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Master's Thesis

Master's Thesis by Sa Wu

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In [1] Mallet-Paret introduces a Fredholm Alternative theorem for asymptotically hyperbolic linear functional differential equations of mixed type and first order and uses it in [2] to apply the implicit function theorem to certain one dimensional nonlinear functional differential equations of first order. In this thesis, we try to use these results for some two dimensional or order two equations.

Contents

Nomenclature

$\bar{x}^F_{\pm}$	two stable equilibria $\pm(1,1)$ of (3.1)	28
$.^F$	something pertaining to a function F	2
$\cdot_{c,x,\rho}$	something pertaining to linearizations at (c,x,ρ)	2
Δ	characteristic function	7
$\imath(A,B)$	index of an asymptotically hyperbolic operator with limits A, B	12
$\operatorname{ind} A$	index of a Fredholm operator A, (2.8)	8
ker	kernel of some operator	2
Λ	a linear fde operator	2
Λ^*	quasidual to Λ, (2.14)	11
A_i	coefficients of a linear fde	2
E	eigenvalues	3
$M(\xi)$	perturbation of some constant coefficient operator L_0, (2.7)	7
q	unstable equilibrium of (2.19)	15
r_i	some distinct shifts in $\mathbb{R}$, $r_1 = 0$	1

Numbers

1 Introduction

1.1 Problem

When investigating solutions to nonlinear function differential equations of mixed type as

$$\begin{aligned} 0 &= \mathcal{G}(c, x, \rho)(\xi) \\ &= -cx^{(1)}(\xi) - F\Big(x(\xi + r_1), \cdots, x(\xi + r_N), \rho\Big) \end{aligned} \tag{1.1}$$

for $d \in \mathbb{N}, N \in \mathbb{N}\backslash\{1, 2\},\ c \neq 0, \rho \in \mathbb{R}, x, F \in \left(\mathbb{R}^d\right)^{\mathbb{R}}$ sufficiently smooth and $r_1 = 0, r_2, \cdots r_N \in \mathbb{R}$ distinct, one might ask for existence of solutions near known ones for a varying parameter $\rho \in V$ taken from an open subset V of some Banach space W, i.e. if the implicit function theorem may be applied.

Such equations may arise from various sources such as the search for traveling wave solutions $f(z, t) = \varphi(\big\langle z, v\big\rangle - ct)$ of some differential equation

$$0 = -\dot{f}(x, t) - F(f(x + x_0, t), \cdots, f(x + x_N, t), \rho)$$

with some $x, x_0, \cdots, x_N, z, v \in \mathbb{R}^d; \varphi \in \left(\mathbb{R}^d\right)^{\mathbb{R}}, f \in \left(\mathbb{R}^d\right)^{\mathbb{R}^d \times \mathbb{R}}$ sufficiently smooth.

For use of implicit function theorem for solutions of (1.1) around some (c_0, x_0, ρ_0) we need G to be at least Fréchet C^1 in some open neighborhood of (c_0, x_0, ρ_0) and $D_{c,x}G$ to be an isomorphism. Fréchet differentiability is often only a technicality, if F is sufficiently smooth. The isomorphism condition on $D_{c,x}(c_0, x_0, \rho_0)$ however might not be so easy to show.

However if $D_{c,x}(c_0, x_0, \rho_0)$ is *asymptotically hyperbolic*, then parts of proving bijectivity can be done as roughly outlined in the following section.

1.2 Agenda

Taking, dependent on (c, x, ρ) with some fixed $\rho \in V$

$$A_i = A_{i,x,\rho}(\xi) = \mathcal{D}_{x(\cdot + r_j)}\, F(x(\xi + r_1), \cdots, x(\xi + r_N), \rho) \tag{1.2}$$

$$(\Lambda y)(\xi) = (\Lambda_{c,x,\rho} y)(\xi) = -cy^{(1)}(\xi) - \sum_{i=1}^{N} A_{i;x,\rho}(\xi) y(\xi + r_i) \tag{1.3}$$

the linearization of G about some (c_0, x_0, ρ_0), where x_0 is not an equlibrium of (1.1), with respect to (c_0, x_0) takes the form

$$\begin{aligned}\Big(\big(D_{c,x}G\big)(c_0,x_0,\rho_0)\Big)(d,u)(\xi) =&\Big(-x_0^{(1)}(\xi)\Big)d+\\ &\Big(-c_0u^{(1)}(\xi)-\sum_{i=1}^{N}A_i(\xi;x_0,\rho_0)u(\xi+r_i)\Big)\\ =&\Big(\big(-x_0^{(1)}d\big)+\big(\Lambda_{c_0,x_0,\rho_0}u\big)\Big)(\xi).\end{aligned} \tag{1.4}$$

Since we consider multiple equations in the later parts, we will add a supscript to differentiate the origins of operators, coefficients etc like $A^F_{i,x,\rho}$.

Studying $D_{c,x}\mathcal{G}(c_0, x_0, \rho_0)$ naturally involves studying Λ_{c_0,x_0,ρ_0} and its relation to the linear span of $x_0^{(1)}$. Naturally if (c_0, x_0, ρ_0) is some solution to (1.1) with x_0 not being an equilibrium then $\forall \xi \in \mathbb{R}$:

$$\begin{aligned}0 &= \frac{d}{d\xi}\mathcal{G}(c_0,x_0,\rho_0)(\xi)\\ &= -c\big(x_0^{(1)}\big)^{(1)}(\xi) - \sum_{i=1}^{N}\mathcal{D}_{x(\cdot+r_j)}F(x_0(\xi+r_1),\cdots,x_0(\xi+r_N),\rho)x_0^{(1)}(\xi+r_i)\\ &= \Big(\Lambda(c_0,x_0,\rho_0)x_0^{(1)}\Big)(\xi)\end{aligned}$$

so $u \equiv x_0^{(1)}$ solves

$$0 = \Lambda_{c_0,x_0,\rho_0}u. \tag{1.5}$$

This implies $x_0^{(1)} \in \mathfrak{K}_{c_0,x_0,\rho_0} := \ker \Lambda_{c_0,x_0,\rho_0}$ and $\dim \mathfrak{K}_{c_0,x_0,\rho_0} \geq 1$. Hence, assuming that $G\colon \mathbb{R}\times W^{1,\infty}\times V \to L^\infty$, any restriction $D_{c,x}\mathcal{G}(c_0, x_0, \rho_0)\colon \mathbb{R}\times X \to L^\infty$ to a subspace $W^{1,\infty} \supseteq X \ni x_0$, on which Λ_{c_0,x_0,ρ_0} is injective and whose range does not contain $x_0^{(1)}$, is injective. Surjectivity can then be established by just setting $Y := x_0^{(1)}\mathbb{R}\oplus\Lambda_{c_0,x_0,\rho_0}(X)$. To sum up we need to

1. establish $x_0 \notin \mathfrak{K}_{c_0,x_0,\rho_0}$, (1.6a)
2. find $W^{1,\infty} \supseteq X \ni x_0$ such that $X \cap \mathfrak{K}_{c_0,x_0,\rho_0} = \{0\}$, (1.6b)
3. establish $x_0^{(1)} \notin \Lambda_{c_0,x_0,\rho_0}(X)$, (1.6c)
4. set $Y := \Lambda_{c_0,x_0,\rho_0} \oplus x_0^{(1)}\mathbb{R}$, (1.6d)
5. and use the implicit function theorem on $G\colon \mathbb{R}\times X \times V \to Y$. (1.6e)

f $\mathfrak{K}_{c_0,x_0,\rho_0}$ splits $W^{1,\infty}$, then a natural choice for X is a indexsplit topological complement of the kernel or any closed subspace thereof.

