MATRIX-DEPENDENT MULTIGRID-HOMOGENIZATION FOR DIFFUSION PROBLEMS

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Abstract. For problems with strongly varying or discontinuous diffusion coefficients we present a method to compute coarse-scale operators and to approximately determine the effective diffusion tensor on the coarse-scale level. The approach is based on techniques that are used in multigrid, such as matrix-dependent prolongations and the construction of coarse-grid operators by means of the Galerkin approximation. In numerical experiments we compare our multigrid-homogenization method with continuous homogenization, renormalization and simple averaging approaches. [1]

Key words. homogenization, multigrid, matrix-dependent prolongation, Galerkin approximation, Schur complement

AMS subject classification. 35B27, 65N55, 65N30

1. Introduction. Solutions for problems which model locally strong varying phenomena on a micro-scale level require that all length scales present in the problem are completely resolved. However, due to storage requirements and numerical complexity, the grid for numerical simulation cannot be chosen fine enough to meet this requirement. Fortunately, in many practical applications the fine-scale details of the solution are not of interest, but merely a coarse-scale solution is sought. Therefore it is sufficient to work with averaged equations, which describe the fine-scale behaviour of the problem under consideration directly. Thus, the first step in the direction of an accurate numerical treatment of these problems is the determination of an approximate mathematical model, that captures the influence of the unresolved fine scales of the medium and hence effectively disconnects the mesh size of the computational grid from the size of the heterogeneities. This "upscaling" or homogenization procedure results in equations with so called effective coefficients that vary on a coarse scale. The mechanics of disperse media, filtration, composite materials and porous media are examples from branches of physics where this homogenization procedure has been studied extensively [BLP87, BP84, Chr, SW91].

A typical example of such a problem is the diffusion equation

$$\begin{array}{cl} -\nabla(D\nabla u) = f & \text{in } \Omega = [0,1]^n \\ u = 0 & \text{on } \delta\Omega \end{array} \quad \text{where } D(x) \in \mathbb{R}^{n\times n}, \tag{1}$$

with D the hydraulic conductivity, the permeability or diffusion tensor. In porous media flow, under the assumption $\vec{g} = 0$ (zero gravity), the conservation of mass, the slight incompressibility assumption $u = u_0 e^{cp}$ and the restriction to the stationary case, this equation results from Darcy's law $\vec{q} = -\frac{D}{\mu}(\nabla p - u\vec{g})$ [ABT88].[2] In general the diffusion coefficient D is discontinuous, anisotropic and highly varying on a scale far below the

[1] S. Knapek, Matrix-dependent multigrid-homogenization for diffusion problems, SIAM J. Sci. 20(2), pp. 515-533, 1999.

[2] Here, p denotes the pressure, u the density, μ the dynamic viscosity of the fluid, $\vec{g}$ describes the

resolution of the computational grid. In petroleum reservoir models for example, the basic geophysical properties are usually generated by geostatistical reservoir description algorithms. These result in fine scale descriptions of reservoir properties, such as permeability, on grids that are far too fine to be used as a computational grid. Typically, the geostatistical and the computational grids differ in size by two, or more, orders of magnitude [Chr].

For the diffusion equation (1) an effective model is obtained from the determination of an effective diffusion tensor $\tilde{D}$, satisfying the following criteria:

- $\tilde{D}$ is slowly varying or even constant,
- $\nabla \tilde{D} \nabla$ is a direct description of the problem on the coarse scale,
- $\tilde{D}$ provides a good approximation of the influence that the unresolved fine scales have on the coarse-scale solution.

Replacing D by $\tilde{D}$ then gives a continuous coarse-scale problem, whose solution directly describes u on a coarse scale.

In the beginning of numerical simulation of reservoir performance this upscaling process was done by simply averaging the (measured or randomly generated) fine-scale diffusion coefficients using the arithmetic, geometric or harmonic mean (component-wise). However, these methods are not very accurate – especially in the case of strongly varying diffusion coefficients – and therefore are of little or even no use [Bou77, ABK91]. In the one-dimensional case it can be shown that the exact homogenized diffusion coefficient is given by the harmonic average [San80]. However, this is no longer true in multiple dimensions. Hence, in practice averaging techniques are used to approximate the exact homogenized diffusion tensor. In [Kin89] a renormalization procedure is applied to compute an approximation of the effective diffusion coefficient. This method is much more accurate than the simplistic means, but as with the simplistic means, the resulting $\tilde{D}$ is always a diagonal tensor. However, for a diffusion coefficient involving layers that are not oriented with the coordinate axes (which still can be described on the fine scale by a diagonal D) this is inadequate, as $\tilde{D}$ must contain non-diagonal entries which describe diffusion in the diagonal direction. Thus in the case of a dense diffusion tensor, the direction of the flow is never computed correctly.

The method of homogenization applies an asymptotic analysis to define the homogenized diffusion tensor by rigorous mathematical means, e.g. [Bab76, BP84, Hol95, JKO94]. In contrast to simple averaging and to renormalization, the homogenized diffusion tensor $\tilde{D}$ is in general no longer a diagonal matrix. For periodic problems this procedure reduces to solving a local problem, which is a linear partial differential equation on a basic cell with periodic boundary conditions [BP84, JKO94]. However, this approach is in general too expensive.

Similar problems arise in the development of multi-level iterative solvers. Especially in multigrid methods, discretizations on successively coarser grids are needed. To obtain them, aside from direct discretization on these grids, the Galerkin coarse-grid discretization method is used frequently [Hac85, Bra84]. It involves the discretization on the finest grid and produces the operators and equations on the coarser levels by using inter-grid transfer operators between successive grids. The efficiency of the resulting multigrid method depends heavily on the choice of inter-grid transfer operators. In most cases

gravity and $\vec{q}$ the volume of the fluid transported per time unit and area unit, f denotes a sink or source term.

standard inter-grid transfer operators (i.e. bilinear interpolation and full weight restriction), which correspond to an arithmetic averaging of the fine-grid equation, are used. Consequently a "physically meaningful" coarse-grid operator is not obtained in general, resulting in a deterioration of the convergence rate of the multigrid solver. Moreover the multigrid method with standard interpolation as prolongation is not robust, that is the convergence rate is dependent on the fine-grid structure and jumps in the diffusion coefficient [ABDP81]. The use of a combination of arithmetic and harmonic mean to define the coarse-grid operators in [ABDP81] and a related method in [ER93] utilizing the renormalization procedure of [Kin89] led to more efficient, but still not robust methods. There exist methods to determine the interpolation involved in the Galerkin coarsening in a *matrix-dependent* way [Den82, Den83, Zee90]. This results in stable and "physically meaningful" coarse-grid operators which are necessary for good convergence rates of the corresponding multigrid method also in the case of interface problems and singular perturbed operators.

