *and smooth vector field* $X : \Omega \to T\Omega$ *such that,*

$$0 < \beta < \inf_{\xi \in \partial f(x)} \langle \xi, L_{yx}(X(y)) \rangle, \quad \textit{for all } x, y \in \Omega, d(x,y) \leq \theta.$$

Proof. Set $c = c(B(M_{ab}, 1))$. We claim that there exist numbers $0 < \varepsilon \leq 1$ and $\alpha > 0$ such that for all $x \in B(M_{ab}, \varepsilon)$, the set

$$A(x, \alpha) := \{(y,u) \in TM : d(y,x) < c/3, \ \alpha < \inf_{\xi \in \partial f(x)} \langle L_{yx}(u), \xi \rangle\}$$

is nonempty. On the contrary, assume that it is not true. Then there exists a sequence $\{x_q\}$ such that for all $q, x_q \in B(M_{ab}, 1/q)$ and the set $A(x_q, 1/q)$ is empty. Since the set $B(M_{ab}, 1)$ is compact, without loss of generality, we can suppose that $\{x_q\}$ converges to some element $\bar{x}$, which clearly belongs to M_{ab}. Since 0 does not belong to $\partial f(\bar{x})$, a nonempty, closed, convex subset of $T_{\bar{x}}M^*$, by a separation theorem, there exists $(\bar{x}, \bar{u})$ in $T_{\bar{x}}M$ and a real number γ such that $0 < \gamma < \inf_{\xi \in \partial f(\bar{x})} \langle \bar{u}, \xi \rangle$. It follows from Remark 3.2 that the set

$$\{x \in B(\bar{x}, c/3) : \ \gamma < \inf_{\xi \in \partial f(x)} \langle L_{\bar{x}x}(\bar{u}), \xi \rangle\}$$

is open and contains $\bar{x}$. Hence, for q large enough, the set $A(x_q, 1/q)$ is nonempty. Thus we arrive at a contradiction which ends the proof of the claim.

We now let $\varepsilon > 0$ and $\alpha > 0$ be defined as in the above claim, we also let $\Omega := \operatorname{int} B(M_{ab}, \varepsilon)$, and for $(y,u) \in TM$, we define

$$\begin{aligned} A^{-1}(y,u) &:= \{x \in M : (y,u) \in A(x, \alpha)\} \\ &= \{x \in B(y, c/3) : 0 < \alpha < \inf_{\xi \in \partial f(x)} \langle L_{yx}(u), \xi \rangle\}. \end{aligned}$$

By Remark 3.2, $A^{-1}(y,u)$ is an open subset of M. From the above claim,

$$B(M_{ab}, \varepsilon) \subseteq \cup_{(y,u) \in TM} A^{-1}(y,u)$$

and, from the compactness of $B(M_{ab}, \varepsilon)$, there exists a finite number of elements such that

$$B(M_{ab}, \varepsilon) \subseteq V := \cup_{i=1}^{n} A^{-1}(y_i, u_i).$$

Hence, there exists a C^∞ partition of unity $\{\lambda_i : \ i = 1, ..., n\}$ subordinate to open covering $\{A^{-1}(y_i, u_i) : i = 1, ..., n\}$ of V. We now define, for all i, the set

$$K_i = B(M_{ab}, \varepsilon) \cap \operatorname{supp} \lambda_i,$$

where $\operatorname{supp} \lambda_i := \overline{\{x \in V : \lambda_i(x) > 0\}}$. Then K_i is compact and is a subset of $A^{-1}(y_i, u_i)$.

Since $A^{-1}(y_i, u_i)$ is an open set, for each $x \in A^{-1}(y_i, u_i)$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subseteq A^{-1}(y_i, u_i)$. Hence, $K_i \subseteq \cup_{x \in A^{-1}(y_i,u_i)} B(x, \delta_x/2)$. Extracting a finite subcover by the compactness of K_i, we find points $\{x^i_j : j = 1, ..., n\}$ such that $K_i \subseteq \cup_{j=1}^n B(x^i_j, \delta_{x^i_j}/2)$. Therefore, considering $\eta := \frac{1}{2} \min\{\delta_{x^i_j}\}$, and using the continuity of the distance function on the compact set K_i, one has

$$B(K_i, \eta) \subset A^{-1}(y_i, u_i), \quad \text{for } i = 1, \ldots, n.$$

Since M_{ab} is compact it follows that for $x \in K_i \subseteq A^{-1}(y_i, u_i)$, $i \in \{1, \ldots, n\}$, there exists $s \in M_{ab}$ such that $d_{M_{ab}}(x) = d(x, s)$. Hence

$$d_{M_{ab}}(y_i) \leq d(s, y_i) \leq d(x, y_i) + d(x, s) \leq \frac{c}{3} + \varepsilon.$$

Without any loss of generality one can suppose that $\varepsilon + \frac{c}{3} < 1$, then $y_i \in B(M_{ab}, 1)$ and $B(y_i, c)$ is convex. Now for any $i \in \{1, \ldots, n\}$, let $y \in B(y_i, c/3)$, $v_i = \exp^{-1}_{y_i}(y)$ and let $\gamma_i(t) = \exp_{y_i}(tv_i)$ be the unique geodesic in $B(y_i, c/2)$ joining y_i to y. Then the map $g_i : B(y_i, c/2) \to TM$ defined by $g_i(y) := L_{y_i y}(u_i)$ obtained by parallel translation of u_i from y_i to y along γ_i is smooth; see [19, p. 148]. Hence, the map $X : V \to TM$ defined by

$$X(y) = \sum_{i=1}^{n} \lambda_i(y) L_{y_i y}(u_i),$$

is a smooth vector field.

Since $\bar{\Omega}$ is compact and f is locally Lipschitz on M, it follows that there exists $k > 0$ such that for every $x \in \Omega$ and every $\xi \in \partial f(x)$ one has $\|\xi\|_* < k$. Let $m := \max\{\|u_i\| : i = 1, ..., n\}$ and choose $\varepsilon_0 > 0$ satisfying $\varepsilon_0 < \min\{1, \frac{\alpha}{2mk+2m}\}$. Then by Remark 4.2, there exists $\eta_1 > 0$ such that for each $i \in \{1, ..., n\}$,

$$\|L_{yx} L_{y_i y} - L_{y_i x}\| \leq \varepsilon_0 \ \text{ provided that } d(x, y) < \eta_1 \text{ and } x, y \in A^{-1}(y_i, u_i). \qquad (2)$$

Now let $x, y \in \Omega$ with $d(x, y) < \theta := \min\{\eta, \eta_1\}$ and let $\mathcal{I}(y) := \{i : \lambda_i(y) > 0\}$. Then for $i \in \mathcal{I}(y)$, we have $x \in B(K_i, \eta) \subset A^{-1}(y_i, u_i)$. Hence $(y_i, u_i) \in A(x, \alpha)$, which implies $\alpha < \inf_{\xi \in \partial f(x)} \langle L_{y_i x}(u_i), \xi \rangle$. It follows from (2) that for each $\xi \in \partial f(x)$,

$$|\langle \xi, L_{yx} L_{y_i y}(u_i) - L_{y_i x}(u_i) \rangle| < \|\xi\|_* \|L_{yx} L_{y_i y} - L_{y_i x}\| \|u_i\| < k\varepsilon_0 m.$$