We remark that a restriction to some closed subspace as proposed in (1.6e) does not restrict validity of the implicit function theorem, i.e if the implicit function theorem were applicable for a larger space yielding some unique solution, then it would have to coincide with a unique solution obtained from the smaller space.

f Λ_{c_0,x_0,ρ_0} is a Fredholm operator, i.e. $\dim \mathfrak{K}_{c_0,x_0,\rho_0} < \infty$ and $\operatorname{codim} \Lambda_{c_0,x_0,\rho_0}(W^{1,\infty}) < \infty$, then a standard result from functional analysis, e.g. Standard Example 17 in subsection 3.9.4 from [4], guarantees that $\mathfrak{K}_{c_0,x_0,\rho_0}$, since it's finite dimensional, splits $W^{1,\infty}$. Then we only need to show $x_0^{(1)} \notin \Lambda_{c_0,x_0,\rho_0}(X)$. Λ_{c_0,x_0,ρ_0} being Fredholm also implies

$$\Lambda_{c_0,x_0,\rho_0} = \mathfrak{K}^*_{\perp L^\infty}$$

where $\mathfrak{K}^*$ is usually the kernel of Λ_{c_0,x_0,ρ_0}'s dual operator and $B_{\perp L^\infty}$ denotes B's annihilator in L^∞. Hence knowledge of $\mathfrak{K}^*$, i.e. just one $u^* \in \mathfrak{K}^*$ such that $f(x_0^{(1)}) \neq 0$, proves $x_0^{(1)} \notin \Lambda_{c_0,x_0,\rho_0}$.

If Λ_{c_0,x_0,ρ_0} really were Fredholm then its index

$$\operatorname{ind} \Lambda_{c_0,x_0,\rho_0} = \dim\ker \Lambda_{c_0,x_0,\rho_0} - \operatorname{codim} \Lambda_{c_0,x_0,\rho_0} = \dim\ker \Lambda_{c_0,x_0,\rho_0} - \dim \mathfrak{K}^* \qquad (1.7)$$

together with knowledge of $\mathfrak{K}_{c_0,x_0,\rho_0}$, of which we already know that it contains $x_0^{(1)}\mathbb{R}$, can help us to establish dimension of $\mathfrak{K}^*$ and hence find some $u^* \in \mathfrak{K}^*$ satisfying $\left\langle x_0^{(1)}, u^* \right\rangle = u^*(x_0^{(1)}) \neq 0$.

For the last few arguments to be applicable we of course have to establish that Λ_{c_0,x_0,ρ_0} is really a Fredholm operator. Normally this would already require explicit knowledge of kernel and range of this operator but theory established in [1] and reexamined in [3] allows us to forego such explicit knowledge and prove Fredholmness as long as Λ_{c_0,x_0,ρ_0} is an *asymptotically hyperbolic* linear operator, which is basically an operator that joins two limiting linear constant coefficients operators without imaginary eigenvalues.

In particular this theory from originally [1] establishes Fredholmness for all asymptotically hyperbolic linear operators $W^{1,\infty} \to L^\infty$. Since the $\operatorname{ind}\colon F(W^{1,\infty}, L^\infty) \to \mathbb{N}$ is a continuous function from the set of Fredholm operators to the natural numbers in discrete topology, the index is constant along continuous paths in $F(W^{1,\infty}, L^\infty)$, i.e.

$$\forall A\colon [0,1] \to F(W^{1,\infty}, L^\infty) \text{ continuous } : \rho \to \operatorname{ind} A(\rho) \equiv \text{const.}$$

. So continuous transformations of asymptotic hyperbolic operators, which preserve asymptotic hyperbolicity, also preserve Fredholmness and hence the index. This knowledge can enable us to establish the index of Λ_{c_0,x_0,ρ_0} by just finding homotopies in the space of asymptotically hyperbolic operators that join operators we investigate and operators with known indices. Morever we know from this theory, that hyperbolic operators are isomorphisms and hence operators with known index zero.

Furthermore theory from [1] tells us that we need not concern ourselves with dual spaces and dual operators for determination of $x_0^{(1)} \notin \Lambda_{c_0,x_0,\rho_0}$ via Fredholm Alternative, but can instead use an asymptotically hyperbolic quasidual operator

$$\Lambda^*_{c_0,x_0,\rho_0}\colon W^{1,1} \to L^1$$

for the dual correspondencies normally associated with Fredholm operators. In particular, similar to Fredholm Alternatives via *Dual Pairs* in section 5.10 of [4], we have

$$\Lambda_{c_0,x_0,\rho_0}(W^{1,\infty}) = \mathfrak{K}^*_{\perp L^\infty} = \{f \in L^\infty | \forall u^* \in \ker \Lambda^*_{c_0,x_0,\rho_0} \colon \int_{\mathbb{R}} \langle f, u^* \rangle = 0\}$$

which might further simply the search for some $u^* \in \mathfrak{K}^* \colon u^*(x_0^{(1)}) \neq 0$.

In any case we would have to establish asymptotical hyperbolicity of Λ_{c_0,x_0,ρ_0} for the preceding arguments to be valid. If x_0 is a nonequlibirum solution that joins two stable equilibria, say $\bar{x}_\pm$ of the main nonlinear equation (1.1), then smoothness requirements imposed on F tell us that asymptotical hyperbolicity is equivalent to hyperbolicity of the linearizations at the equilibria it joins, $\Lambda_{c_0,\bar{x}_\pm,\rho_0}$. Then these linearizations also form the limiting operators of Λ_{c_0,x_0,ρ_0}.