Thus, on one hand, methods that are used in the modelling of homogenized operators such as renormalization and asymptotic analysis can be used for the determination of coarse-grid operators in multigrid methods, discretizing the homogenized equations directly to obtain coarse-grid operators [EL93, ER93, Neu96]. Conversely, matrix-dependent Galerkin approximations, which are used in some multigrid methods to obtain robust solvers, lead to coarse-grid operators that describe directly the behaviour of the coarse-scale solution [Kna95]. These discrete coarse-grid operators then lead to approximations of the effective diffusion equation by interpreting them as discrete coarse-scale operators. This is illustrated in figure 1.

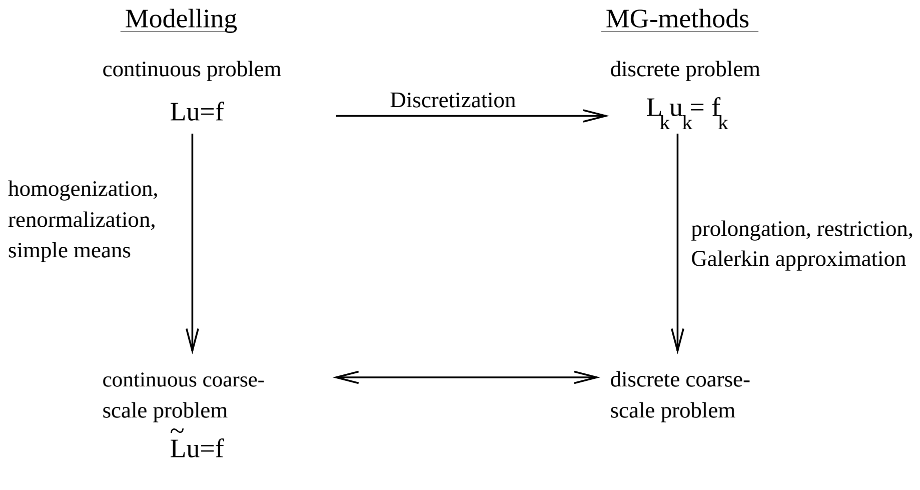

FIG. 1. *Relation between mathematical modelling methods and multigrid coarsening*

In this paper we present the interpretation of matrix-dependent Galerkin coarsening methods as discrete homogenization methods and we derive a technique for modelling the effective coarse-scale diffusion coefficient. Thus in section 2.1 we review the essential components of the multigrid method (e.g. prolongation and restriction). In section 2.2 we describe matrix-dependent prolongations and Schur complement approximations, demonstrating that these may be viewed as energy dependent averaging procedures that lead to averaged equations. In section 2.3 we present our method to define the diffusion coefficient from the stencil of a given coarse-grid operator and we derive the corresponding homogenization procedure. Finally (section 3) we give some results from numerical experiments for the diffusion equation and compare our new method with simple means, renormalization and exact homogenization. This approach gives results that are better than the ones obtained with standard averaging or with renormalization and nearly as accurate as those obtained with homogenization. A physically motivated technique related to the approach presented here has been studied in [DHM94]. It is a local cell-based method which is complementary to the vertex-based mathematically motivated method proposed here.

2. A multigrid homogenization method. Our approach for the computation of a coarse-scale operator and an approximation to the effective diffusion coefficient is based on methods that have been used for some time in robust multigrid algorithms. Specifically, the coarse-grid operators that are constructed with matrix-dependent prolongations by means of the Galerkin approximation may be viewed as discrete homogenized operators (see also figure 1). From the coarse-grid limit operator approximations of the effective diffusion coefficient can be determined, which are in the 1D case the exact homogenized diffusion coefficients and in 2D qualitatively good approximations of the homogenized tensor.

In the following we review variational coarsening and discuss matrix-dependent prolongations. We give a general principle on which schemes for matrix dependent prolongations can be based. This shows their relationship to incomplete factorization and approximate block Gaussian elimination. We show how (by applying repeated Galerkin coarsening) an approximation of the effective diffusion coefficient can be computed.

2.1. Multigrid discretization. Multigrid is an efficient method for solving partial differential equations, in which a sequence of grids

$$\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_{k-1} \subset \Omega_k$$

is defined, and hence mappings between the associated grid function spaces are needed. In the following discussion we will consider standard coarsening, as this suffices for our applications.

A popular and robust alternative to direct discretization on level l is the (recursive application of a) Galerkin approximation. If we let R_l^{l-1} denote the restriction from level l to $l-1$ and P_{l-1}^l denote the prolongation from level $l-1$ to l, then given a discretization L_k on the finest level k, coarse-grid operators $L_l, l = k-1, ...0$, are to be defined. Specifically, L_{l-1} is computed from L_l, P_{l-1}^l and R_l^{l-1} by the application of restriction and prolongation:

$$L_{l-1} = R_l^{l-1} L_l P_{l-1}^l, \text{ where (for symmetric } L_l\text{) } R_l^{l-1} := (P_{l-1}^l)^T \tag{2}$$

This approach is frequently referred to as variational coarsening as it follows upon the equivalent variational formulation of the linear system as minimization problem [Bra84].

For our model problem (1) a (bi-)linear finite element basis for test and trial space is the proper choice. This suggests the use of (bi-)linear interpolation as prolongation. However, it is well known that for singular perturbed problems and operators with highly variable or discontinuous coefficients, the performance and robustness of a multigrid solver depends on the coarse-grid operators, and hence, in the context of (2), on the fine-grid discretization and the choice of the prolongation [ABDP81]. For example, in the presence of jumps in the diffusion coefficient D, standard prolongation and restriction lead to a deterioration of the convergence rate. A remedy to this problem is the choice of matrix-dependent prolongations. They lead via (2) to coarse-grid operators, which provide a more accurate approximation of the problem's coarse-scale behaviour.

