The effect of sparsity and relaxations thereof in certain function approximation problems

Sebastian Mayer

Institute for Numerical Simulation, University of Bonn

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Institute for Numerical Simulation
Rheinische Friedrich-Wilhelms-Universität Bonn

A (de-)motivating example

Consider the class F_d of **infinitely differentiable** functions

$$f : [0,1]^d \to \mathbb{R}$$

such that

$$\|f\| := \sup_{\alpha} \|D^{\alpha} f\|_{\infty} \leq 1.$$

Minimal deterministic worst-case error:

$$\text{error}(n,d) := \inf_{A_n} \sup_{f \in F_d} \|f - A_n f\|_{\infty}$$

A_n: deterministic algorithm **adaptively** using n arbitrary **linear functionals** (e.g. Fourier coefficients, line integrals, derivatives)

A (de-)motivating example

For any $r > 0$, we have

$$\mathrm{error}(n, d) \leq C_{r,d}\; n^{-r}.$$

Information complexity:

$$n(\varepsilon, d) := \min\{n \in \mathbb{N} : \mathrm{error}(n, d) \leq \varepsilon\}.$$

Theorem ([Novak, Woźniakowski (2009)])

For all $\varepsilon \in (0, 1)$ and all $d \in \mathbb{N}$, we have

$$n(\varepsilon, d) \geq 2^{\lfloor d/2 \rfloor}.$$

The curse of dimensionality despite of smoothness

Corollary

The L_∞-approximation of infinitely differentiable functions from F_d suffers from

the curse of dimensionality*.*

Remarkable:

- **Smoothness** does not guarantee tractability in high dimensions!
- **Asymptotic decay rates** give no indication for the complexity!

How to overcome the curse?

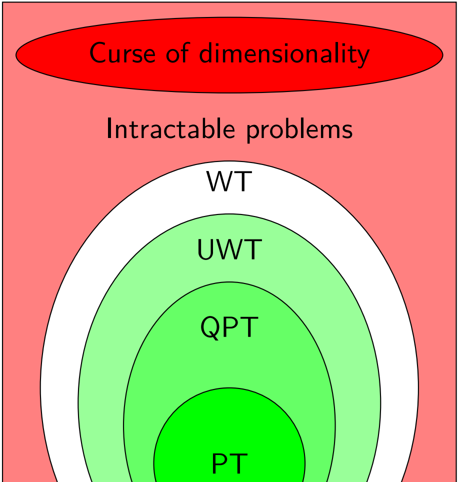
What additional **a priori** knowledge guarantees tractability?

- Sparsity?
- Structured functions?

What determines the complexity?

asymptotic decay rates vs. **preasymptotics**

Notions of tractability



Weak tractability (WT):

$$\lim_{1/\varepsilon+d\to\infty} \frac{\log n(\varepsilon, d)}{1/\varepsilon + d} = 0.$$

Uniform weak tractability (UWT):

for all $\alpha, \beta > 0$

$$\lim_{1/\varepsilon+d\to\infty} \frac{\log n(\varepsilon, d)}{1/\varepsilon^{\alpha} + d^{\beta}} = 0.$$

Quasi-polynomial tractability (QPT):

there exist constants $C, p, q > 0$ such that

$$n(\varepsilon, d) \leq C\,(1/\varepsilon)^{p(1+\log d)}\, d^{q}$$

Sobolev spaces of periodic functions

$\mathbb{T}^d = [0, 2\pi)^d$

The classical Sobolev space $H^m(\mathbb{T}^d)$ consists of all

periodic functions $f \in L_2(\mathbb{T}^d)$

such that

$$D^\gamma f \in L_2(\mathbb{T}^d) \qquad \text{for all } \gamma \in \mathbb{N}_0^d : \|\gamma\|_1 \leq m.$$

What norm to equip $H^m(\mathbb{T}^d)$ with?