For actual equations we now try to work our way backwards in our preceding arguments to enable us to use the implicit function theorem. Accordingly, if we have the goal of using the implicit function theorem to find solutions around some solution x_0 of (1.1) joining the stable equilibrium $\bar{x}_-$ at $-\infty$ to $\bar{x}_+$ at ∞ and forego the use of $\Lambda_{c_0,x_0,\rho}$'s dual by just using the quasidual, we might accomplish this if we

1. establish hyperbolicity of $\Lambda_{c_0,\bar{x}_\pm,\rho_0}$, (1.8a)
2. calculate $\operatorname{ind} \Lambda_{c_0,x_0,\rho_0}$, (1.8b)
3. derive $\mathfrak{K}_{c_0,x_0,\rho_0}$, and $\dim \mathfrak{K}_{c_0,x_0,\rho_0}$ (1.8c)
4. set $X \ni x_0$ as closed subspace of the topological complement of $\mathfrak{K}_{c_0,x_0,\rho_0}$. (1.8d)
5. use (1.8b) and (1.8c) to find $u^* \in \mathfrak{K}^* := \ker \Lambda^*_{c_0,x_0,\rho_0}$, (1.8e)
6. establish $\int_{\mathbb{R}} \langle x_0^{(1)}, u^* \rangle \neq 0$ for some $u^* \in \mathfrak{K}^*$ (1.8f)
7. set $Y := x_0^{(1)} \mathbb{R} \oplus \Lambda_{c_0,x_0,\rho_0}(X)$ (1.8g)
8. and finally use the implicit function theorem at (c_0, x_0, ρ_0) for

$$G \colon \mathbb{R} \times X \times V \to Y, \tag{1.8h}$$

which now has an isomorphic $D_{c,x}\mathcal{G}(c_0, x_0, \rho_0)$ by construction.

Later we might choose to drop some of the subscript $\cdot_{c_0,x_0,\rho_0}$ when the context should leave no doubt about the origins of things or when considering the general class of linear equations with the properties of Λ_{c_0,x_0,ρ_0} as arising from linearizations about known solutions.

2 Already known

To be able to follow the steps noted in (1.8) we now reproduce some definitions, notations and basic results taken from [1], [2], [3].

The first section establishes notation and basic definitions and notation. Some of the definitions might not be optimal for a context more general than just the discussion of implicit function theorem use for solutions of (1.1) but might keep the notation coherent throughout the remainder of this thesis.

Afterwards we present two results from [1]. The first concerns the relation between Fredholmness and asymptotic hyperbolicity and the dual pairing of asymptotically hyperbolic $W^{1,p} \to L^p$ and $W^{1,q} \to L^q$ operators. The second result allows one to calculate the Fredholm index of asymptotically hyperbolic. The third one gives us asymptotics information on solutions to asymptotically hyperbolic linear solutions based on the eigenvalues of the limiting operators.

Finally, to complete our review and presentation of already known thing, we present some of the main results from [2], reexamined in [3], that are needed in the last chapter of this thesis to analyze a simple 2d equation composed of two components solving the equation discussed in [2].

Since this chapter mainly contains reviews and reproductions we only sketch the general ideas of proofs for just copied results.

2.1 Some definitions

We start with some definitions and notation as used in [1] for theory of asymptotically hyperbolic linear operators $\Lambda\colon W^{1,p}(\mathbb{R}, \mathbb{C}^d) \to L^p(\mathbb{R}, \mathbb{C}^d)$ for $1 \leq p \leq \infty$.

Definition 2.1.1
We will use the common notations

$$L^p := L^p(\mathbb{R}, C^d)$$
$$W^{k,p} := \{f \in L^p | \forall 0 \leq j \leq k : f^{(k)} \in L^p\}$$

for L^p and Sobolev spaces, $Y^X := \{f : X \to Y\}$ for functions between any two sets X, Y and $B(X, Y) \subseteq Y^X$ as the normed space of all bounded linear operators $X \to Y$, $X^* = B(X, \mathbb{C})$ for the dual space of a normed space X.

Moreover for any open interval $J \subseteq \mathbb{R}$ we write

$$C^k(J) := C^k(J, C^d)$$
$$C^k_B(J) := \left(\{f \in C^k(J) | \|f\|_{C^k_B(J)} < \infty\}, \|\cdot\|_{C^k_B(J)}\right)$$

with derivatives generally to be assumed in the weak sense,

$$\|f\|_{C^k_B(J)} \coloneqq \sup_{0\le j\le k} \sup \xi \in J \|f^{(i)}(\xi)\|$$

and

$$C^k(\overline{J}) \coloneqq \{f \in \left(C^d\right)^{\overline{J}} | (f|_J) \in C^k(J) \wedge \forall 0 \le i \le k : f^{(i)} \text{ continuously extendable to } \overline{J}\}$$
$$C^k_B(\overline{J}) \coloneqq \{f \in C^k(\overline{J}) | \|f\|_{C^k_B(J)} < \infty\}$$

for the closure $\overline{J}$ of a relatively compact interval J.

For $A \in B(X, Y)$ we shall use

$$\ker A \coloneqq \{x \in X : Ax = 0\} \qquad A(X) \coloneqq \{Ax | x \in X\}$$

for kernel and range of a bounded linear operator.

For any normed space X and $A \subseteq X, B \subseteq X^*$ we write

$$B_\perp = B_{\perp A} \coloneqq \{x \in A | \forall f \in B \colon f(x) = 0\}$$

for a subsets annihilator and omit the A in $\perp A$ when the setting is clear.

Most of these notations can be found in textbooks on general functional analysis like [4] or books on PDEs or Sobolev spaces.

Morever we use the $\|\cdot\|_2$ with $\mathbb{C}^d$'s and $\|(a, b)\|_{X\times Y} \coloneqq \|a\|_X + \|B\|_Y$ for the product of two normed spaces $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y$.

Definition 2.1.2
We already introduced a special case of

$$0 = \big(\Lambda x\big)(\xi) = \big(\Lambda_c x\big)(\xi) = -cx^{(1)}(\xi) - \sum_{i=1}^{N} A_i(\xi) x(\xi + r_i) \tag{2.1}$$

in (1.5), with $c \in \mathbb{R}\setminus\{0\}, x \in W^{1,\infty}$ and A_is uniformly bounded and measurable.

For convenience we introduce the minimal and maximal shifts

$$r_- \coloneqq \min\{r_i | 1 \le i \le N\} \qquad r_+ \coloneqq \max\{r_i | 1 \le i \le N\} \tag{2.2}$$

and some shift operator for $\xi \in \mathbb{R}$

$$\begin{aligned} \tau_\xi \colon \left(\mathbb{C}^d\right)^{\mathbb{R}} &\to \left(\mathbb{C}^d\right)^{\mathbb{R}} \\ (\tau_\xi f)(\eta) &= f(\eta - \xi) \end{aligned} \tag{2.3}$$

to describe equation (2.1) with a family of operators

$$\begin{aligned} L(\xi) \colon C_B([r_-, r_+]) &\to \mathbb{C}^d \\ \varphi &\mapsto \sum_{i=1}^{N} A_i(\xi)\varphi(r_i) \end{aligned} \tag{2.4}$$

with $L(\xi) = L_{x_0,\rho_0}(\xi)$ in the setting of the linearization (1.5) to introduce the notation

$$0 = \big(\Lambda_{c,L}x\big)(\xi) = x^{(1)}(\xi) - L(\xi)\big((\tau_{-\xi}x)|_{[r_-,r_+]}\big)$$

for (2.1), with $(\tau_{-\xi}x)|_{[r_-,r_+]} \in C_B([r_-,r_+)$ being guaranteed by the Sobolev embedding $W^{k,p} \to C_B^{k-1}$.