Hence

$$\alpha < \inf_{\xi \in \partial f(x)} \langle \xi, L_{y_i x}(u_i) \rangle < \inf_{\xi \in \partial f(x)} \langle \xi, L_{yx} L_{y_i y}(u_i) \rangle + k\varepsilon_0 m.$$

Due to the choice of ε_0, if we let $\beta := \alpha - 2m\varepsilon_0 k - \varepsilon_0 m - \varepsilon_0^2 m$, then we get

$$0 < \beta < \inf_{\xi \in \partial f(x)} \langle \xi, L_{yx} L_{y_i y}(u_i) \rangle.$$

Now, for each $\xi \in \partial f(x)$, we have $\langle \xi, L_{yx}(X(y)) \rangle = \sum_{i=1}^{n} \lambda_i(y) \langle \xi, L_{yx} L_{y_i y}(u_i) \rangle$. Hence

$$\begin{aligned} 0 < \beta = \sum_{i=1}^{n} \lambda_i(y)\beta &< \sum_{i=1}^{n} \lambda_i(y) \inf_{\xi \in \partial f(x)} \langle \xi, L_{yx} L_{y_i y}(u_i) \rangle \\ &\leq \inf_{\xi \in \partial f(x)} \langle \xi, L_{yx}(X(y)) \rangle. \end{aligned}$$

As required. □

Let $X : \Omega \to T\Omega$ be defined as in Lemma 4.6 and for $x \in \Omega$, let $\phi(., x)$ denote the maximal integral curve of X passing through x, that is, $\phi(., x)$ is the solution of

$$\dot{x}(t) = X(x(t)), \qquad x(0) = x, \tag{3}$$

with the maximal interval of definition $I(x)$.

Lemma 4.7. *Let Ω and $\beta > 0$ be as in Lemma 4.6. Then for all $x \in \Omega$ and for all $t_1, t_2 \in I(x)$ with $t_2 \geq t_1$*

$$f(\phi(t_2, x)) - f(\phi(t_1, x)) \geq \beta(t_2 - t_1).$$

Proof. Since the map ϕ is C^∞, for all $x \in \Omega$ the function $t \mapsto g(t) := f(\phi(t, x))$ is locally Lipschitz on $I(x)$. It follows from [22, Theorem 3.3] that for all $t_1, t_2 \in I(x)$ with $t_2 > t_1$,

$$g(t_2) - g(t_1) \in \{u(t_2 - t_1) : u \in \partial g(t),\ t \in (t_1, t_2)\}.$$

On the other hand by [22, Theorem 3.2] we have that

$$\begin{aligned} \partial g(t) &\subseteq \overline{\text{co}}\{\partial \langle u, \exp^{-1}_{\phi(t,x)} \circ \phi(., x)\rangle(t) :\ u \in \partial f(\phi(t, x))\} \\ &\subseteq \overline{\text{co}}\{\langle u, \dot{\phi}(t, x)\rangle :\ u \in \partial f(\phi(t, x))\} \\ &= \{\langle u, \dot{\phi}(t, x)\rangle :\ u \in \partial f(\phi(t, x))\} \\ &= \{\langle u, X(\phi(t, x)) :\ u \in \partial f(\phi(t, x))\rangle\}. \end{aligned}$$

Therefore by Lemma 4.6 $\partial g(t) \subset [\beta, , +\infty)$, which implies

$$g(t_2) - g(t_1) = f(\phi(t_2, x)) - f(\phi(t_1, x)) \geq \beta(t_2 - t_1). \qquad \square$$

Lemma 4.8. *There exists a compact neighborhood K of M_{ab} and a Lipschitz function $\tau : K \to \mathbb{R}$ such that,*

(*i*) $\tau(x) \in I(x)$ *and* $|\tau(x)| \leq |a - f(x)|/\beta$,

(*ii*) $f(\phi(\tau(x), x)) = a$, *for all* $x \in K$.

Proof. Applying the same argument as the one in [5, p. 325] one can find numbers a', b' such that $a' < a < b < b'$ and the set $K = M_{a'b'} \cap \Omega$ is closed in M. Since M is complete the latter implies that K is compact. Also, one can prove that, for all $x \in K$, there exists a unique real number $\tau(x)$ in $I(x)$ such that $|\tau(x)| \leq |a - f(x)|/\beta$ and $f(\phi(\tau(x), x)) = a$.

It remains to show that the function τ is Lipschitz. First, we prove that the restriction of the function $\tau : K \to \mathbb{R}$ to the compact set $K_+ = \{x \in K : f(x) \geq a\}$ is locally Lipschitz. By Remark 4.4, for every $x \in K$, there is a convex ball $B(x, \varepsilon)$ such that X is Lipschitz on $B(x, \varepsilon)$ of rank

$$k_2 = \sup \frac{\|L_\gamma(X(\gamma(0))) - X(\gamma(1))\|}{\operatorname{length} \gamma},$$

where $\gamma : [0, 1] \to M$ varies over all geodesics, and L_γ is the parallel transport along γ, and $K \subset \bigcup_{j=1}^{l} B(x_j, \varepsilon_j/2)$. Let $\epsilon = \min\{\varepsilon_j/2, c(K_+)/2\}$ then for each $y \in K_+$, there exists $B(y, \delta_y)$ such that if $x_1, x_2 \in B(y, \delta_y)$ and $t \in I(x_1) \cap I(x_2)$, then $d(\phi(t, x_1), \phi(t, x_2)) < \epsilon$.

Let x_1, x_2 be two arbitrary points in $B(y, \delta_y) \cap K_+$ and we let $t_1 = \tau(x_1)$ and $t_2 = \tau(x_2)$. Since $f(\phi(\tau(x_1), x_1)) = a \leq f(x_1) = f(\phi(0, x_1))$, by Lemma 4.7 $\tau(x_1) = t_1 \leq 0$, similarly is t_2. Without any loss of generality, we can suppose that $t_1 \leq t_2 \leq 0$, so that $|\tau(x_1) - \tau(x_2)| = t_2 - t_1$. Since $f(\phi(t_1, x_1)) = f(\phi(t_2, x_2)) = a$, from Lemma 4.7 one gets

$$\beta|\tau(x_2) - \tau(x_1)| \leq f(\phi(t_2, x_1)) - f(\phi(t_1, x_1)) = f(\phi(t_2, x_1)) - f(\phi(t_2, x_2)).$$