Additionally we note that with finite elements we get the stiffness matrix entries (with $\{\psi_j^k\}_{j=1}^{n_k}$, $\{\varphi_j^k\}_{j=1}^{n_k}$ as trial and test functions on Level k)

$$(L_k)_{ij} = \int_\Omega \nabla\varphi_j^k D \nabla\psi_i^k dx.$$

With

$$R_k^{k-1} := (r_{ij})_{ij} \text{ and } P_{k-1}^k := (p_{ij})_{ij}$$

we have

$$(L_{k-1})_{ij} = (R_k^{k-1} L_k P_{k-1}^k)_{ij} = \sum_{l,m} r_{jl} p_{mj} (L_k)_{lm} = \int_\Omega \nabla(\sum_l r_{jl}\varphi_l^k) D \nabla(\sum_m p_{mi}\psi_m^k) dx, \quad (3)$$

and with the new basis functions on level $k-1$ defined by

$$\varphi_j^{k-1} := \sum_l r_{jl}\varphi_l^k \text{ and } \psi_j^{k-1} := \sum_m p_{mj}\psi_m^k \quad (4)$$

we get the stiffness matrix entries

$$(L_{k-1})_{ij} = \int_\Omega \nabla\varphi_j^{k-1} D \nabla\psi_i^{k-1} dx. \quad (5)$$

Therefore prolongation and restriction are not only mappings between grid function spaces but may be viewed, via (4), as part of the discretization on coarse grids. Note, that in this way the Galerkin approximation can be interpreted as a discrete method for calculating averaged equations, which also model the influence of the scales smaller than the grid size of the actual level.

2.2. Matrix-dependent prolongations. The standard prolongations

$$P_{l-1}^{l\ *} = \begin{bmatrix} 1/2 & 1 & 1/2 \end{bmatrix} \text{ and } P_{k-1}^{k\ *} = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

(here in stencil notation for the 1D and 2D case [Zee90]) involving (bi-)linear interpolation on a uniform grid, correspond to (bi-)linear nodal basis functions on every level. A short calculation using (4) and (5) shows that this choice of interpolation in the Galerkin approach (2), yields an *arithmetic averaging* of D. As mentioned previously, arithmetic averaging is not suitable for the computation of the effective diffusion coefficient in the case of strongly varying or discontinuous D. Therefore, we will not use standard prolongation in the multigrid coarsening process. Instead we will utilize the information that is stored in the matrix L_l to construct a *locally weighted* prolongation P_{l-1}^l that resembles an operator-induced interpolation, which leads, via (2), or equivalently (4) and (5), to an implicitly weighted averaging of D.

Matrix-dependent prolongations P_{l-1}^l can be derived using a general principle: They can be interpreted as transformations to perform (approximately) a block Gaussian elimination of the operator L_l, which will be explained in the following. Let there be given a discrete problem $L_l u_l = f_l$ on a grid Ω_l. Partitioning of L_l in coarse grid and fine-without-coarse grid parts gives

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{6}$$

where u_1 denotes the unknowns in $\Omega_l \setminus \Omega_{l-1}$ and u_2 the unknowns in the next coarser grid Ω_{l-1}. The *optimal* matrix-dependent prolongation arises from the *exact* block Gaussian elimination of the outer diagonal blocks L_{12} and L_{21}, i.e. from transforming L_l to

$$\begin{pmatrix} I & -L_{11}^{-1}L_{12} \\ 0 & I \end{pmatrix}^T \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} I & -L_{11}^{-1}L_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} - L_{21}L_{11}^{-1}L_{12} \end{pmatrix}. \tag{7}$$

The associated matrix-dependent prolongation operator between grid $l-1$ and l is then defined by

$$P_{l-1}^l = \begin{pmatrix} -L_{11}^{-1}L_{12} \\ I \end{pmatrix} \tag{8}$$

and the coarse-grid operator obtained via the Galerkin approach (2) yields

$$L_{l-1} = (P_{l-1}^l)^T L_l P_{l-1}^l := L_{22} - L_{21}L_{11}^{-1}L_{12}, \tag{9}$$

which is just the well known Schur complement of L_l with respect to Ω_{l-1}.

What is done implicitly here is a multilevel-splitting of the underlying finite element basis. If we use the standard nodal basis functions on the finest grid, which we denote by $\{\phi_i\}_{i\in N_k}$ where $N_k := \{i : x_i \in \Omega_k\}$, the Gaussian elimination is nothing else than a splitting of $span(\{\phi_i\}_{i\in N_k})$ into the spaces spanned by the nodal basis functions in the grid points in $\Omega_k \setminus \Omega_{k-1}$ and its a-orthogonal complement, where a is the scalar product induced by the model problem:

$$a(u,v) = \int_\Omega \nabla v D \nabla u dx \text{ with } u,v \in H^1(\Omega)$$

More precisely the space V_k spanned by the standard nodal basis functions on the fine grid is split according to

$$V_k = W_k \oplus_a \tilde{V}_{k-1}$$

with $W_k := span(\{\phi_i\}_{i\in N_k \setminus N_{k-1}})$ and $\tilde{V}_{k-1} := span(\{\psi_i\}_{i\in N_{k-1}})$, where ψ_i is given by $[\psi_i] = P_{k-1}^k (\begin{array}{cc} 0 & I \end{array})[\phi_i]$ for $i \in N_{k-1}$ (in coordinates given by $\{\phi_i\}_{i\in N_k}$). Applying this approach recursively leads to a multilevel splitting of the finite element space into a-orthogonal subspaces

$$V_k = W_k \oplus_a W_{k-1} \oplus_a \ldots \oplus_a W_1 \oplus_a \tilde{V}_0$$

and the coarse-grid operator from (9) is a restriction of the fine-grid operator to the subspace $\tilde{V}_{l-1} = W_{l-1} \oplus_a \ldots \oplus_a W_1 \oplus_a \tilde{V}_0$.

Of course, this approach is impractical in general as the inverse of the fine-grid block, L_{11}^{-1}, is not a local operator, and therefore the sparsity structure of L_{l-1} is lost. However, in the 1D case things are different [Hac85]: Let L_l be given by the 3-point stencil

$$L_l^*(x_i) = [\begin{array}{ccc} L_{-1}(x_i) & L_0(x_i) & L_1(x_i) \end{array}]. \tag{10}$$

The partitioning of the associated matrix according to (6) now results in just a diagonal matrix L_{11}. Then L_{11}^{-1} is diagonal and consequently the prolongation P_{l-1}^{l} and the coarse-grid operator L_{l-1} are local operators. In stencil notation we obtain from (8)

$$P_{l-1}^{l}{}^{*}(x_{i+1}) = \left[\ -\tfrac{L_1(x_i)}{L_0(x_i)} \quad 1 \quad -\tfrac{L_{-1}(x_{i+2})}{L_0(x_{i+2})}\ \right] \tag{11}$$

(with $x_{i+1} = x_i + h_l$, h_l the mesh width on level l) and by the Galerkin approach (9) we have

$$L_{l-1}{}^{*}(x_{i+1}) = \left[\ -\tfrac{L_{-1}(x_i)L_{-1}(x_{i+1})}{L_0(x_i)} \quad \tfrac{L_{-1}(x_i)L_{-1}(x_{i+1})}{L_0(x_i)} + \tfrac{L_1(x_{i+1})L_1(x_{i+2})}{L_0(x_{i+2})} \quad -\tfrac{L_1(x_{i+1})L_1(x_{i+2})}{L_0(x_{i+2})}\ \right]$$

with $x_{i+1} \in \Omega_{l-1}$.