Whenever a Λ_0 as in (2.1) and hence some L_0 as in (2.4) has constant coefficients A_i, we may introduce a characteristic function

$$\begin{aligned}\Delta = \Delta_{c,L_0} &: \mathbb{C} \to \mathbb{C}\\ \lambda &\mapsto -c\lambda - \sum_{i=1}^{N} A_i e^{\lambda r_i}\end{aligned} \tag{2.5}$$

We call the constant coefficient system/equation (2.1)/operator Λ_{c,L_0}/operator $\frac{1}{c}L_0$ *hyperbolic* if

$$\forall b \in \mathbb{R}\colon \Delta_{c,L_0}(\mathrm{i}b) \neq 0 \tag{2.6}$$

and the values $\{\lambda \in \mathbb{C}\,|\Delta(\lambda) = 0\}$ *eigenvalues*, the term arising from the context of the point spectrum of the operator

$$\begin{aligned}A : \{\varphi \in C^1(\mathbb{R},\mathbb{C}^d) | \forall \xi \in \mathbb{R}\colon \varphi^{(1)}(\xi) = L_0\tau_{-\xi}\} &\to C(\mathbb{R},\mathbb{C}^d)\\ \varphi &\mapsto \varphi^{(1)}\end{aligned}$$

which coincides with the zeros of Δ_{c,L_0}. If a hyperbolic operator arises from linearizations about an equilibrium we also call this equilibrium hyperbolic.

We call the elements in the generalized eigenspaces of A to each such eigenvalue λ *eigensolutions*. Note however that while these functions do solve $\varphi(\xi) = L_0\tau_{-\xi}\varphi$ with L_0 being applicable here, eigensolutions x are not per se solutions to $\Lambda x = 0$ since the setting is generally

$$\Lambda\colon W^{1,\infty} \to L^\infty$$

in addition to the smoothness requirements of the linearization and implicit function theorem setting. Moreover these eigensolutions all take the form $\xi \mapsto e^{\lambda\xi}v_0$ for some $v_0 \in C^d$. Note that the term eigenvalues might be misleading in regards to the point spectrum of $L(\xi)$ for various ξ or $\Lambda_{c,L}$ to which they have no connection. We shall write E^Λ for the eigenvalues of Λ.

If we can express $L(\xi)$ as a perturbation of some constant coefficient operator L_0, i.e.

$$L(\xi) = L_0 + M(\xi) \qquad\qquad \equiv \forall i\colon A_i(\xi) = A_i + B_i(\xi) \tag{2.7}$$

with some B_is, uniformly bounded and measurable like the A_is, and

$$\begin{aligned}&\lim_{\xi\to\infty} \|M(\xi)\| = \|M(\xi)\|_{B(C_B^1([r_-,r_+],\mathbb{C}^d),\mathbb{C}^d)} = 0\\ \Leftrightarrow &\lim_{\xi\to\infty} \|B_i(\xi)\| = 0 \text{ uniformly with respect to } i\end{aligned}$$

then we call $L(\xi)$ or Λ or (2.1) *asymptotically autonomous* at ∞. We call it/them asymptotically autonomous at $-\infty$ it the same conditions are fulfilled for $\xi \to -\infty$. If it is/they are asymptotically autonomous at both $\pm\infty$ we call it/them asymptotically autonomous.

If L is asymptotically autonomous with limiting operators $L_\pm, \Lambda_{c,\pm}$ at $\pm\infty$ and furthermore $\Lambda_{c,\pm}$ are hyperbolic then we call (2.1),L,$\Lambda_{c,L_\pm}$ *asymptotically hyperbolic.*

In the context of the asymptotic autonomy we write $A_{i,+,\cdots}$ etc for the coefficients and other things related to the constant coefficient operator at ∞ and $A_{i,-,\cdots}$ etc for the constant coefficient operator at $-\infty$.

Definition 2.1.3

Let X, Y be two Banach spaces over $\mathbb{C}$. $A \in B(X,Y) \subseteq L(X,Y)$, with $B(X,Y)$ denoting the space of continuous linear operators and $L(X,Y)$ the space of all linear operators $X \to Y$, is called a *Fredholm Operator* or simply *Fredholm* iff

$$\dim\ker A < \infty \qquad\qquad \operatorname{codim} A(X) < \infty.$$

Then the *index* or *Fredholm index* of A is defined as the integer

$$\operatorname{ind} A := \dim\ker A - \operatorname{codim} A(X) \tag{2.8}$$

Let $F(X,Y) \subseteq B(X,Y)$ denote the closed space of all Fredholm operators and let $A \in F(X,Y)$. Then, with $X*$ denoting Xs dual and A^T its dual operator, the following properties are met.

1. $A(X)$ is closed (2.9a)
2. $A(X) = (\ker A^T)_\perp$ and $A^T(Y^*) = (\ker A)^\perp$ (2.9b)
3. $\operatorname{codim} A^T(Y^*) = \dim\ker A$ (2.9c)
4. $\operatorname{codim} A(X) = \dim\ker A^T = \dim\ker A - \operatorname{ind} A$ (2.9d)
5. $A^T : Y^* \to X^*$ is Fredholm and $\operatorname{ind} A^T = -\operatorname{ind} A$ (2.9e)
6. $\operatorname{ind} : L(X,Y) \to \mathbb{N}$ is continuous (2.9f)

(2.9b), (2.9c), (2.9d) represent a so called *Fredholm Alternative.*

The interested reader is referred to [4]. For (2.9f) in particular, which is the basis of relatively simple calculation of the Fredholm index for asymptotically hyperbolic linear equations as (1.3), see Proposition 1 of section 5.8 of [4].

Definition 2.1.4

With solution, e.g. to (1.1), we generally mean a function $x \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^d)$ satisfying (1.1) in L^p sense. Let $J \subseteq \mathbb{R}$ be some interval. Then by a solution on J we mean a function $x\colon J^\# := J + \{r_i | 1 \le i \le N\}$ satisfying

1. $x \in$ "$C(J^\#, \mathbb{C}^d) \cap L^p(J^\#, \mathbb{C}^d)$" (2.10a)

2. $x|_J \in W^{1,p}(J, \mathbb{C}^d)$ (2.10b)

3. $\forall \xi \in J\colon : 0 = -cx^{(1)}(\xi) - F(x(\xi + r_1), \cdots, x(\xi + r_N), \rho)$ (2.10c)

with $C(\mathbb{R}, \mathbb{C}^d) \cap L^p$ understood in usual sense of L^p functions with continuous representatives. Equivalent definitions hold for equations with $F = F(\xi, x(\xi), \cdots)$ etc.