From Lemma 4.7, $\phi(t_2, x_2) \in M_{a'b'} \cap \Omega \subset K$ and also $\phi(t_2, x_1) \in M_{ab'} \cap \Omega \subset K$, $a = f(\phi(t_1, x_1)) \leq f(\phi(t_2, x_1)) \leq f(\phi(0, x_1)) \leq b'$. Since f is locally Lipschitz on M and K is compact, then f is Lipschitz on K with constant k_1. consequently,

$$\beta|\tau(x_2) - \tau(x_1)| \leq k_1 d(\phi(t_2, x_1), \phi(t_2, x_2)),$$

also there exists $B(x_j, \varepsilon_j/2)$ such that $\phi(t, x_1), \phi(t, x_2) \in B(x_j, \varepsilon_j)$, we end the proof by showing that

$$d(\phi(t_2, x_1), \phi(t_2, x_2)) \leq k\, d(x_1, x_2),$$

with

$$k = \exp[k_2.(b' - a)/\beta].$$

Indeed, since $t_1 \leq t_2 \leq 0$, for all $t \in [t_2, 0]$, from Lemma 4.7 both $\phi(t, x_2)$ and $\phi(t, x_1)$ belong to $M_{ab'} \cap \Omega \subset K$, also $d(\phi(t, x_1), \phi(t, x_2)) < c(K_+)/2$. Define

$$\psi(t) = d^2(\phi(t, x_2), \phi(t, x_1)) \exp(2k_2 t).$$

Then, there exists $v \in T_{\phi(t,x_1)}M$ such that $\exp_{\phi(t,x_1)}(v) = \phi(t, x_2)$ and $\gamma(t) = \exp_{\phi(t,x_1)}(tv)$ is the unique minimal geodesic connecting $\phi(t, x_1)$ and $\phi(t, x_2)$. By

[2, Lemma 6.5],

$$L_{\phi(t,x_2)\phi(t,x_1)}\left(\frac{\partial d}{\partial y}(\phi(t,x_1),\phi(t,x_2))\right) = -\frac{\partial d}{\partial x}(\phi(t,x_1),\phi(t,x_2)),$$
$$L_{\phi(t,x_2)\phi(t,x_1)}\left(\frac{\partial d}{\partial y}(\phi(t,x_1),\phi(t,x_2))\right) = \frac{v}{\|v\|}.$$

Now, $\frac{d}{dt}d^2(\phi(t,x_1),\phi(t,x_2))$ is equal to

$$2d(\phi(t,x_1),\phi(t,x_2))\left(\frac{-v}{\|-v\|}, L_{\phi(t,x_1)\phi(t,x_2)}\frac{v}{\|v\|}\right)(X(\phi(t,x_1)),X(\phi(t,x_2)))$$
$$= 2d(\phi(t,x_2),\phi(t,x_1))\left\langle \frac{v}{\|v\|}, L_{\phi(t,x_2)\phi(t,x_1)}(X(\phi(t,x_2)) - X(\phi(t,x_1))\right\rangle.$$

Since X is k_2-Lipschitz on $B(x_j,\varepsilon_j)$, so

$$\left|\left\langle \frac{v}{\|v\|}, L_{\phi(t,x_2)\phi(t,x_1)}(X(\phi(t,x_2))) - X(\phi(t,x_1))\right\rangle\right| \le k_2\, d(\phi(t,x_2),\phi(t,x_1)).$$

Thus,

$$\frac{d}{dt}\psi(t) \ge \exp(2k_2t).\, 2d^2(\phi(t,x_1),\phi(t,x_2))(-k_2)$$
$$+ \exp(2k_2t)\, 2k_2d^2(\phi(t,x_1),\phi(t,x_2)),$$

which means, $\frac{d}{dt}\psi(t) \ge 0$, so that $\psi(t)$ is increasing and

$$d^2(\phi(t_2,x_1),\phi(t_2,x_2)) \le \exp(-2k_2t_2)d^2(\phi(0,x_1),\phi(0,x_2)) \le k^2\, d^2(x_1,x_2),$$

that proves τ is locally Lipschitz on compact set K_+, so is Lipschitz on K_+. Similarly, one shows that the restriction of τ to the compact set $K_- = \{x \in K : f(x) \le a\}$ is also Lipschitz. Consequently, τ is Lipschitz by the following lemma. □

Lemma 4.9. *Let M and N be two Riemannian manifolds. Assume that A_1, A_2 are two closed subsets of M and $g_1 : A_1 \to N$, $g_2 : A_2 \to N$ are two locally Lipschitz functions. Moreover, $g_1(x) = g_2(x)$ provided that $x \in A_1 \cap A_2$. If $A_1 \cap A_2 \subset \operatorname{int}(A_1 \cup A_2)$, then $g : A_1 \cup A_2 \to N$ defined by*

$$g(x) = \begin{cases} g_1(x) & x \in A_1 \\ g_2(x) & x \in A_2, \end{cases}$$

is a locally Lipschitz function.

Proof. We give the proof only for the case that y is a boundary point of A_1 and A_2, other cases are left to the reader. For this purpose, set $\delta := \min\{\delta_1,\delta_2,\delta_3\}$, where

g_1 is Lipschitz on $A_1 \cap B(y,\delta_1)$, g_2 is Lipschitz on $A_2 \cap B(y,\delta_2)$, and $B(y,\delta_3) \subset A_1 \cup A_2$, also $B(y,\delta)$ is convex. We proceed to show that g is Lipschitz on $B(y,\delta)$. Suppose that z, w are two arbitrary points in $B(y,\delta)$ and $\gamma : [0,1] \to B(y,\delta)$ is the unique minimal geodesic which connects z and w. Therefore, $\lim_{t\to 1^-} \gamma(t) = z$ and $\lim_{t\to 0^+} \gamma(t) = w$. For the cases, $z, w \in A_1$ or $z, w \in A_2$, the proof is straightforward. Assume that, $z \in A_2 \setminus A_1$ and $w \in A_1 \setminus A_2$, then $B(y,\delta) \cap A_1^c$ and $B(y,\delta) \cap A_2^c$ are open sets respectively containing z and w. If we set

$$\Theta_1 = \sup\{\theta_1 > 0 : \ \gamma(t) \in B(y,\delta) \cap A_1^c \text{ for } -\theta_1 < t-1 < 0\},$$
$$\Theta_2 = \sup\{\theta_2 > 0 : \ \gamma(t) \in B(y,\delta) \cap A_2^c \text{ for } 0 < t < \theta_2\},$$

then,

$$\gamma(t) \in A_1 \cap A_2 \cap B(y,\delta) \ \text{ provided that } \Theta_2 \le t \le -\Theta_1 + 1.$$

It is sufficient to use $z_0 = \gamma(t_0)$ in $A_1 \cap A_2 \cap B(y,\delta)$, then

$$\begin{aligned} d(g(z), g(w)) &\le d(g(z), g(\gamma(t_0))) + d(g(w), g(\gamma(t_0))) \\ &= d(g_2(z), g_2(\gamma(t_0))) + d(g_1(w), g_1(\gamma(t_0)) \\ &\le k_1 d(\gamma(1), \gamma(t_0)) + k_2 d(\gamma(0), \gamma(t_0)) \\ &\le \ \max\{k_1,k_2\} \left(\int_0^{t_0} \|d\gamma(t)\| dt + \int_{t_0}^1 \|d\gamma(t)\| dt \right) \\ &= \ \max\{k_1,k_2\} \int_0^1 \|d\gamma(t)\| dt = \max\{k_1,k_2\} d(z,w), \end{aligned}$$

where for $i = 1, 2$, k_i is Lipschitz constant of g_i on $B(y,\delta)$. □

We now come to the proof of Theorem 4.5.