A family of equivalent norms on $H^m(\mathbb{T}^d)$

The classical norm:

$$\|f\|^2 = \sum_{\|\gamma\|_1 \leq m} \|D^\gamma f\|_2^2 \asymp_m \sum_{k \in \mathbb{Z}^d} (1 + \sum_{i=1}^{d} |k_i|^2)^m \, |\hat{f}(k)|^2$$

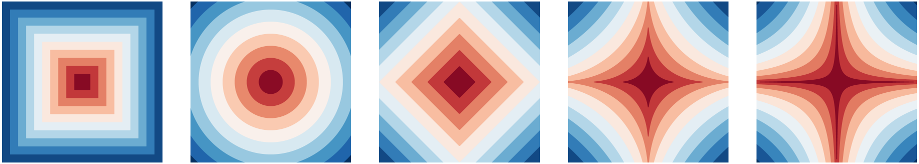
A family of equivalent norms on $H^m(\mathbb{T}^d)$

$$\|f\|_{H^{m,p}(\mathbb{T}^d)} := \left(\sum_{k \in \mathbb{Z}^d} w_{m,p}(k)^2 \, |\hat{f}(k)|^2 \right)^{1/2}$$

Weights:

$$w_{m,p}(k) := (1 + \sum_{i=1}^{d} |k_i|^p)^{m/p}$$

$$w_{m,\infty}(k) := \max\{1, |k_1|, \dots, |k_d|\}^m$$



$p = \infty$ $\quad$ $p = 2$ $\quad$ $p = 1$ $\quad$ $p = 0.5$ $\quad$ $p = 0.1$

$p < 1$: **compressibility constraint** on frequency vectors

Approximation numbers

Sobolev space of fractional smoothness $\alpha > 0$:

$$H^{\alpha,p} := \{f \in L_2(\mathbb{T}^d) : \|f\|_{H^{\alpha,p}(\mathbb{T}^d)} < \infty\}$$
$$F_d := \{f \in L_2(\mathbb{T}^d) : \|f\|_{H^{\alpha,p}(\mathbb{T}^d)} \leq 1\}$$

Minimal non-adaptive worst-case error
(aka. **approximation numbers**):

$$\begin{aligned} \mathrm{error}(n,d) &= \inf_{A_n} \sup_{\|f\|_{H^{\alpha,p}(\mathbb{T}^d)} \leq 1} \|f - A_n f\|_2 \\ &= a_{n+1}(\mathrm{Id} : H^{\alpha,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \end{aligned}$$

Only non-adaptive algorithms!

Historic perspective

In 1967, J. W. Jerome proved

$$c_{\alpha,d}\ n^{-\alpha/d} \leq \mathrm{error}(n,d) \leq C_{\alpha,d}\ n^{-\alpha/d}$$

Result holds true for **any** of the norms $\|\cdot\|_{H^{\alpha,p}(\mathbb{T}^d)}$.

Where is the compressibility information gone?

All information is in the weights

$(\sigma_n)_{n\in\mathbb{N}}$: **non-increasing rearrangment** of $(1/w_{\alpha,p}(k))_{k\in\mathbb{Z}^d}$

Optimal algorithm:

- Order frequency vectors $k \in \mathbb{Z}^d$ according to $(\sigma_n)_{n\in\mathbb{N}}$:

$$k^{(1)}, k^{(2)}, k^{(3)}, \dots$$

- Take

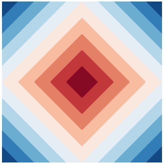
$$A_n^* f(x) = \frac{1}{(2\pi)^{d/2}} \sum_{j=1}^{n} \hat{f}(k^{(j)})\, e^{ik^{(j)}\cdot x}$$

$$\mathrm{error}(n,d) = \sup_{\|f\|_{H^{\alpha,p}(\mathbb{T}^d)}\le 1} \|f - A_n^* f\|_2 = \sigma_{n+1}$$