Before beginning to recall theory established in [1] and [2] we just add a standard lemma to make sure $\mathcal{G}$ is really Fréchet C^1.

Lemma 2.1.5
Assume that $F = F(v, \rho)$ in (1.1) is C^1 in $(\mathbb{R}^d)^N \times V$. and that $D_v F(v, \rho)$ is locally Lipschitz with respect to v.

Then $\mathcal{G}$ is C^1 on the open set

$$U \coloneqq \big(\mathbb{R} \setminus \{0\}\big) \times \{x \in W^{1,\infty} | x^{(1)} \neq 0\} \times V \subseteq \mathbb{R} \times W^{1,\infty} \times W \tag{2.11}$$

and the partial Fréchet derivative $D_{c,x}\mathcal{G}(c, x, \rho)$ takes the form (1.5).

Proof. Fix $(c, x, \rho) \in U$. We set $\kappa(x, \xi) \coloneqq (x(\xi + r_1), \cdots, x(\xi + r_N))$

$$\begin{aligned} L(c, x, \rho)\colon\ & \mathbb{R} \times W^{1,\infty} \times W \to L^\infty \\ & (d, y, \sigma) \mapsto -dx^{(1)} + \Lambda_{c,x,\rho} y - \frac{\partial}{\partial \rho} F(\kappa(x, \cdot), \rho)\sigma \end{aligned} \tag{2.12}$$

Obviously $(c, x, \rho) \mapsto L(c, x, \rho)$ is continuous in U.

We claim that L is $\mathcal{G}$'s Fréchet derivative.

Since F is C^1 in $(\mathbb{R}^d)^N \times V$ we have $\forall (v, \rho) \in (\mathbb{R}^d)^N \times V\colon\ \forall \epsilon > 0\colon\ \exists \delta_0(\epsilon) > 0$ such that $\forall \delta_0 > \|(w, \sigma)\|\colon$

$$\frac{1}{\|(w, \sigma)\|} \Big(F(v + w, \rho + \sigma) - F(v, \rho) - D_{v,\rho} F(v, \rho)(w, \sigma) \Big) \leq \epsilon.$$

From the definition of κ we get $\forall \xi \in \mathbb{R}\colon\ \forall x \in W^{1,\infty}\colon$

$$\begin{aligned} \|\kappa(x, \xi)\| &= \|(x(\xi + r_1), \cdots, x(\xi + r_N))\| \\ &\leq \sum_{i=1}^{N} \|x(\xi + r_i)\| \leq N \|x\|_{L^\infty} \\ &\leq N \|x\|_{W^{1,\infty}} \end{aligned}$$

Since

$$\frac{|d| \|y\|_{W^{1,\infty}}}{\|(d, y)\|} = \frac{|d| \|y\|_{W^{1,\infty}}}{|d| + \|y\|_{W^{1,\infty}}}$$

it follows that if $\|d, y\| < \frac{1}{N}\epsilon \eqqcolon \delta_1$ we have

$$\frac{|d| \|y\|_{W^{1,\infty}}}{\|(d, y)\|} < \frac{1}{2N}\epsilon.$$

With $\|(d,y,\sigma)\| < \max\{\frac{1}{2N}\delta_0(\frac{1}{N}\epsilon), \delta_1\} =: \delta$ we now have

$$\begin{aligned}
&\frac{1}{N\|(d,y,\sigma)\|}\|\big(\mathcal{G}(c+d,x+y,\rho+\sigma)-\mathcal{G}(c,x,\rho)\\
&\qquad\qquad - L(c,x,\rho)(d,y,\sigma)\big)(\xi)\|\\
\leq\ &\frac{1}{\|d,(\kappa(y,\xi),\sigma)\|}\|\Big(-(c+d)(x+y)^{(1)}(\xi)+cx^{(1)}(\xi)\\
&\qquad - F(\kappa(x+y,\xi),\rho+\sigma)+F(\kappa(x,\xi),\rho)\\
&\qquad + dx^{(1)}(\xi)+cy^{(1)}(\xi)+D_{x,\rho}\big(F(\kappa(x,\xi),\rho)\big)(\kappa(y,\xi),\sigma)\Big)\|\\
\leq\ &\frac{1}{\|(\kappa(y,\xi),\sigma)\|}\Bigg(|d|\|y^{(1)}\|_{W^{1,\infty}}+\|\Big(F(\kappa(x,\xi)+\kappa(y,\xi),\rho+\sigma)-F(\kappa(x,\xi),\rho)\\
&\qquad - D_{v,\rho}F(\kappa(x,\xi),\rho)(\kappa(y,\xi),\sigma)\Big)\|\Bigg)\\
\leq\ &\frac{1}{N}\epsilon.
\end{aligned}$$

Taking the (essential) supremum over ξ and multiplying with N yields $\forall\|(d,y,\sigma)\| < \delta$:

$$\frac{1}{\|(d,y,\sigma\|}\|\mathcal{G}(c+d,x+y,\rho+\sigma)-\mathcal{G}(c,x,\rho)-L(c,x,\rho)(d,y,\sigma)\|_{L^\infty}\leq\epsilon$$

Thus $\mathcal{G}$ has the Fréchet derivative L and since L is continuous $\mathcal{G}$ is C^1 in U.

Proposition 8 from section 4.2 of [4] applied to (2.12) together with the first part of the proof yields

$$\begin{aligned}
\big(D_{c,x}\mathcal{G}(c,x,\rho)\big)(d,y) &= \big(D_{c,x,\rho}\mathcal{G}(c,x,\rho)\big)(d,y,0)\\
&= \big(L(c,x,\rho)\big)(d,y,0)\\
&= -x^{(1)}d+\Lambda_{c,x,\rho}y+0.
\end{aligned}$$

This proves the second part of the lemma. □

2.2 Linear Fredholm Theory

Using the above definitions and notation we now p pr As advertised we present two results, which establish Fredholmness for asymptotically hyperbolic linear operators with duality for $W^{1,p}\to L^p$ through a dual pairing with $W^{1,q}\to L^q$ for conjugated $1\leq p,q\leq\infty$ and means to calculate the index when the limiting operators at $\pm\infty$ are related or can be joined in a hyperbolicity preserving way. A third one establishes a relation between general solutions and eigensolutions of (2.1).

A Fredholm Alternative

First we reproduce a theorem established as Theorem A in [1] and reexamined as Theorem 3.2.3 in [3]. Using this result we can later establish step (1.8a) of our agenda, that is, establish Fredholmness of Λ_{c_0,x_0,ρ_0}.