Proof. (a) Let Ω, a', b' be defined as in Lemma 4.8, and let $\widetilde{M} = \{x \in M : \ f(x) \le a\} \cup \{x \in \Omega : \ a \le f(x) \le b'\} = M_a \cup K_+$. Hence $\widetilde{M}$ is a neighborhood of M_b. Define $r : \widetilde{M} \to M_a$ by

$$r(x) = \begin{cases} \phi(\tau(x), x) & \text{if } x \in K_+ \\ x & \text{if } f(x) \le a. \end{cases}$$

Since $M_a \cap K_+ = \{x \in M : \ f(x) = a\} \subset \operatorname{int} \widetilde{M}$, it follows from Lemma 4.8 and Lemma 4.9 that r is locally Lipschitz and the proof of part (a) is complete.

(b) For ε_0 in the proof of Lemma 4.6, there exists η' such that (see [22, Theorem 2.10])

$$L_{yx}(\partial f(y)) \subseteq \partial f(x) + \varepsilon_0 B_{T_xM^*} \ \text{ provided that } d(x,y) < \eta'.$$

Now, for each $x \in K$, there exists δ_x with $0 < \delta_x \le c(K)/2$,

$$\|L_{xp}L_{qx} - L_{qp}\| < \varepsilon_0/3 \ \text{ provided that } d(x,p) < \delta_x, \tag{4}$$

$$\|L_{xp}(d(\exp_x^{-1})(q)) - d\exp_p^{-1}(q)\| < \varepsilon_0/3 \ \text{ provided that } d(x,p) < \delta_x, \tag{5}$$

$$\|L_{xp}(d(\exp_x^{-1}(q))) - L_{xp}L_{qx}\| < \varepsilon_0/3 \ \text{ provided that } d(x,q) < \delta_x. \tag{6}$$

By the compactness of K, there exists $\{x_i : i = 1,..,n\}$ such that $K \subseteq \bigcup_{i=1}^n B(x_i, \delta_{x_i}/2)$. Let $\delta \leq \min\{\delta_{x_i}/2\}$. If $p, q \in K$ with $d(p,q) < \delta$, then there exists x_i such that $p, q \in B(x_i, \delta_{x_i})$. Thus, by (4), (5), (6), we obtain

$$\|L_{qp} - d(\exp_p^{-1})(q)\| < \varepsilon_0. \tag{7}$$

Now, let $\eta'' = \min\{\eta', \theta, \delta\}$ where θ has been obtained in Lemma 4.6. It is straightforward to show, there exists $\varepsilon \in (0, b-a)$ such that if $x, y \in f^{-1}([a, a+\varepsilon])$ and $r(x) = r(y)$, then $d(x,y) < \eta''$. Now, let $x \in f^{-1}((a, a+\varepsilon])$, $y \in f^{-1}([a, a+\varepsilon])$ with $r(x) = r(y)$ and let $\xi \in \partial f(y)$. We consider the mapping ψ from $[\tau(x), 0]$ to $\mathbb{R}$ defined by

$$\psi(t) := \langle -\exp_x^{-1}(\phi(t,x)),\ L_{yx}(\xi)\rangle.$$

The function ψ is well defined, since $\phi(t,x) \in f^{-1}[a, a+\varepsilon]$ and $r(y) = r(x) = \phi(\tau(x), x) = r(\phi(t,x))$, so $d(\phi(t,x), x) < \eta'' \leq c(K)/2$. Moreover, $d(\phi(t,x), x) < \eta'' \leq \delta$, thus (7) implies

$$\|d(\exp_x^{-1})(\phi(t,x)) - L_{\phi(t,x)x}\| < \varepsilon_0,$$

by Lemma 4.6,

$$\alpha - k\varepsilon_0 m < \langle L_{\phi(t,x)x}(X(\phi(t,x))), \xi_x\rangle,$$

where $L_{yx}(\xi) = \xi_x + \varepsilon_0 v$ with $v \in B_{T_xM^*}$, $\xi_x \in \partial f(x)$ and α, k, m are defined in the proof of Lemma 4.6.

Now,

$$\begin{aligned}
\frac{d}{dt}\psi(t) &= \left\langle -d\exp_x^{-1}(\phi(t,x))\left(\frac{\partial\phi}{\partial t}(\phi(t,x))\right), L_{yx}(\xi)\right\rangle \\
&= \left\langle -d\exp_x^{-1}(\phi(t,x))\left(\frac{\partial\phi}{\partial t}(\phi(t,x))\right), (\xi_x + \varepsilon_0 v)\right\rangle \\
&\leq \langle -L_{\phi(t,x)x}(X(\phi(t,x))), \xi_x\rangle + \langle\langle -L_{\phi(t,x)x}(X(\phi(t,x))), \varepsilon_0 v\rangle + \varepsilon_0 mk + \varepsilon_0^2 m \\
&< \ -\alpha + 2\varepsilon_0 mk + \varepsilon_0 m + \varepsilon_0^2 m = -\beta < 0,
\end{aligned}$$

where β is defined in the proof of Lemma 4.6. Consequently, ψ is decreasing. Clearly, $\psi(0) = 0$, $\psi(\tau(x)) = \langle -\exp_x^{-1} r(x), L_{yx}(\xi)\rangle$. So,

$$\langle \exp_x^{-1} r(x), L_{yx}(\xi)\rangle < 0.$$

Hence the proof of part (b) is complete. □

Let us introduce the notion of an $\mathcal{L}$-retract in Riemannian manifolds; see [4]. A subset S of Riemannian manifold M is said to be an $\mathcal{L}$-retract if there exist a neighborhood V of S in M, a retraction $r : V \to S$ i.e. $r(x) = x, x \in S$, and a constant $L > 0$ such that