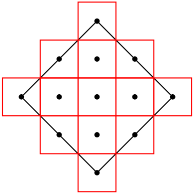
Counting via entropy

Consider, for instance, $p = 1$

$$\|f\|_{H^{\alpha,p}(\mathbb{T}^d)} = \left(\sum_{k \in \mathbb{Z}^d} (1 + \|k\|_p^p)^{2\alpha/p} \, |\hat{f}(k)|^2 \right)^{1/2}$$



Counting: $\sharp\{k \in \mathbb{Z}^d \mid \|k\|_p \le m\} \longleftrightarrow$



$1/m$-covering of B_p^d in ℓ_∞^d

Entropy numbers $\varepsilon_n(B_p^d, \ell_\infty^d)$:
minimal number of ℓ_∞^d-balls of radius ε which cover B_p^d

Characterization by entropy numbers

$$\mathrm{error}(n,d) = \inf_{A_n} \sup_{\|f\|_{H^{\alpha,p}(\mathbb{T}^d)}\le 1} \|f - A_n f\|_2$$

Theorem ([Kühn, M., Ullrich (2015)])

For all $\alpha > 0$ and all $0 < p \le \infty$, we have

$$\mathrm{error}(n,d) \asymp \varepsilon_{n+1}(B_p^d, \ell_\infty^d)^\alpha.$$

(With constants independent of n and d)

Characterization by entropy numbers

$\varepsilon_n(B_p^d, \ell_q^d)$: behavior in n and d **completely understood**!

For $p = \infty$, we obtain

$$\mathrm{error}(n,d) \asymp_{\alpha,p} \begin{cases} 1 & , \ 1 \le n \le 2^d \\ n^{-\alpha/d} & , \ n \ge 2^d \end{cases}$$

$\longrightarrow$ *curse of dimensionality*

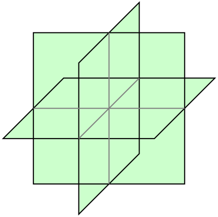
For $p < \infty$, we obtain

$$\mathrm{error}(n,d) \asymp_{\alpha,p} \begin{cases} 1 & , \ 1 \le n \le d \\ \left(\frac{\log(1+d/\log n)}{\log n}\right)^{\alpha/p} & , \ d \le n \le 2^d \\ d^{-\alpha/p} n^{-\alpha/d} & , \ n \ge 2^d \end{cases}$$

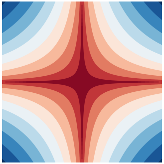
$\longrightarrow$ *weak tractability iff* $\alpha > p$ **strong compressibility** $\longrightarrow$ *almost uniform weak tractability*

Relaxations of sparsity

Sparsity

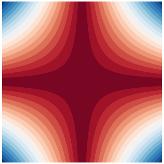


Compressibility



$w_{\alpha,p}(k) = (1 + \sum_{i=1}^{d} |k_i|^p)^{\alpha/p}$

Hyperbolic cross



$w_{\alpha,p}(k) = \prod_{i=1}^{d}(1 + |k_i|^2)^{\alpha/2}$

Sobolev spaces of dominating mixed smoothness

$$\mathrm{error}(n,d) = \inf_{A_n} \sup_{\|f\|_{H^{\alpha}_{\mathrm{mix}}(\mathbb{T}^d)}\le 1} \|f - A_n f\|_2$$

Theorem ([Kühn, Sickel, Ullrich (2015)])

For $1 \le n \le 4^d$, we have

$$\mathrm{error}(n,d) \le \left(\frac{e^2}{n}\right)^{\frac{\alpha}{2+\log d}} . \longrightarrow \textit{quasi-polyomial tractability}$$

For $n \ge 4^d$, we have

$$\mathrm{error}(n,d) \asymp n^{-\alpha}(\log n)^{\alpha(d-1)}$$

(with constants decaying super-exponentially in d).