For the special case of the diffusion equation $(d(x)u(x)_x)_x = f(x)$, where $d(x)$ is assumed to be piecewise constant on the finest grid Ω_k, having jumps only at the grid points, we obtain by discretizing with finite differences or finite elements on Ω_k the stencil

$$L_k{}^{*}(x_{IF+1}) = \frac{1}{h_k}\left[\ -d_{IF+1}^k \quad d_{IF+1}^k + d_{IF+2}^k \quad -d_{IF+2}^k\ \right] \tag{12}$$

with $d_{IF+1}^k = d(x_{IF+1} - h_k/2)$, compare figure 2. Here we use for coarse-grid values the index IC and for fine-grid values the index IF.

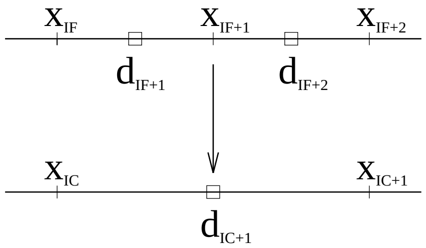

Fig. 2: 1D case: fine and coarse grid

Using the corresponding prolongation (11)

$$P_{k-1}^{k}{}^{*}(x_{IC}) = \left[\ \tfrac{d_{IF}^k}{d_{IF-1}^k + d_{IF}^k} \quad 1 \quad \tfrac{d_{IF+1}^k}{d_{IF+1}^k + d_{IF+2}^k}\ \right] \text{ for } x_{IC} \in \Omega_{k-1}, \tag{13}$$

we obtain

$$L_{k-1}{}^{*}(x_{IC}) = \frac{2}{h_{k-1}}\left[\ -\tfrac{d_{IF-1}^k d_{IF}^k}{d_{IF-1}^k + d_{IF}^k} \quad \tfrac{d_{IF-1}^k d_{IF}^k}{d_{IF-1}^k + d_{IF}^k} + \tfrac{d_{IF+1}^k d_{IF+2}^k}{d_{IF+1}^k + d_{IF+2}^k} \quad -\tfrac{d_{IF+1}^k d_{IF+2}^k}{d_{IF+1}^k + d_{IF+2}^k}\ \right]. \tag{14}$$

Comparing (14) with (12) gives the coarse-grid diffusion coefficient on the interval $[x_{IC}, x_{IC+1}]$:

$$d_{IC+1}^{k-1} = 2\frac{d_{IF+1}^k d_{IF+2}^k}{d_{IF+1}^k + d_{IF+2}^k} \tag{15}$$

This is just the harmonic mean of the two corresponding fine-grid values. In other words: Direct discretization on the coarse grid Ω_{k-1} using the *local harmonic average* of the fine-grid diffusion coefficient would give the same coarse-grid operator. Applying this approach recursively for the sequence of successively coarser grids, we finally obtain on Ω_0 the global harmonic mean as effective diffusion coefficient. That is, we obtain the exact result of homogenization [San80].

In the two-dimensional case L_{11} is tridiagonal, and hence it is no longer practical to employ the exact matrix-dependent prolongation, as this would involve the computation of L_{11}^{-1}, which is in general a full matrix. Therefore approximations to the optimal matrix-dependent prolongation are used, which result in local operators. Such approximations can be obtained by substituting the set of equations associated with the grid points $\Omega_k \setminus \Omega_{k-1}$ by an altered set of equations, i.e. substitute $\begin{pmatrix} L_{11} & L_{12} \end{pmatrix}$ by some $\begin{pmatrix} \tilde{L}_{11} & \tilde{L}_{12} \end{pmatrix}$, which meets the following demands [Re93a, Fuh94, DE88, Kna95]:

- $\tilde{L}_{11}$ is 'easy' to invert,
- $\tilde{L}_{11}^{-1}\tilde{L}_{12}$ is an approximation of $L_{11}^{-1}L_{12}$ and
- $\begin{pmatrix} -\tilde{L}_{11}^{-1}\tilde{L}_{12} \\ I \end{pmatrix}$ is a local operator.

The associated prolongation operator $\tilde{P}_{k-1}^{k}$ is defined by

$$\tilde{P}_{k-1}^{k} = \begin{pmatrix} -\tilde{L}_{11}^{-1}\tilde{L}_{12} \\ I \end{pmatrix} \tag{16}$$

and the resulting coarse-grid operator L_{k-1} is given by

$$L_{k-1} = (\tilde{P}_{k-1}^{k})^t L_k \tilde{P}_{k-1}^{k}.$$

Thus, the approximation of the optimal matrix-dependent prolongation is the optimal matrix-dependent prolongation of an altered discretization, namely

$$\tilde{L}_k = \begin{pmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ L_{21} & L_{22} \end{pmatrix}. \tag{17}$$

We consider two simple examples how to construct an approximate matrix-dependent prolongation. To this end, we proceed as follows: If $l > 0$, then every grid Ω_l can be divided in four disjunct grids:

$$\begin{array}{lcl} \Omega_l^{00} & := & \{(x,y) \in \Omega_l : x = ih_l, y = jh_l, i,j \in \mathrm{Z}, i \text{ even}, j \text{ even}\} = \Omega_{l-1} \\ \Omega_l^{10} & := & \{(x,y) \in \Omega_l : x = ih_l, y = jh_l, i,j \in \mathrm{Z}, i \text{ odd}, j \text{ even}\} \\ \Omega_l^{01} & := & \{(x,y) \in \Omega_l : x = ih_l, y = jh_l, i,j \in \mathrm{Z}, i \text{ even}, j \text{ odd}\} \\ \Omega_l^{11} & := & \{(x,y) \in \Omega_l : x = ih_l, y = jh_l, i,j \in \mathrm{Z}, i \text{ odd}, j \text{ odd}\}. \end{array} \tag{18}$$

Then we can partition the matrix L_l as

$$L_l = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \left(\begin{array}{cc|c} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ \hline l_{31} & l_{32} & l_{33} \end{array}\right). \tag{19}$$

Here $\begin{pmatrix} l_{11} & l_{12} & l_{13} \end{pmatrix}$ corresponds to the equations belonging to the unknowns associated with the points in Ω^{11}, $\begin{pmatrix} l_{21} & l_{22} & l_{23} \end{pmatrix}$ corresponds to the equations belonging to the unknowns associated with the points in $\Omega^{10} \cup \Omega^{01}$ and $\begin{pmatrix} l_{31} & l_{32} & l_{33} \end{pmatrix}$ corresponds to the equations belonging to the unknowns associated to the points in $\Omega^{00} = \Omega_{l-1}$. If L_l can be described by a 9-point stencil then l_{11} and l_{33} are diagonal matrices.