$$d(x, r(x)) \leq L d_S(x), \quad \text{for all } x \in V.$$

Remark 4.10. It is proved in Theorem 4.5 that if S is a compact epi-Lipschitz subset of a complete Riemannian manifold M, then there exists a locally Lipschitz retraction for S, i.e. there are an open neighborhood U of S and a retraction $r : U \to S$ which is locally Lipschitz. Hence, for each $x \in U$, there exists $\epsilon(x) > 0$ such that the restriction of r to the open ball $B(x, 2\epsilon(x))$ is Lipschitz with constant $L(x) > 0$. By the compactness of S, there are $x_1, \dots, x_k \in S$ such that $S \subseteq \bigcup_{i=1}^k B(x_i, \epsilon(x_i))$. Now set $L := \max\{L(x_i) : i = 1, \dots, k\}$, and $V := \bigcup_{i=1}^k B(x_i, \epsilon(x_i))$. For each $x \in V$, the Hopf-Rinow theorem implies that there exists $y \in S$ such that $d_S(x) = d(y, x)$. Moreover, there exists $x_i \in S$ such that $x \in B(x_i, \epsilon(x_i))$ and $d(x, y) \leq d(x, x_i) < \epsilon(x_i)$. Hence, $x, y \in B(x_i, 2\epsilon(x_i))$ and $d(r(x), r(y)) \leq L(x_i) d(x, y)$. Therefore,

$$d(r(x), x) \leq d(x, y) + d(r(x), r(y)) \leq (L + 1) d(x, y).$$

Hence, every compact epi-Lipschitz subset of a complete Riemannian manifold is $\mathcal{L}$-retract.

We conclude this section by the following theorem which will not be used here but is worth pointing out, see [1] for another proof.

Theorem 4.11. *Let M be a complete Riemannian manifold and let $f : M \to \mathbb{R}$ be a locally Lipschitz function. Assume that a, b are real numbers, $a < b$ such that the set M_{ab} is nonempty and compact and for all $x \in M_{ab}$, 0 does not belong to $\partial f(x)$. Then there exists a continuous mapping $H : M \times [0, 1] \to M$ such that*

(*i*) *for every $s \in [0, 1]$, $H_s : M \to M$ is a homeomorphism, where $H_s(x) = H(s, x)$.*
(*ii*) *H_0 is identity.*
(*iii*) *$H_1(M_a) = M_b$*

Proof. For $c, d \in \mathbb{R}$, we define $M_{(c,d)} := f^{-1}(c, d)$. Since there exists an open set Ω containing M_{ab}, we may choose $\varepsilon > 0$ such that for all x in $M_{(a-\varepsilon,a)}$ and $M_{(b,b+\varepsilon)}$ we have $X(x) \neq 0$ where X is defined in Lemma 4.6. Hence by [7, Theorem 3.14], for $p \in M_{(a-\varepsilon,a)}$ there exists a chart (V, ψ) at p such that $V \subset M_{(a-\varepsilon,a)}$ and the flow θ of X is given by

$$\theta(t, y_1, y_2, \dots y_n) = (y_1, y_2, \dots, y_n + t) \quad \text{for all } t \in \mathbb{R}.$$

Moreover, $\psi(p) = (0, 0, \dots, 0)$ and $\psi_*(X) = \frac{\partial}{\partial y_n}$ at every point of V. We claim that there exists $T_0 \in \mathbb{R}$ such that $\theta(T_0, 0, \dots, 0) \in M_{(b,b+\varepsilon)}$. Indeed, $f(\theta(0, 0, \dots, 0)) = f(0, 0, \dots, 0) < a$ and by the theorem of Rademacher (see [2, Theorem 5.7]) f is differentiable almost everywhere. Hence by Lemma 4.6, at the points where f is differentiable we have $X(f) > 0$. Thus almost everywhere on $\psi(V)$ we have $\frac{\partial f}{\partial y_n} > \beta$. Therefore if we choose $T_0 = \frac{b-a}{\beta}$ then

$$f(T_0) - f(0) = \int_0^{T_0} \frac{\partial f}{\partial y_n} dy_n > b - a.$$

This means $f(\theta(T_0, 0, \ldots, 0)) = f(T_0) > b$. Now by the continuity of θ one can find an open neighborhood W of $p, W \subset V$ such that for all $q \in W, \theta(T_0, q) \in M_{(b,b+\varepsilon)}$. Thus, for $q \in W$ and $0 \le t \le T_0$ we have $\theta(t, q) \in M_{(a-\varepsilon,b+\varepsilon)}$.

By [23, Corollary 3.3.5] there is a Lipschitz change of coordinate (φ, U) such that $p \in U \subset W$ and $\varphi(y_1, y_2, \ldots, y_{n-1}, s) = (y_1, y_2, \ldots, f(y_1, y_2, \ldots, y_{n-1}, s))$. In this coordinate we have

$$f(\theta(t, q)) = f(q) + t, \quad q \in U,\ 0 \le t \le T_0.$$

Now the proof can be completed along the same lines as [37, Corollary, 4.27]. □

5. Topological degree

This section is devoted to the definition of the topological degree of set valued mappings defined from an n-dimensional oriented complete manifold M to $2n$-dimensional manifold cotangent bundle TM^* or tangent bundle TM.

Let D be an open bounded subset of an n-dimensional complete oriented Riemannian manifold M, we denote by $C(\mathrm{cl}\, D, TM, 0)$ the set of all set valued mappings $F : \mathrm{cl}\, D \rightrightarrows TM$ satisfying: (i) F is upper semicontinuous, with nonempty, convex, and compact values, (ii) $0 \notin F(x)$ if $x \in \partial D$, and (iii) if $x \in \mathrm{cl}\, D$, then $F(x) \subset T_xM$. Note that since $F^{-1}(0)$ is compact in M, there exist charts (φ_i, U_i), $i = 1, \ldots, l$, such that $F^{-1}(0) \subseteq \bigcup_{i=1}^{l} U_i$. Now for every $F \in C(\mathrm{cl}\, D, TM, 0)$, we denote by $\deg(F, 0)$ the topological degree of F which is defined as follows; see [21, p. 133].

Definition 5.1. Let $\varphi_i : U_i \to \mathbb{R}^n$ be a chart (preserving orientation). Consider $T\varphi_i : TU_i \to T\mathbb{R}^n$, defined by $T\varphi_i(p, v) = d\varphi_{ip}(v)$. Then $T\varphi_i \circ F \circ \varphi_i^{-1}$ defines a set-valued upper semicontinuous mapping $g_i : \varphi_i(U_i) \rightrightarrows \mathbb{R}^n$, $g_i(\varphi_i(x)) = (g_{i,1}(\varphi_i(x)), ..., g_{i,n}(\varphi_i(x)))$. It is easy to see that if $\{E_k\}_{k=1}^n$ is a local basis on U_i, then $F(x) = \sum_{k=1}^n g_{i,k}(\varphi_i(x))E_k(x)$ on U_i. Now $\deg(F, 0) := \sum_{i=1}^l \deg(g_i, 0)$.

In the previous definition we used the Cellina-Lasota definition in order to define $\deg(g_i, 0)$. Hence, in the case of Euclidean spaces Definition 5.1 coincides with the Cellina-Lasota definition, see [11].