Sparsified Sobolev spaces

$$H^{\alpha}(\mathbb{T}^d, s) := \{ f \in H^{\alpha,\infty}(\mathbb{T}^d) : \hat{f}(k) = 0 \text{ if } \operatorname{supp}(k) > s \}$$

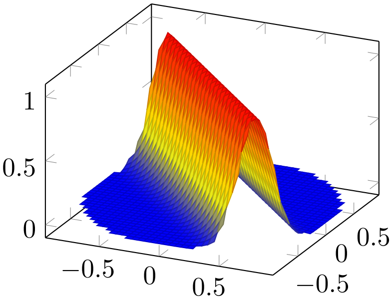
Theorem ([Kühn, M., Ullrich (2015)])

$$\operatorname{error}(n, d) \asymp \begin{cases} 1 & n \leq 2^s \binom{d}{s}, \\ d^s n^{-\alpha/s} & n > 2^s \binom{d}{s}. \end{cases}$$

$\longrightarrow$ *polynomial tractability*

(this time the **asymptotic decay** is crucial!)

Ridge functions



$$f: \quad \Omega \subseteq \mathbb{R}^d \to \mathbb{R},$$

$$f(x) = g(a \cdot x) = g(\sum_{i=1}^{d} a_i x_i).$$

$a \in \mathbb{R}^d$: **ridge direction**, g: univariate **profile**

Ridge functions on the Euclidean unit ball

For $\alpha > 0$ and $0 < p < 2$, ($p < 1$: **compressibility**)

$$\mathcal{R}_d^{\alpha,p}(B_2^d) = \left\{ g(a\cdot) : B_2^d \to \mathbb{R},\ \|g\|_{\mathrm{Lip}(\alpha)} \leq 1, \|a\|_p \leq 1 \right\}$$

$$\alpha = s + \beta, \quad \|g\|_{\mathrm{Lip}_\alpha} = \max\{\|g\|_\infty, \|g^{(1)}\|_\infty, \dots, \|g^{(s)}\|_\infty, |g^{(s)}|_\beta\}$$

Minimal deterministic worst-case sampling error
(aka. **sampling numbers**):

$$\mathrm{error}(n, d) = \inf_{S_n} \sup_{f \in \mathcal{R}_d^{\alpha,p}} \|f - S_n f\|_\infty$$

S_n: deterministic algorithm **adaptively** using n function values

Lipschitz spaces

Consider general **multivariate Lipschitz functions**

$$f : B_2^d \to \mathbb{R}.$$

Then,

$$\mathrm{error}(n, d) = \inf_{S_n} \sup_{\|f\|_{\mathrm{Lip}(\alpha)\leq 1}} \|f - S_n f\|_\infty \asymp \begin{cases} 1 & : n \leq 2^d \\ n^{-\alpha/d} & : n > 2^d \end{cases}$$

For **univariate Lipschitz functions**

$$g : [-1, 1] \to \mathbb{R},$$

it is well-known that

$$\mathrm{error}(n, d) = \inf_{S_n} \sup_{\|g\|_{\mathrm{Lip}(\alpha)\leq 1}} \|g - S_n g\|_\infty \asymp n^{-\alpha}$$

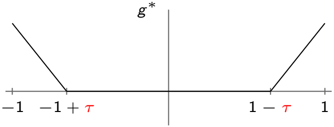
Why should recovery of ridge functions be hard?

Concentration of measure phenomenon:

$a \cdot x$ will be small with high probability for large d!

Fooling profile:

In case $\alpha = 1$, consider $f(x) = g^*(a \cdot x)$.