Theorem 2.2.1

Consider an asymptotically hyperbolic equation as in (2.1). Let $1 \leq q \leq \infty$ be conjugated to some $1 \leq p \leq \infty$. Then the quasidual operator defined by

$$\begin{aligned} &\Lambda^* : W^{1,q} \to L^q \\ &\big(\Lambda^* y\big)(\xi) = c y^{(1)}(\xi) + \sum_{i=1}^{N} A_i^T(\xi - r_i) y(\xi - r_i) \end{aligned} \tag{2.14}$$

Λ^* is also asymptotically hyperbolic and

1. hyperbolic, constant coefficient Λ are isomorphisms (2.15a)
2. Λ, Λ^* are Fredholm operators (2.15b)
3. $\ker \Lambda, \ker \Lambda^*$ are independent from p, q. (2.15c)
4. $$\begin{aligned} \Lambda(W^{k,p}) &= \big(\ker \Lambda^*\big)_{\perp L^p} \\ \Lambda^*(W^{k,q}) &= \big(\ker \Lambda\big)_{\perp L^q} \end{aligned} \tag{2.15d}$$
5. $$\begin{aligned} \operatorname{codim} \Lambda(W^{k,p}) &= \dim \ker \Lambda^* \\ \operatorname{codim} \Lambda^*(W^{k,q}) &= \dim \ker \Lambda \\ \operatorname{ind} \Lambda &= -\operatorname{ind} \Lambda^* \end{aligned} \tag{2.15e}$$

using $\ker \Lambda$ for kernel, $\Lambda(W^{k,p}$ for image of the operator Λ and $X_{\perp Y}$ for the annihilator of X as subspace of Y.

Proof. (2.15a) is Theorem 4.1 from [1] or Theorem 2.1.2 from [3]. The proof involves viewing (2.1) as an equation of temepered distributions and solving it by transforming Λ into multiplication operator via Fourier transform. This gives rise to a Green's function for Λ. The equation holding for tempered distributionally for a distribution induced by a proper function implies that the equation also holds for weak derivatives and hence in the $W^{1,p}$ and L^p sense.

(2.15b) to (2.15e) follow by first investigating small perturbations of the isomorphic hyperbolic constant coefficient operators as in (2.15a). In this case the von Neumann series can establish bijectivity and existance of a Green's function for such perturbations.

Asymptotic autonomy implies that asymtptically hyperbolic operators can be viewed as such perturbations for large ξ. This enables us to use the results on small perturbations of hyperbolic constant coefficient operators to establish asymptotics of general asymptotically hyperbolic linear operators. These asymptotics combined with the Arzelà Ascoli theorem then enable us to prove relative compactness of $W^{1,p}$ sets with relative compact range under Λ in L^p. This fact leads to finite dimensionality of the kernel via compactness of the unit ball and closedness of the range. With this and the defintion of Λ^* the remaining results follow. □

In the context of traveling wave solutions and implicit function theorem as in (1.1) we have $c \neq 0$ so Λ_{c0,x_0,ρ_0} for solutions x_0 joining hyperbolic equilibria is applicable.

Computation of the Fredholm Index

The next result is Theorem B of [1] with slight addendums. We are specifically interested in the implication, that if the hyperbolic linearizations $\Lambda_{c0,\bar{x}_\pm,\rho_0}$ about equilibria $\bar{x}_\pm$ can be connected by a simple homotopy (in fact any but we only provide the proof for the line connecting them) that preserves hyperbolicty, then linearizations about any solution x_0 joining these two equilibria have index zero and thus finishing step (1.8b).

All the results on the Fredholm index are based on continuity of $\mathrm{ind} : F(X, Y) \to \mathbb{N}$ with $\mathbb{N}$ in discrete topology. Hence homotopies in $F(X, Y)$ preserve the index and we need only find such between the operators we want to study and operators having the index we need. However such homotopies $\rho \mapsto \Lambda$ only impose continuity restrictions on $\Lambda(\rho)$, i.e. $A_i(\rho)$, $A_i(\xi)$ need still only be measurable and uniformly bounded.

Theorem 2.2.2

Consider an asymptotically hyperbolic linear operator Λ as in (2.1). Assume that the limiting operators at $\pm\infty$ are Λ_{pm}.

Then the Fredholm index of Λ only depends on $\Lambda_\pm$, i.e.

$$\mathrm{ind}\,\Lambda = \imath(\Lambda_-, \Lambda_+). \tag{2.16a}$$

Furthermore for any triple $\Lambda_1, \Lambda_2, \Lambda_3$ of hyperbolic constant coefficient operators

$$\imath(\Lambda_1, \Lambda_1) = 0 \tag{2.16b}$$

and

$$\imath(\Lambda_1, \Lambda_2) + \imath(\Lambda_2, \Lambda_3) = \imath(\Lambda_1, \Lambda_3) \tag{2.16c}$$

the latter being dubbed the *Cocycle property*.

Furthermore, if

$$\forall \rho \in [0, 1] : (1 - \rho)\Lambda_1 + \rho\Lambda_2 \text{ is hyperbolic,}$$

then

$$\imath(\Lambda_1, \Lambda_2) = 0 \tag{2.16d}$$

Proof. First we show that the index only depends on the limiting operators. Let Λ^0, Λ^1 be two asymptotically hyperbolic linear operators as in (2.1) with the same limiting operators $\Lambda_\pm$ at $\pm\infty$.

For ρ in $[0, 1]$ set

$$\Lambda^\rho = (1 - \rho)\Lambda^0 + \rho\Lambda^1.$$

Then for each ρ the operator Λ^ρ again is asymptotically hyperbolic with limiting operators $\Lambda_\pm$ and hence according to 2.2.1 is a Fredholm operator. Thus $\rho \mapsto \Lambda^\rho$ is a continuous function $\mathbb{R} \to F(W^{1,p}, L^p)$ so $\operatorname{ind}\Lambda^0 = \operatorname{ind}\Lambda^1$, proving (2.16a).

Since each Λ_i is hyperbolic, each on of these operators is an isomorphism according to 2.2.1. Hence they all have index 0.

Consider an operator Λ with both limiting operators being Λ_1. Then

$$\Lambda^\rho = (1-\rho)\Lambda + \rho\Lambda_1$$

for $\rho \in [0,1]$ yields asymptotically hyperbolic operators with limiting operator Λ_1 at both $\pm\infty$ for each ρ. $\rho \to \Lambda^\rho$ is again a continuous function through $F(W^{1,p}, L^p)$, so $\operatorname{ind}\Lambda = \operatorname{ind}\Lambda_1 = 0$, proving (2.16b)