Perhaps the simplest approximation is to replace L_{11} by its diagonal $diag\{L_{11}\}$. Another possibility, which we suspect is more accurate, is to choose

$$\tilde{L}_{11} = \begin{pmatrix} l_{11} & l_{12} \\ 0 & \tilde{l}_{22} \end{pmatrix} \tag{20}$$

(remember that l_{11} is diagonal!) with $\tilde{l}_{22}$ a diagonal matrix or an upper triangular matrix. Then the inverse is given by

$$\tilde{L}_{11}^{-1} = \begin{pmatrix} l_{11}^{-1} & -l_{11}^{-1} l_{12} \tilde{l}_{22}^{-1} \\ 0 & \tilde{l}_{22}^{-1} \end{pmatrix} \tag{21}$$

and the prolongation can be defined by

$$\tilde{P}_{l-1}^{l} = \begin{pmatrix} -\tilde{L}_{11}^{-1} \begin{pmatrix} l_{13} \\ \tilde{l}_{23} \end{pmatrix} \\ I \end{pmatrix} = \begin{pmatrix} -l_{11}^{-1}(l_{13} - l_{12}\tilde{l}_{22}^{-1}\tilde{l}_{23}) \\ -\tilde{l}_{22}^{-1}\tilde{l}_{23} \\ I \end{pmatrix}.$$

Note that this is the exact prolongation of the altered discretization

$$\tilde{L}_{l} = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ 0 & \tilde{l}_{22} & \tilde{l}_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix}. \tag{22}$$

Further prolongations can easily be constructed in a similar way. See for example [ABDP81, Den82, DE88, FG93, Fuh94, Re93a, Re93b] and [AV89, AV90, Zee90]. We state that all the matrix-dependent schemes we know of can be based on the principle mentioned above and can be denoted also in form of a substitution (17). This shows their relationship to incomplete factorization and approximate block Gaussian elimination.

As an example consider the following substitutions, which involve 'directional' lumping of fine-grid stencils [Den82]. Let

$$\begin{bmatrix} L_{-1,1} & L_{0,1} & L_{1,1} \\ L_{-1,0} & L_{0,0} & L_{1,0} \\ L_{-1,-1} & L_{0,-1} & L_{1,-1} \end{bmatrix}$$

denote a stencil of the fine-grid operator L_k. The stencils in points of Ω^{10} and Ω^{01} are changed in the following way:

$$\begin{bmatrix} L_{-1,1} & L_{0,1} & L_{1,1} \\ L_{-1,0} & L_{0,0} & L_{1,0} \\ L_{-1,-1} & L_{0,-1} & L_{1,-1} \end{bmatrix} \longrightarrow \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline L_{-1,0} & L_{0,0} & L_{1,0} \\ +L_{-1,-1} & +L_{0,-1} & +L_{1,-1} \\ +L_{-1,1} & +L_{0,1} & +L_{1,1} \\ \hline 0 & 0 & 0 \end{array}\right] \text{ in } \Omega^{10}, \tag{23}$$

$$\begin{bmatrix} L_{-1,1} & L_{0,1} & L_{1,1} \\ L_{-1,0} & L_{0,0} & L_{1,0} \\ L_{-1,-1} & L_{0,-1} & L_{1,-1} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & L_{0,1} + L_{-1,1} + L_{1,1} & 0 \\ 0 & L_{0,0} + L_{-1,0} + L_{1,0} & 0 \\ 0 & L_{0,-1} + L_{-1,-1} + L_{1,-1} & 0 \end{bmatrix} \text{ in } \Omega^{01}. \tag{24}$$

Then $\tilde{l}_{22}$ becomes a diagonal matrix.

Since for our simple model problem (1) most of these methods coincide with this scheme, we will restrict ourselves to it in the numerical experiments. With this choice the resulting coarse-grid operator L_{l-1} built from a 9-point stencil matrix L_l and the associated matrix-dependent prolongation P_{l-1}^{l} by means of the Galerkin approximation (2) are again given by 9-point stencils and the matrix-dependent coarsening procedure can be repeated recursively on coarser grids.

2.3. The multigrid homogenized diffusion coefficient. Variational coarsening results in coarse-grid operators, which can be viewed as homogenized operators via (4),(5). We are interested in an approximation of the effective diffusion coefficient. Thus we present a technique to extract a diffusion tensor from a given coarse-grid stencil. We apply a vertex centered approach to compute a diffusion tensor for every stencil of the coarse-grid operator, that is for every grid point. This is in contrast to the cell centered approach given in [DHM94], where a diffusion tensor is computed for every cell of the coarse grid. Moreover we give an algorithm for our homogenization procedure (considering the periodic case) and a possible extension to the non-periodic case.

Let

$$\begin{bmatrix} L_{-1,1} & L_{0,1} & L_{1,1} \\ L_{-1,0} & L_{0,0} & L_{1,0} \\ L_{-1,-1} & L_{0,-1} & L_{1,-1} \end{bmatrix}$$

be a stencil of a given coarse-grid operator. Then this stencil can be interpreted as a linear combination of the finite element stencils

$$id = \frac{1}{36}\begin{bmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{bmatrix},$$

$$\partial_x = \frac{1}{12h}\begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix}, \quad \partial_y = \frac{1}{12h}\begin{bmatrix} 1 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & -4 & -1 \end{bmatrix},$$

$$\partial_{xx} = \frac{1}{6h^2}\begin{bmatrix} 1 & -2 & 1 \\ 4 & -8 & 4 \\ 1 & -2 & 1 \end{bmatrix}, \quad \partial_{yy} = \frac{1}{6h^2}\begin{bmatrix} 1 & 4 & 1 \\ -2 & -8 & -2 \\ 1 & 4 & 1 \end{bmatrix},$$

$$\partial_{xy} = \frac{1}{4h^2}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad \partial_{xxy} = \frac{1}{2h^3}\begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix},$$

$$\partial_{xyy} = \frac{1}{2h^3}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{bmatrix}, \quad \partial_{xxyy} = \frac{1}{h^4}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

which form a basis for an arbitrary 9-point stencil. Assuming $L_k 1_k = 0_k$ and $\tilde{L}_k 1_k = 0_k$ (zero row sums) we have

$$\begin{pmatrix} \tilde{L}_{11} & \tilde{L}_{12} \end{pmatrix} 1_k = 0_k \Longleftrightarrow \tilde{P}^k_{k-1} 1_{k-1} = 1_k$$

and

$$L_{k-1} 1_{k-1} = (\tilde{P}^k_{k-1})^t L_k \tilde{P}^k_{k-1} 1_{k-1} = (\tilde{P}^k_{k-1})^t L_k 1_k = (\tilde{P}^k_{k-1})^t 0_k = 0_{k-1}.$$