To be sensible definition, the degree of F should not depend on the representative g_i of F. There is a slight flaw which will become apparent when we examine what happens if we change the ordered basis $\{E_1(x), ..., E_n(x)\}$ of T_xM to the ordered basis $\{\bar{E}_1(x), ..., \bar{E}_n(x)\}$. If A is the transition matrix corresponding to the change of bases $E_1(x), ..., E_n(x) \to \bar{E}_1(x), ..., \bar{E}_n(x)$. Then,

$$\deg(Ag_i, 0) = \mathrm{sgn}(\det A) \deg(g_i, 0).$$

This means that the degree function defined in this way depends on the choice of the basis. The problem is solved by using the widely used terminology of differential geometry. The bases $\{E_1(x), ..., E_n(x)\}$ and $\{\bar{E}_1(x), ..., \bar{E}_n(x)\}$ of T_xM have the same orientation, if the matrix A has the positive determinant. When we fix an ordered basis $\{E_1(x), ..., E_n(x)\}$ of T_xM, the definition of the orientation gives us an equivalence relation in the set of all bases of T_xM, with exactly two equivalence classes. Hence, for an n-dimensional complete oriented Riemannian manifold M, the topological degree of F is well defined.

A similar definition holds for the set valued mapping F defined on the closure of an open bounded subset of an n-dimensional complete oriented Riemannian manifold M to the cotangent bundle TM^*.

We denote by $C(\partial D, TM, 0)$ the set of all set valued mappings $F : \partial D \rightrightarrows TM$, satisfying: (i) F is upper semicontinuous, with nonempty, convex, and compact values, (ii) for $x \in \partial D$, $0 \notin F(x)$ and $F(x) \subset T_xM$.

Definition 5.2. Let $F \in C(\partial D, TM, 0)$, then the topological degree of F, also denoted by $\deg(F, 0)$, is defined by;

$$\deg(F, 0) := \deg(\bar{F}, 0),$$

where $\bar{F}$ is an arbitrary set valued mapping in $C(\operatorname{cl} D, TM, 0)$ which extends F to $\operatorname{cl} D$.

A similar definition holds for the set valued mapping F defined on the boundary of an open bounded subset of an n-dimensional complete oriented Riemannian manifold M to the cotangent bundle TM^*.

Note that if M is parallelizable, then there exists $f_k : \operatorname{cl} D \rightrightarrows \mathbb{R}^n$, $k = 1, ..., n$, such that for every $x \in \operatorname{cl} D$, $F(x) = \sum_{k=1}^{n} f_k(x)E_k(x)$. Hence $\deg(F, 0) := \deg(f, 0)$, more precisely, $\deg(F, 0) := \sum \deg(f \circ \varphi_i^{-1}, 0)$, where φ_i is a local chart of $x_i \in F^{-1}(0)$.

6. The Euler characteristic of an Epi-Lipschitz subset of a complete Riemannian manifold

Now, we define the Gauss set valued mapping of an epi-Lipschitz set in a complete oriented Riemannian manifold as follows.

Definition 6.1. Let M be a complete oriented Riemannian manifold and S be a closed epi-Lipschitz set in M. The set valued mapping $G_S : \partial S \rightrightarrows TM^*$ defined by

$$G_S(x) = \operatorname{co}[N_S(x) \cap \mathbb{S}_x],$$

where $\mathbb{S}_x$ is the unit sphere of T_xM^*, is called the Gauss mapping of S.

The following theorem gives us some properties of the Gauss mapping G_S. The proof of it can be obtained along the same lines as [14, Proposition 3.1].

Theorem 6.2. *Let S be a closed epi-Lipschitz set in a complete oriented Riemannian manifold M. Then the Gauss set valued mapping G_S is upper semicontinuous with nonempty, convex, compact values and for every $x \in \partial S$, $0 \notin G_S(x)$. Moreover, there is an upper semicontinuous set valued mapping with nonempty, convex and compact values which is an extension of G_S on S.*

Using Theorem 6.2 and Definition 5.2 we define the Euler characteristic of a nonempty compact, epi-Lipschitz subset S of an n-dimensional complete oriented Riemannian manifold M, by $\chi(S) = \deg(G_S, 0)$.

Following [14, Propositions 3.2 and 3.3], the next theorem and its subsequent corollary can be proved for epi-Lipschitz subsets of complete parallelizable Riemannian manifolds.

Theorem 6.3. *Let S be an epi-Lipschitz and nonempty compact set in a complete parallelizable Riemannian manifold M, and let $G : \partial S \rightrightarrows T^*M$ be an upper semicontinuous set valued mapping, with nonempty, convex, compact values such that for every $x \in \partial S$, $G(x) \subset T_xM^*$ and $G(x) \cap -N_S(x) = \emptyset$. Then $\chi(S) = \deg(G, 0) = \deg(\bar{G}, 0)$, where $\bar{G} : S \rightrightarrows TM^*$ is every upper semicontinuous set valued mapping with nonempty, convex, compact values, which extends G to S and for every $x \in S$ satisfies $\bar{G}(x) \subset T_xM^*$.*

Corollary 6.4. *Let S be an epi-Lipschitz and nonempty compact set in a complete parallelizable Riemannian manifold M, and let $G : \partial S \rightrightarrows T^*M$ be an upper semicontinuous set valued mapping, with nonempty, convex, compact values such that for every $x \in \partial S$, $G(x) \subset T_xM^*$ and $G(x) \cap N_S(x) \neq \emptyset$, and $0 \notin G(x)$. Then $\chi(S) = \deg(G, 0) = \deg(\bar{G}, 0)$, where $\bar{G} : S \rightrightarrows TM^*$ is every upper semicontinuous set valued mapping with nonempty, convex, compact values, which extends G to S and for every $x \in S$ satisfies $\bar{G}(x) \subset T_xM^*$.*

In the following theorem an equivalent definition of the Euler characteristic is obtained.

Theorem 6.5. *Let S be an epi-Lipschitz and nonempty compact set in a complete parallelizable Riemannian manifold M.*

(a) *There exists a continuous vector field $X : S \to TM$ such that for every $x \in \partial S$*
$$X(x) \in \operatorname{int} T_S(x) \subset T_S(x) \setminus \{0\}.$$

(b) *If $F : S \rightrightarrows TM$ is an upper semicontinuous set valued mapping with nonempty, convex, and compact values such that*
$$F(x) \subset T_xM \quad \textit{for every } x \in S, \tag{8}$$
and
$$F(x) \subset T_S(x) \setminus \{0\} \quad \textit{for every } x \in \partial S, \tag{9}$$
then $\chi(S) = \deg(-F, 0)$.