For (2.16c) set

$$R(\rho) = \begin{pmatrix} \cos(\frac{\pi\rho}{2})I_d & \sin(\frac{\pi\rho}{2})I_d \\ -\sin(\frac{\pi\rho}{2})I_d & \cos(\frac{\pi\rho}{2})I_d \end{pmatrix}$$

and consider the family of $2d$ operators defined by

$$\left(\Lambda^\rho \begin{pmatrix} x \\ y \end{pmatrix}\right)(\xi) = \begin{cases} \begin{pmatrix} \Lambda_1 x \\ \Lambda_2 y \end{pmatrix}(\xi) & \xi < 0 \\ \left(R(\rho)\begin{pmatrix} \Lambda_2 \\ \Lambda_3 \end{pmatrix} R(-\rho)\begin{pmatrix} x \\ y \end{pmatrix}\right)(\xi) & \xi \geq 0 \end{cases}.$$

with I_d denoting the $d \times d$ identity matrix. With this definition Λ^ρ is asymptotically hyperbolic for all $\rho \in [0,1]$. Its limit at $-\infty$ is (Λ_1, Λ_2) and $-\infty$ it's $\big(R(\rho)(\Lambda_2, \Lambda_3)R(-\rho)\big)$. For $\rho = 0$ and $\rho = 1$ the system decouples and the index can be explicitly calculated as the sum of the indices of the x and y equations. Furthermore $\rho \mapsto \Lambda^\rho$ is again a continuous map $[0,1] \to F(W^{k,p}, L^p)$ and hence, using (2.16a)

$$\imath(\Lambda_1, \Lambda_2) + \imath(\Lambda_2, \Lambda_3) = \operatorname{ind}\Lambda^0 = \operatorname{ind}\Lambda^1 = \imath(\Lambda_2, \Lambda_3) + \underbrace{\imath(\Lambda_2, \Lambda_2)}_{=0} = \imath(\Lambda_1, \Lambda_3)$$

By (2.16a) it suffices to show, that just one asymptotically hyperbolic linear operator with limits Λ_1, Λ_2 has index zero to prove (2.16d). Consider the family of operators defined by

$$\big(\Lambda^\rho x\big)(\xi) = \begin{cases} \big(\Lambda_1 x\big)(\xi) & \xi < 0 \\ \big((1-\rho)\Lambda_1 + \rho\Lambda_2\big)(\xi) & \xi \geq 0 \end{cases}$$

Then $\Lambda^0 = \Lambda_1$ is an index 0 operator and Λ^1 is asymptotically hyperbolic with Λ_1, Λ_2 as limits at $\pm\infty$. Furthermore Λ_ρ is asymptotically hyperbolic with limiting operators Λ_1 at $-\infty$ and $(1-\rho)\Lambda_1 + \rho\Lambda_2$ at $+\infty$ for each $\rho \in [0,1]$ and hence, according to Theorem 2.1.2, is again Fredholm. Since $\rho \mapsto \Lambda^\rho$ is continuous it follows that $\operatorname{ind}\Lambda^1 = \operatorname{ind}\Lambda^0 = \operatorname{ind}\Lambda_1 = 0$ and (2.16d) is shown. □

Eigensolutions

The final result, Proposition 7.2 from [1], we present establishes that solutions of the linear equations (2.1) are asymptotically close to eigensolutions.

Theorem 2.2.3

Let that x is a solution of (2.1) on some interval $[\tau, \infty)$. Assume that this linear equation is asymptotically autonomous at ∞ and use the notations established in Defition 2.1.2, in particular the decomposition (2.7). Assume $\exists a > 0, k > 0$ such that

$$x(\xi) \in O(\xi \mapsto e^{-a\xi}) \qquad \|M(\xi)\| \in O(\xi \mapsto e^{-k\xi}) \qquad \xi \to \infty \tag{2.17}$$

Then either one of the following hold.

1. $\exists b \geq a, \epsilon > 0$ and a nontrivial eigensolution y of the limiting equation at ∞ corresponding to the nonempty set of eigenvalues λ with $\Re\lambda = -b$ such that
$$x(\xi) - y(\xi) \in O(\xi \mapsto e^{-(b+\epsilon)\xi}) \tag{2.18a}$$
2. for each $b \in \mathbb{R}$ we have
$$\lim_{\xi \to \infty} e^{b\xi} x(\xi) = 0 \tag{2.18b}$$

We remark that we have not required hyperbolicity of the limiting operator at ∞.

Proof. Set
$$b = \sup\{a_0 \geq a | x(\xi) \in O(\xi \mapsto e^{-a_0\xi}) \text{ as } \xi \to \infty\}$$

If $b = \infty$ then (2.18b) holds.

If $b < \infty$ then, we can rewrite the linear equation (2.1) to an inhomogeneous constant coefficient equation
$$(\Lambda_0 x)(\xi) = M(\xi)(\tau_{-\xi} x)$$
with the right hand side satisfying a growth condition $O(\xi \mapsto e^{(k+b-\epsilon)\xi})$ for some $\epsilon > 0$. This growth condition together with the growth conditions (2.17) allow us to Laplace transform the inhomogeneous constant coefficient equation. Because of the growth conditions we thus obtain holomorphy of the Laplace transforms of $(\Lambda_0 x)$ and $\xi \mapsto M(\xi)$ on some half planes $\Re s > d$ for some $d \in \mathbb{R}$. The Laplace transform transforms Λ_0 into a multiplication operator with the associated characteristic function Δ_0, which is almost a polynomial. For $\Delta_0 \cdot \tilde{x}$ to be holomorphic on some half plane, $\tilde{x}$ has to be at least meromorphic in this half plane. Thus, using the inverse Laplace transform to get x from its Laplace transform $\tilde{x}$, shifting the path of integration, and using the residue theorem, we get
$$x(\xi) = y(\xi) + w(\xi)$$
where $y(\xi)$ is a sum of residuals, which is an eigensolution and w is the reconstruction along the shifter path. Investigation of w with its integral formula obtained from w's definition as shifted inverse Laplace transform of $\tilde{x}$ establishes $w \in O(\xi \mapsto e^{-(b+\epsilon)\xi})$. Finally triviality of y would contradict the definition of b which would finish the proof. □

2.3 A 1d equation

These section is just a summary sections 4 and the beginning of section 6 of [2] or chapter 4 from [3]. We try to only reproduce the results that can be used for a 2d equation with two decoupled components solving the following 1d equation and only give outlines of the proofs and relations between the results.

The interest in [2] was the equation

$$0 = \mathcal{G}(c, x, \rho)(\xi) = -cx^{(1)}(\xi) - F(x(\xi + r_1), \cdots, x(\xi + r_N), \rho) \tag{2.19}$$

as in (1.1) with $G\colon \mathbb{R} \times W^{1,\infty}(\mathbb{R}, \mathbb{R}) \times V \to \mathbb{R}$ and F satisfying

1. F satisfies

$$\begin{aligned} &F\colon \mathbb{R}^N \times V \to \mathbb{R}, (v, \rho) \mapsto F(v, \rho) \text{ is } C^1 \\ &D_v F\colon \mathbb{R}^N \times V \to B(\mathbb{R}^N, \mathbb{R}^N) \text{ is locally Lipschitz in } v \end{aligned} \tag{2.20a}$$