Hence the row sums of the coarse-grid operators (defined via the Galerkin approach) equal zero and the coefficient of id in the linear combination is zero. Note that $\tilde{P}^k_{k-1} 1_{k-1} = 1_k$ does hold for Dendy's scheme. That is we have a linear combination

$$\alpha_x \partial_x + \alpha_y \partial_y + \alpha_{xx} \partial_{xx} + \alpha_{yy} \partial_{yy} + \alpha_{xy} \partial_{xy} + \alpha_{xxy} \partial_{xxy} + \alpha_{xyy} \partial_{xyy} + \alpha_{xxyy} \partial_{xxyy}$$

for the 9-point stencil. Then a diffusion tensor $\tilde{D}$ can be defined:

$$\tilde{D} = \frac{1}{h^2} \begin{pmatrix} \alpha_{xx} & \frac{\alpha_{xy}}{2} \\ \frac{\alpha_{xy}}{2} & \alpha_{yy} \end{pmatrix}$$

It should be mentioned, that it is in general not possible to describe a stencil of a coarse-grid operator as a linear combination of the finite element stencils of ∂_{xx}, ∂_{yy} and ∂_{xy} only, as we will see in the following section.

Now we give the algorithm for our homogenization procedure. We first consider the periodic case and then give a possible extension to the non-periodic case.

Periodic case: Let D be a periodic diffusion coefficient on an infinitely large domain, and imagine the diffusion coefficient to be completely resolved by the numerical grid. We use bilinear finite elements to obtain a discrete operator, which can be described by a nine point stencil at every grid point. We apply the matrix dependent Galerkin coarsening procedure recursively until we have (due to the periodicity of D) at every coarse-grid point the same diffusion tensor $\tilde{D}$. Eventually after several more coarsening steps, the resulting $\tilde{D}$ is invariant under further Galerkin coarsening steps. This diffusion tensor is our so-called MG-homogenized diffusion tensor D^{MG-hom}. It is obvious, that due to the periodicity of D it suffices to compute the coarse-grid operator for a finite domain, that is for only a few periods of the diffusion coefficient.

In summary, we obtain the following scheme:

1. Discretize the diffusion equation on a domain Ω of several periods (the number of periods depends on the problem, see section 3) of D
2. Apply a matrix dependent Galerkin coarsening step on the fine-grid operator
3. Interpret the resulting stencils as linear combination of the finite element stencils associated to ∂_x, ∂_y, ∂_{xx}, ∂_{yy}, ∂_{xy}, ∂_{xxy}, ∂_{xyy} and ∂_{xxyy} and read of the entries of $\tilde{D}$ which are defined as the coefficients of ∂_{xx}, ∂_{yy} and ∂_{xy} in the linear combination
4. Repeat step 2,3 until in every grid point the same $\tilde{D}$ (invariant under variational coarsening) is obtained; this defines D^{MG-hom}

Non-periodic case: For completeness we propose a possible extension to the non-periodic case, but do not investigate its performance. In the non-periodic case, the domain can be repeated to tile $\mathbb{R}^2$, removing the influence of boundary conditions. Then we can proceed as in the periodic case.

In the case of a randomly distributed diffusion coefficient, we apply no periodicity trick in the coarsening process, but apply Galerkin coarsening recursively on the sequence of grids $\Omega_k, \Omega_{k-1}, \dots \Omega_0$. We then determine the corresponding approximation of the effective diffusion coefficient from the operator on the second coarsest level Ω_1.

We get the following scheme:

1. Generate N samples of the random distribution
2. For every sample compute the MG-homogenized diffusion tensor as described above
3. Compute an average of the N MG-homogenized diffusion coefficients

Up to now we used for simplicity in 3. an arithmetic averaging of the diffusion coefficients (component wise), which would coincide with standard coarsening applied to the N MG-homogenized operators.

Remark: Note that if the diffusion coefficient D can not be completely resolved on the finest grid, then the result of MG-homogenization is dependent on the size of the finest grid, as the discretization on this grid is in itself some kind of averaging.

3. Numerical examples. In the following discussion we present experiments that provide insight into the behaviour of MG-homogenization. Here, unless indicated otherwise, we consider the periodic setting for the computation of the approximation of the effective diffusion coefficient. We start from the discretization with conforming bilinear finite elements, repeat the Galerkin coarsening process using *Dendy's* matrix-dependent prolongations and compute the MG-homogenized diffusion coefficient D^{MG-hom} as described in the preceding section. In order to compare our results with those from other methods, we also compute the approximations D^{ren} using renormalization according to [Kin89], D^{arith}, D^{geo} and D^{harm} using arithmetic, geometric and harmonic averaging, respectively, and D^{hom} using homogenization. In all of the following examples we assume, that the diffusion coefficient D is completely resolved on the finest grid used, that is, the jumps of the diffusion coefficient coincide with grid lines.

The first problem considered is a layered medium with the layers aligned with the coordinate axes (i.e. an essentially one-dimensional problem). We then consider an example with a square internal inhomogeneity, a layered medium with the layering oriented in the diagonal direction, and a problem with diffusion coefficient of checkerboard type. Finally, we discuss two cases of randomly distributed diffusion coefficients.