Proof. (a) The proof is along the same lines as [14, Proposition 3.4]. However, we must prove that Michael's selection theorem holds when S is a subset of n-dimensional parallelizable Riemannian manifold M and $T : S \rightrightarrows TM$ is a lower semicontinuous mapping with nonempty, convex values with $T(x) \subset T_xM$, for every $x \in S$. In this case, let $t : S \rightrightarrows \mathbb{R}^n$ be such that $T(x) = \sum_{i=1}^n t_i(x)E_i(x)$ where $\{E_i(x)\}_{i=1}^n$ is an ordered basis of T_xM. Since M is a fully normal space, Micheal's selection theorem holds for the set valued mapping t and there is a continuous function $f : S \to \mathbb{R}^n$ such that $f(x) \in t(x)$ for every $x \in S$. We define $\bar{f} : S \to TM$ by $\bar{f}(x) = \sum_{i=1}^n f_i(x)E_i(x)$ where $f(x) = (f_1(x), ..., f_n(x))$. Then $\bar{f}$ is continuous and $\bar{f}(x) \in T(x)$ for each $x \in S$.

(b) Suppose that $F(x) = \sum_{i=1}^n f_i(x)E_i(x)$, where $\{E_i(x)\}_{i=1}^n$ is an ordered basis of T_xM for each $x \in S$. We define $G : \partial S \rightrightarrows TM^*$ by $G(x) = -\sum_{i=1}^n f_i(x)w_i(x)$, where $\{w_i(x)\}_{i=1}^n$ is an ordered basis of T_xM^* for every $x \in S$. Moreover, for all $x \in S$,

$$\langle E_i(x), w_j(x)\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

We claim that

$$G(x) \cap -N_S(x) = \emptyset \ \text{ for all } x \in \partial S.$$

Then we conclude from Theorem 6.3 that, $\chi(S) = \deg(G, 0) = \deg(-F, 0)$. We proceed by contradiction. Let $y = \sum_{i=1}^n y_i w_i(x) \in G(x) \cap -N_S(x)$ then from (9) and the relationship between F and G, $-\sum_{i=1}^n y_i E_i(x) \in T_S(x) \setminus \{0\}$. So that $\sum_{i=1}^n y_i^2 \leq 0$ which means $y = 0$ and this contradiction completes the proof. □

Theorem 6.6. *Let S be a convex and compact set in a complete parallelizable Riemannian manifold M with a nonempty interior. Then $\chi(S) = 1$.*

Proof. Let $\bar{x} \in \operatorname{int} S$ and define $f : S \to TM$ by

$$f(x) := \exp_x^{-1}(\bar{x}).$$

By Theorem 3.5, the function f satisfies the conditions of Theorem 6.5, so that $\chi(S) = \deg(-f, 0)$. By [2, Lemma 6.5], if $\exp_x^{-1}(\bar{x}) = \sum_{i=1}^n v_i(x)E_i(x)$ then $\exp_{\bar{x}}^{-1}(x) = -L_{x\bar{x}}(\exp_x^{-1}(\bar{x})) = -\sum_{i=1}^n v_i(x)L_{x\bar{x}}(E_i(x))$ where for every $x \in M$, $\{E_i(x)\}_{i=1}^n$ is an ordered basis of T_xM. On the other hand, $d\exp_{\bar{x}}^{-1}(\bar{x}) : T_{\bar{x}}M \to T_{\bar{x}}M$ is identity function. So $\chi(S) = \deg(-f, 0) = 1$. □

Let us point out the following theorem which is a characterization of epi-Lipschitz subsets of complete Riemannian manifolds, (see [22] for a proof).

Theorem 6.7. *Let M be a complete Riemannian manifold and S be a closed subset of M. Then the following assertions are equivalent.*

(a) *S is epi-Lipschitz.*

(b) *There is a locally Lipschitz function $\varphi : M \to \mathbb{R}$ such that*
  - (i) $S = \{x \in M : \varphi(x) \leq 0\}$,
  - (ii) *If* $\varphi(x) = 0$, *then* $0 \notin \partial\varphi(x)$,
  - (iii) $\partial S = \partial(\operatorname{int} S) = \{x \in M : \varphi(x) = 0\}$.

We are going to investigate the relationship between the Euler characteristic of epi-Lipschitz subsets of complete Riemannian manifolds and a function φ which satisfies in the conditions of Theorem 6.7.

Theorem 6.8. *Let S be a nonempty and compact epi-Lipschitz set in a complete parallelizable Riemannian manifold M. Let φ be a function satisfying the conditions of Theorem* 6.7 *such that if $\varphi(x) > 0$ then $0 \notin \partial\varphi(x)$. Then $\chi(S) = 1$.*

It is worth pointing out that Theorem 4.5 will be fundamental to the proof of Theorem 6.8. Under the assumptions of Theorem 6.8, let $\gamma : M \to [0,1]$ be a smooth function such that $\gamma(x) = 0$ if $x \in B(\partial S, 1/2)$ and $\gamma(x) = 1$ if $x \notin \operatorname{int} B(\partial S, 1)$. We define the function $\tilde{\varphi} : M \to \mathbb{R}$ by

$$\tilde{\varphi}(x) = \begin{cases} (1-\gamma(x))\varphi(x) + \gamma(x)\operatorname{sgn}\varphi(x) & \varphi(x) \neq 0 \\ 0 & \varphi(x) = 0, \end{cases}$$

denoting $\operatorname{sgn} t = t/|t|$ if $t \in \mathbb{R} \setminus \{0\}$. Then $\varphi^{-1}[0,1) \subset \tilde{\varphi}^{-1}[0,1) \subset \operatorname{int} B(\partial S, 1)$. Therefore $\varphi^{-1}[0,1]$ is compact and by the compactness of S, for every $\epsilon > 0$, $\varphi^{-1}[-\epsilon, 1]$ is compact. Also, there is $\epsilon' > 0$ such that for every $x \in \varphi^{-1}[-\epsilon', 0]$, zero does not belong to $\partial\varphi(x)$. Hence by Theorem 4.5, there exist $\varepsilon > 0$, a neighborhood $\widetilde{M}$ of $\{x \in M :\ \varphi(x) \leq 1\}$ and a locally Lipschitz mapping r from $\widetilde{M}$ to $M_\varepsilon := \{x \in M :\ \varphi(x) \leq -\varepsilon\}$ satisfying the following conditions,

(i) $r(x) = x$ for $x \in M_\varepsilon$,
(ii) if $x \in \partial S$, then $\sup\{\langle \xi, \exp_x^{-1}(r(x))\rangle \mid \xi \in \partial\varphi(x)\} < 0$.

We now come to the proof of Theorem 6.8.