For n sampling points,

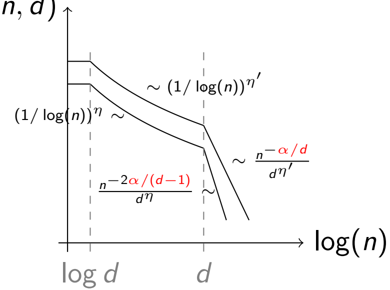
$$\tau = \varepsilon_n(\mathbb{S}_p^{d-1}, \ell_2^d)^2.$$

Characterization by entropy numbers

Theorem ([M., Ullrich, Vybíral (2014)])
For all $\alpha > 0$ and all $0 < p \leq 2$

$$\varepsilon_n(\mathbb{S}_p^{d-1}, \ell_2^d)^{2\alpha} \lesssim \mathrm{error}(n,d) \lesssim \varepsilon_{n/\binom{d+s}{s}}(B_2^d, \ell_{p'}^d)^{\alpha}$$

(p': dual index of p)



$$\eta = \alpha(1/p - 1/2)$$
$$\eta' = \alpha(1/2 - 1/p')$$

Tractability results (Euclidean ball)

Theorem ([M., Ullrich, Vybíral (2014)])

L_∞-reconstruction of ridge functions from the class $\mathcal{R}_d^{\alpha,p}(B_2^d)$ in the deterministic setting

- *suffers from the **curse of dimensionality** iff*

$$p = 2,$$

- *is **intractable** if $p < 2$ and*

$$\alpha \leq \frac{1}{1/p - 1/2},$$

- *is **weakly tractable** if $p < 2$ and*

$$\alpha > \frac{1}{1/\max\{1, p\} - 1/2},$$

Ridge functions on the cube

For $\alpha > 0$ and $0 < p \leq 1$, ($p < 1$: **compressibility**)

$$\mathcal{R}_d^{\alpha,p}([-1,1]^d) = \Big\{ g(a\cdot) : [-1,1]^d \to \mathbb{R},\ \|g\|_{\mathrm{Lip}(\alpha)} \leq 1, \|a\|_p \leq 1 \Big\}$$

Probabilistc worst-case sampling error:

$$\text{prob-error}(n,d) := \inf_{S_n} \sup_{f \in \mathcal{R}_d^\alpha} \inf\{\varepsilon > 0 : \mathbb{P}(\|f - S_n f\|_\infty \leq \varepsilon) \geq 1/2\}$$

Theorem ([Doerr, M., Rudolf])

Let $\alpha > 1$ *and* $0 < p < 1$.

1. *For* $d < n \leq \exp(d/8)/2$, *we have*

$$\left[\frac{1}{\log(n)}\right]^{\alpha(1/p-1)} \lesssim \text{prob-error}(n,d) \lesssim \left[\frac{1}{\log(n-d)}\right]^{\alpha(1/p-1)}$$

2. *For* $n \geq 2^d + d$, *we have*

$$n^{-\alpha} \lesssim \text{prob-error}(n,d) \lesssim 2^{\alpha d}(n-d)^{-\alpha}.$$

Tractability results (cube)

Theorem ([Doerr, M., Rudolf])

L_∞-reconstruction of ridge functions from the class $\mathcal{R}_d^{\alpha,p}([-1,1]^d)$ in the probabilistic setting

1. *suffers from the* ***curse of dimensionality*** *iff*

$$p = 1,$$

2. *is* ***intractable*** *iff $0 < p < 1$ and*

$$\alpha \leq \frac{1}{1/p - 1},$$

3. *is* ***weakly tractable*** *iff $0 < p < 1$ and*

$$\alpha > \frac{1}{1/p - 1}.$$

Summary

	Sobolev	ridge(ball)	ridge(cube)
CURSE	$p = \infty$	$p = 2$	$p = 1$
WT	$\alpha > p$	$\alpha > (\frac{1}{\max\{1,p\}} - \frac{1}{2})^{-1}$	$\alpha > (\frac{1}{p} - 1)^{-1}$
(almost) UWT	strong compressibility ($p \ll 1$)		
QPT	hyperbolic cross	?	
PT	sparsity	$\lvert g'(0)\rvert > \kappa$	sparsity, $\lvert g'(0)\rvert > \kappa$, $a \geq 0$

Conclusions

For high-dimensional approximation problems:

- Asymptotic rates can have a very low significance
- Complexity is often determined by **preasymptotics**
- Smoothness can be enough
- Structure does not always help

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