Example 1: Layered medium. We consider a diffusion coefficient which is of tensor product type and depends on the x-coordinate only.

$$D(x,y) = D(x) = \begin{cases} d1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \forall (x,y) \in Y_I \\ d2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \forall (x,y) \in Y_{II}. \end{cases} \quad \text{with } d_1, d_2 \in \mathbb{R}^+.$$

This resembles a layered structure of the medium which is aligned with the y-axis (see figure 3 (left)). By means of periodicity, structures such as that of figure 3 (right) are associated. In this case an analytic result for the homogenized diffusion tensor exists, namely

$$D^{hom} = \begin{pmatrix} \frac{2d_1 d_2}{d_1+d_2} & 0 \\ 0 & \frac{1}{2}(d_1+d_2) \end{pmatrix}. \tag{25}$$

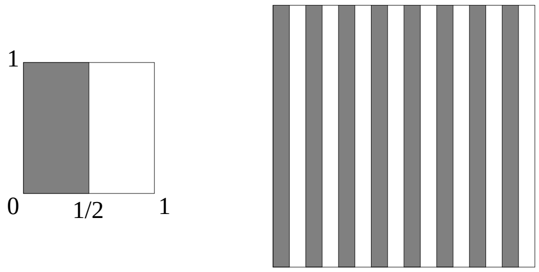

Fig. 3: Structure of the diffusion coefficient for example 1

For the MG-homogenization we have to consider the discretization and the corresponding matrix-dependent prolongation. The stencil of the finite element discretization operator L_k in any point is

$$L_k^* = \begin{bmatrix} L_{-1} & (L_{-1}+L_1)/2 & L_1 \\ L_{-1} & -4(L_{-1}+L_1) & L_1 \\ L_{-1} & (L_{-1}+L_1)/2 & L_1 \end{bmatrix}$$

Plugging this into the formulae of Dendy's approach we observe the following: In the y-direction the corresponding matrix-dependent prolongation reduces to the standard interpolation, and thus results in arithmetic averaging, whereas in the x-direction we obtain the harmonic average of the diffusion coefficient, a result consistent with the 1D example (15). Thus, for this problem MG-homogenization gives the exact homogenized diffusion tensor. We note that renormalization also yields (25).

3.1. Example 2: Domain with inclusions. The diffusion coefficient of the second model problem is defined by

$$D(x,y) = \begin{cases} \alpha \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \forall (x,y) \in Y_I \\ 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \forall (x,y) \in Y_{II}. \end{cases}$$

This is illustrated in figure 4 (left). By means of periodicity structures such as that shown in figure 4 (right) are associated, see also [ABK91, Bou77]. The symmetry of the problem

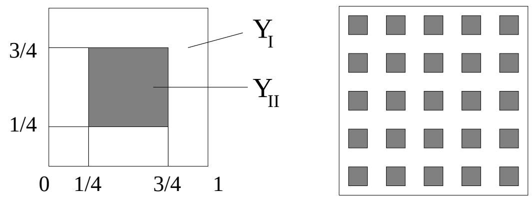

FIG. 4. *Structure of the diffusion coefficient for example* 2

and the diagonal structure of D imply that all methods yield an approximation of the effective diffusion tensor which is a scalar multiple of the identity,

$$\tilde{D} = \gamma \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma \in \mathbb{R}^+.$$

The value for γ depends on the value α and the respective method under consideration. We will denote the different values for γ by d^{MG-hom}, d^{hom} and so on.

With our MG-homogenization method we obtain for $\alpha = 10$ (on a grid of 31×31 inner grid points, 8×8 periods of the diffusion coefficient, 3 coarsening steps) in every coarse-grid point a stencil which is a linear combination of the stencils corresponding to Δ and $\frac{\partial^4}{\partial x^2 \partial y^2}$, namely

$$-6.70088 \cdot h^2 \Delta + \beta \cdot h^4 \frac{\partial^4}{\partial x^2 \partial y^2}$$

with different β for different grid points. The coefficient of Δ is invariant under further Galerkin coarsening steps. Therefore we obtain

$$d^{MG-hom} = 6.70088.$$

Bilinear interpolation as prolongation would result in the arithmetic mean $d^{arith} = \frac{3\alpha+1}{4}$ of d. Other popular choices include the geometric mean, which gives $d^{geo} = \alpha^{3/4}$ and the harmonic mean which is simply $d^{harm} = \frac{4}{3/\alpha+1}$. A comparison for $\alpha = 10$ and $\alpha = 100$ is summarized in Table 1. Again the results are obtained using a grid of 31×31 inner grid points, 8×8 periods of the diffusion coefficient. 3 coarsening steps were necessary to obtain an diffusion tensor which is invariant under further coarsening. We clearly see that the approach based on recursive matrix-dependent Galerkin coarsening gives values which are closest to the exact values obtained for this periodic problem by homogenization (d^{hom}). All other approaches give worse values.

TABLE 1
Problem 1: Comparison of the approximations of the effective diffusion coefficient

α	d^{hom}	d^{MG-hom}	d^{ren}	d^{arith}	d^{geo}	d^{harm}
10	6.52	6.70088	6.16712	7.75	5.6234	3.0769
100	59.2	61.73646	54.24763	75.25	31.6228	3.8835

Note that the computed values do not depend on the grid size of the finest level used, provided the jumps of the diffusion coefficient coincide with the grid lines of some level. For more general problems involving a diffusion coefficient which is not constant on each cell of the finest grid the resulting approximations of the effective diffusion coefficient depend on the fine-grid size.

3.2. Example 3: Diagonally layered structure. In the third example we set the diffusion coefficient to

$$D(x,y) = \begin{cases} 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \forall (x,y) \in Y_I \\ \alpha \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \forall (x,y) \in Y_{II}, \end{cases}$$

where the subdomains Y_I and Y_{II} are indicated in figure 5 (left), see also [SV93]. By means of periodicity layered structures like the one given in figure 5 (right) are built. Because

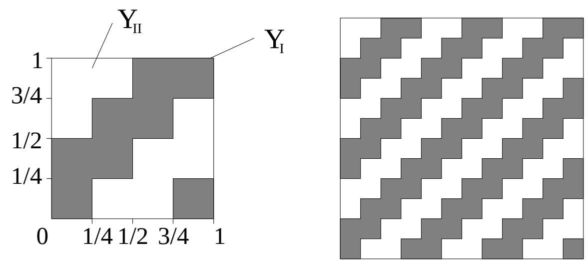

FIG. 5. *Representative cell and sample tiling*

of the layering of the diffusion coefficient in diagonal direction the exact homogenized diffusion coefficient must be a dense matrix, i.e. it contains diffusion in the off-diagonal entries as well.

For $\alpha = 8$ on a grid with 63×63 inner grid points (16×16 periods of the diffusion coefficient) we obtain after 4 coarsening steps the coarse-grid stencil

$$A = \begin{bmatrix} -2.0551 & -1.0416 & -0.6669 \\ -1.0416 & 9.6105 & -1.0416 \\ -0.6669 & -1.0416 & -2.0551 \end{bmatrix},$$

which is a linear combination of the finite element stencils corresponding to ∂_{xx}, ∂_{yy}, ∂_{xy} and ∂_{xxyy}. Reinterpreting this as a continuous equation, we have

$$-\nabla D^{MG-hom} \nabla + 0.1065 \cdot h_k^2 \frac{\partial^4}{\partial x^2 \partial y^2}$$

with

$$D^{MG-hom} = \begin{pmatrix} 3.7636 & 1.3882 \\ 1.3882 & 3.7636 \end{pmatrix} = U \begin{pmatrix} 5.15 & 0 \\ 0 & 2.37 \end{pmatrix} U^t,$$

where h_k is the mesh width and the matrix of eigenvectors

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

defines the principal axes of diffusion.