Proof. Let B be a closed ball containing a neighborhood of the set S. Define $f : B \to TM$ by $f(x) = -\exp_x^{-1}(r(x))$. Now set $K = B \setminus \operatorname{int} S$ which is compact and $f(x) \neq 0$ for all $x \in K$. Employing the properties of the topological degree, one can deduce

$$\deg(f, 0) = \deg(f|_S, 0), \tag{10}$$

where $f|_S$ is the restriction of the mapping f to S. On the other hand, by Theorem 3.5, for every $x \in \partial B$, $-f(x) = \exp_x^{-1}(r(x)) \in T_B(x) \setminus \{0\}$. So Theorem 6.5 implies $\chi(B) = \deg(f, 0)$. We claim that $\chi(S) = \deg(f|_S, 0)$. Then (10) and the convexity property of B imply $\chi(S) = 1$. By [22, Corollary 4.12] $N_S(x) = \bigcup_{\lambda \geq 0} \lambda\partial\varphi(x)$, where φ satisfies the conditions of Theorem 6.7. Also, Theorem 4.5 and the bipolar theorem imply

$$\exp_x^{-1}(r(x)) \in T_S(x).$$

Since $r(x) \in M_\varepsilon$ which does not meet ∂S, so $\exp_x^{-1}(r(x)) \neq 0$ for every $x \in \partial S$. Hence from Theorem 6.5 one can obtain $\chi(S) = \deg(f|_S, 0)$, as required. $\square$

We conclude this section with a direct consequence of Corollary 6.4 to compute the Euler characteristic of a compact epi-Lipschitz set in a complete parallelizable Riemannian manifold which is obtained by using the degree of the set valued mapping $x \mapsto \partial\varphi(x)$, where φ is given in Theorem 6.7.

Theorem 6.9. *Let S be a nonempty and compact epi-Lipschitz set in a complete parallelizable Riemannian manifold M, and let φ be a function which satisfies the conditions of Theorem 6.7, then $\chi(S) = \deg(\partial\varphi(.), 0)$.*

7. Applications of Euler characteristic to equilibrium theory

In this section, we present some equilibrium theorems.

Theorem 7.1. *Let S be an epi-Lipschitz and nonempty compact set in a complete parallelizable Riemannian manifold M such that $\chi(S) \neq 0$, and let $F : S \rightrightarrows T^*M$ be an upper semicontinuous set valued mapping, with nonempty, convex, compact values such that for every $x \in S$, $F(x) \subset T_xM^*$. Then there exists $s \in S$ such that*

$$0 \in F(s) - N_S(s).$$

Proof. On the contrary, assume that for every $s \in S$, $0 \notin F(s) - N_S(s)$. Then Theorem 6.3 implies $\chi(S) = \deg(-F, 0)$. By the properties of degree, there exists $s \in S$ such that $0 \in F(s) \subset F(s) - N_S(s)$, hence one gets a contradiction. □

Theorem 7.2. *Let S be an epi-Lipschitz and nonempty compact set in a complete parallelizable Riemannian manifold M, and let $F : S \rightrightarrows TM$ be an upper semi-continuous set valued mapping, with nonempty, convex, compact values such that for every $x \in S$, $F(x) \subset T_xM$. Moreover,*

$$0 \notin F(x), \;\; F(x) \cap T_S(x) \neq \emptyset \;\; \text{for all } x \in \partial S. \tag{11}$$

Then $\chi(S) = \deg(-F, 0)$.

Proof. Let $x \in \partial S$. Since 0 does not belong to $F(x) \subset T_xM$, by separation theorem there exists $v(x) \in T_xM^*$ such that $\sup_{\xi \in F(x)} \langle \xi, v(x) \rangle < 0$. We can define a covector field v on M with value $v(x)$ at x. The upper semicontinuity of F implies that there are an open neighborhood U_x of x and a positive real number ϵ_x such that

$$\sup_{\xi \in F(y)} \langle \xi, v(y) \rangle < -\epsilon_x \;\; \text{provided that } y \in U_x.$$

Since the set ∂S is compact, there exists a finite subset $\{x_1, ..., x_m\}$ in M such that $\partial S \subset \bigcup_{j=1}^m U_{x_j}$. Let φ_j, $j = 1, ..., m$ be a smooth partition of unity subordinate to the covering U_{x_j}, $j = 1, ..., m$, of ∂S. We define a smooth covector field v on M as follows,

$$v(x) = \sum_{j=1}^{m} \varphi_j(x) \bar{v}_j(x),$$

where $\bar{v}_j$ is the corresponding smooth covector field to U_{x_j} on M. Moreover

$$\sup_{\xi \in F(x)} \langle \xi, v(x) \rangle < -\epsilon \;\; \text{provided that } x \in \partial S, \tag{12}$$

where $\epsilon = \min\{\epsilon_{x_i} : \; i = 1, ..., m\}$. Now (11) and (12) imply $v(x) \notin -N_S(x)$, for every $x \in \partial S$. It follows from Theorem 6.3 that $\chi(S) = \deg(v, 0)$. It is sufficient to prove $\deg(v,0) = \deg(-F,0)$. Let $v(x) = \sum_{i=1}^n v_i(x)w_i(x)$ where $\{w_i(x)\}_{i=1}^n$ is an ordered basis of T_xM^* for every $x \in M$. Define the vector field $\tilde{v}(x) = \sum_{i=1}^n v_i(x)E_i(x)$ where $\{E_i(x)\}_{i=1}^n$ is an ordered basis of T_xM for every $x \in M$. Moreover, for every $x \in S$,

$$\langle E_i(x), w_j(x)\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Then, by (11) for every $x \in \partial S$, $0 \notin \mathrm{co}[\tilde{v}(x) \cup -F(x)]$. Indeed, if we assume that there exist $x \in \partial S$, $h \in F(x)$ and $0 \leq t \leq 1$ such that $0 = (1-t)\tilde{v}(x) - th$. Then from (11), $t \neq 1$ and $\tilde{v}(x) = th/(1-t)$. Hence

$$0 > \langle v(x), h\rangle = \langle \tilde{v}(x), h\rangle \geq 0,$$

which is a contradiction. Thus, the properties of the degree imply $\deg(\tilde{v},0) = \deg(-F,0)$, see [14, Property D.5]. On the other hand, $\deg(\tilde{v},0) = \deg(v,0)$ which ends the proof. □

The following theorem is a direct consequence of previous theorem and the properties of the degree. Moreover, it is a generalization of [5, Theorem 2.2]. Indeed, if S is a nonempty compact subset of a complete parallelizable Riemannian manifold M, such that $S = \{x \in M : f(x) \leq 0\}$, where $f : M \to \mathbb{R}$ is locally Lipschitz, and if $b = \max\{f(x);\ x \in \mathrm{co}\, S\}$, then $f^{-1}([0,b])$ is compact and for all $x \in f^{-1}([0,b])$, $0 \notin \partial f(x)$. Then, Theorem 6.8 implies S is an epi-Lipschitz and compact set in a complete parallelizable Riemannian manifold M such that $\chi(S) \neq 0$.

Theorem 7.3. *Let S be an epi-Lipschitz and nonempty compact set in a complete parallelizable Riemannian manifold M such that $\chi(S) \neq 0$, and let $F : S \rightrightarrows TM$ be an upper semicontinuous set valued mapping, with nonempty, convex, compact values such that for every $x \in S$, $F(x) \subset T_xM$. Moreover, suppose that*

$$F(x) \cap T_S(x) \neq \emptyset \quad \textit{for all } x \in \partial S.$$

Then there exists $s \in S$ such that $0 \in F(s)$.

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