The approximations that we obtained with the different methods for $\alpha = 8$ were

$$D^{hom} = \begin{pmatrix} 2.9897 & 0.9443 \\ 0.9443 & 2.9897 \end{pmatrix} = U \begin{pmatrix} 3.92 & 0 \\ 0 & 2.04 \end{pmatrix} U^t,$$

$$D^{arith} = \begin{pmatrix} 4.5 & 0 \\ 0 & 4.5 \end{pmatrix}, \quad D^{geo} = \begin{pmatrix} 2.83 & 0 \\ 0 & 2.83 \end{pmatrix},$$

$$D^{ren} = \begin{pmatrix} 2.44 & 0 \\ 0 & 2.44 \end{pmatrix}, \quad D^{harm} = \begin{pmatrix} 1.78 & 0 \\ 0 & 1.78 \end{pmatrix}.$$

With our MG-homogenization procedure we obtain the exact principal axes of diffusion, whereas the approaches that are based on renormalization and the arithmetic mean (as well as the other simple averaging schemes) are unable to produce off-diagonal entries in the diffusion tensor and thus give a coarse scale equation that is completely wrong.

3.3. Example 4: Checkerboard diffusion. In this example the diffusion coefficient is chosen in a red-black or checkerboard type, as indicated in figure 6. See also [DHM94]. The diffusion coefficient is set according to

$$D(x,y) = \begin{cases} 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \forall (x,y) \in Y_I \\ \alpha \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \forall (x,y) \in Y_{II} \end{cases} \quad \text{with } d_1, d_2 \in \mathbb{R}^+.$$

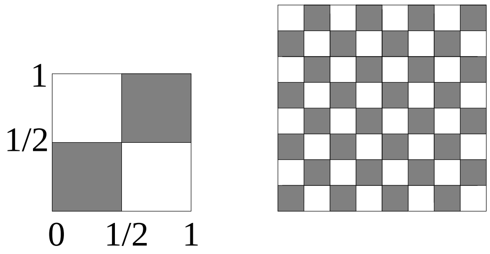

Fig. 6: Structure of the diffusion coefficient for example 4

The exact homogenized diffusion coefficient in this case is [JKO94]

$$\sqrt{d_1 \cdot d_2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

MG-homogenization however reduces to computing the arithmetic mean of d_1 and d_2, and thus we obtain

$$D^{MG-hom} = \frac{d_1 + d_2}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is due to the lumping procedure performed in (23),(24), where the stencils are averaged either in x- or in y-direction. The renormalized diffusion tensor is given by

$$\frac{8(d_1 + d_2)d_1 d_2}{(4d_1 d_2 + 3(d_1 + d_2)^2)} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking $d_1 = \frac{1}{d_2}$ shows that the error of MG-homogenization and of renormalization is unbounded.

3.4. Example 5: Randomly distributed diffusion. The last example presented involves an isotropic diffusion coefficient

$$D(x, y) = d(x, y) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the values of $d(x, y)$ are randomly chosen over Ω and assumed to be piecewise constant on fine-grid cells. This definition of $d(x, y)$ is representative of the statistical models that are used frequently in oil reservoir simulations [Dag86]. We took a 256×256 grid on the finest level and generated 10 samples (i.e. $N = 10$) for every statistic considered. The results that we obtained with the different homogenization methods are given in Table 2 and 3. For comparison we computed an approximation to the exact effective diffusion coefficient by solving the model problem (1) numerically and computing the flux through Ω in x- and in y-direction. In Table 2 the diffusion values were chosen equally distributed from the interval $[a, b]$ and for Table 3 we used a distribution with very large jumps. The diffusion coefficient d is distributed according to

$$d = z^{-ln(a)}, \ z \in \mathcal{X}, \mathcal{X} \text{ equally distributed in } (0, 1).$$

For this special distribution the exact effective diffusion coefficient is the geometric mean of the fine-scale diffusion coefficient, which is given by a [Koz80]. Clearly the new method is superior to the other methods, especially in the case of large jumps in the diffusion coefficient.

TABLE 2

Results for random diffusion coefficients which are equally distributed in $[a, b]$

$[a,b]$	$[1,9]$	$[1,99]$
max	8.999	$9.900e1$
min	1.000	1.000
standard deviation	2.300	$2.811e1$
arith. mean	5.001	49.958
geom. mean	4.313	37.090
harm. mean	3.615	21.140
renormalization	4.284	37.792
MG-homogenization	4.605	44.089
Direct num. simulation	4.543	42.603

TABLE 3

Results for random diffusion coefficients distributed with $d = z^{-ln(a)},\ z \in \mathcal{X}, \mathcal{X}$ *equally distr. in* $(0, 1)$

a	2	5	10	20
max	$1.60e3$	$2.02e8$	$1.11e13$	$3.08e17$
min	1.00	1.00	1.00	1.00
standard deviation	$1.39e1$	$7.95e5$	$4.35e10$	$1.30e15$
arith. mean	3.26	$9.1e3$	$7.5e8$	$5.7e15$
harm. mean	1.68	2.60	3.27	3.80
renormalization	1.88	3.73	5.82	8.76
MG-homogenization	2.12	5.59	11.49	23.63

4. Concluding remarks. In this article we introduced a method to compute an approximation to the effective diffusion tensor. Our technique is based on the Galerkin coarsening approach involving matrix-dependent prolongations and on a local technique for the computation of a diffusion tensor from a given 9-point stencil. We applied the method to problems with a periodic setting and with a random diffusion coefficient. In the periodic case the results were close to the optimal values obtained from exact homogenization and better than that obtained from renormalization or simple averaging. However there are pathological cases such as the checkerboard, where our new method fails. Nevertheless it proved to be suitable and reliable for a variety of cases, including problems with random diffusion coefficients with very large jumps. Generalizations of the proposed homogenization procedure to problems with no periodicity of the diffusion structure are to be considered in the future.

We conjecture that there exist other techniques used to maintain a fast and robust convergence rate in multigrid methods which can be exploited also in many areas of mathematical modelling where multiscale questions are important.

Acknowledgements: The author would like to thank F. Bornemann, A. Bourgeat and the referees for their useful comments. The author is indebted to M. Griebel and to J.D. Moulton for several very helpful suggestions, comments and discussions.

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