2. $\forall \rho \in V\colon \exists U(\rho) \subseteq \{1, \cdots, N\}\colon U(\rho) \neq \emptyset$:

$$\begin{aligned} \forall i \in U(\rho)\colon \forall v \in \mathbb{R}^N : \big(\frac{\partial}{\partial v_i} F\big)(v, \rho) > 0 \\ \forall i \in \{1, \cdots, N\} \setminus U(\rho)\colon \big(\frac{\partial}{\partial v_i} F\big)(v, \rho) = 0 \end{aligned} \tag{2.20b}$$

3. With

$$\begin{aligned} \Phi^F\colon \mathbb{R} \times V &\to \mathbb{R} \\ (y, \rho) &\mapsto F(y, \cdots, y, \rho) \end{aligned}$$

we have $\forall \rho \in V\colon \exists q^F(\rho) \in (-1, 1)$ such that

$$\begin{aligned} \forall y \in (-\infty, -1) \cup (q^F(\rho), 1)\colon \Phi^F(y, \rho) > 0 \\ \forall y \in (-1, q^F(\rho)) \cup (1, \infty)\colon \Phi^F(y, \rho) < 0 \\ \Phi^F(-1, \rho) = \Phi^F(q^y(\rho), \rho) = \Phi^F(1, \rho) = 0 \end{aligned} \tag{2.20c}$$

4. Additionally we require

$$\begin{aligned} D_y \Phi^F(-1, \rho) < 0 \\ D_y \Phi^F(1, \rho) < 0 \\ \Phi^F(q(\rho), \rho) > 0 \end{aligned} \tag{2.20d}$$

and $r_1 = 0, r_2, \cdots, r_N \in \mathbb{R}$ still distinct.

In line with our agenda in the beginning we search for uses of the implicit function theorem around solutions x_0 joining the two equilibria $\bar{x}_- = -1$ at $-\infty$ to $\bar{x}_+ = 1$ at ∞, hence the boundary conditions

$$\lim_{\xi \to -\infty} x(\xi) = \bar{x}_- = -1 \qquad \lim_{\xi \to \infty} x(\xi) = \bar{x}_+ = 1. \tag{2.21}$$

We remark that by (2.20c) these two are the only two stable equilibria. Furthermore the study of solutions joining 1 at $-\infty$ to -1 at ∞ follows trivially from the studies of solutions satisfying (2.21) by a change of variables $\xi \mapsto -\xi$.

Linear

Linearizations of (2.19) are associated with the operator

$$0 = (\Lambda u)(\xi) = -cu^{(1)}(\xi) - \sum_{i=1}^{N} A_i(\xi)u(\xi + r_i). \tag{2.22}$$

as in the decomposition (1.4) with continuous A_i satisfying $\exists \alpha, \beta \in (0, \infty)$ such that $\forall i \in \{2, \cdots N\}$:

1. $A_i \neq 0 \Rightarrow \forall \xi \in \mathbb{R}\colon \alpha \leq A_i(\xi) \leq \beta$ (2.23a)
2. If (2.22) is a constant coefficient equation then

$$\sum_{i=1}^{N} A_i < 0 \tag{2.23b}$$

Condition (2.23a) arises from condition (2.20b). The only constant coefficient equations we study in the context of (2.19) are $\Lambda_{c_0,\pm 1,\rho_0}$. Hence (2.20d) would translate into

$$0 > D_y\Phi^F(\pm_1, \rho) = \sum_{i=1}^{N} \frac{\partial}{\partial u_i} F(\pm 1, \cdots, \pm 1, \rho) = A_{i,\pm 1,\rho_0}$$

(2.23b) for $\Lambda_{c_0,\pm 1,\rho_0}$.

The condition $A_i \neq 0$ is necessary because the shifts r_i are the global shifts even for varying $\rho \in V$ but for distinct ρ F does not have to depend on the same $x(\xi + r_i)$. This is also the reason why there is the set $U(\rho)$ in condition (2.20b) for the nonlinear equation (2.19). Later this will also be important when considering a 2d equation consiting of two (2.19) components whose shifts do not have to align at all.

Before immeadeatly going over to step (1.8a) of our agenda and establishing hyperbolicity of $\bar{x}_\pm = \pm 1$ we first present a results concerning a broader class of equations encompassing both our linear and nonlinear 1d equations (2.22) and (2.19). These allow us to later show that solutions x_0 of (2.19) joining -1 to 1 all satisy $-1 < x_0 < 1$ and that solutions $u \geq 0$ of (2.22) all satisfy $u > 0$. This is useful as $\int_{\mathbb{R}} fg \neq 0$ if $f > 0, g > 0$, which might help when going over step (1.8f) of our agenda.

Consider

$$0 = -cx^{(1)}(\xi) - G(\xi, x(\xi + r_1), \cdots, x(\xi + r_N)). \tag{2.24}$$

with

1. $G\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $(\xi, u) \mapsto G(\xi, u)$ is continuous, locally Lipschitz in u (2.25a)
2. $\forall u \in \mathbb{R}^N\colon \forall i \in \{1, \cdots, N\}\colon \frac{\partial}{\partial u_i} G(\cdot, u) \neq 0 \Rightarrow \forall \xi \in \mathbb{R}\colon \frac{\partial}{\partial u_i} G(\xi, u) > 0$ (2.25b)

For fixed c, ρ both (2.19) and (2.22) are versions of (2.24).

The following Lemma is a summarization of Lemmata 3.1 and 3.3 of [2] or Lemmata 4.1.1 and 4.1.2 of [3]

Lemma 2.3.1
Assume G satisfies the conditions (2.25) and that x_1, x_2 are two solutions of (2.24) on $\mathbb{R}$.

1. If $x_1 \geq x_2$ and $\exists \tau\colon x_1(\tau) = x_2(\tau)$ then

$$c > 0 \Rightarrow x_1|_{[\tau,\infty)} = x_2|_{[\tau,\infty)} \qquad c < 0 \Rightarrow x_1|_{(-\infty,\tau]} = x_2|_{(-\infty,\tau]} \tag{2.26a}$$

2. If $\exists \tau$ such that $x_1|_{[\tau+r_-,\tau+r_+]} = x_2|_{[\tau+r_-,\tau+r_+]}$ then

$$x_1 = x_2 \tag{2.26b}$$

Proof. For (2.26a) the difference $y = x_1 - x_2 \geq 0$ solves the initial value problem

$$\begin{aligned}
-cy^{(1)}(\xi) =& G(\xi, x_2(\xi) + y(\xi), x_1(\xi + r_1), \cdots, x_1(\xi + r_N)) \\
& - G(\xi, x_1(\xi + r_1), \cdots, x_1(\xi + r_N)) \\
=&: H(\xi, y(\xi)) \\
y(\tau) =& 0
\end{aligned}$$

with $\forall \xi \in \mathbb{R}\colon H(\xi, 0) \leq 0$. A standard argument with the Picard Lindelöf theorem establishes (2.26a) for either $(-\infty, \tau]$ or $[\tau, \infty)$ depending on the sign of c.

(2.26b) is shown by assuming $x_1 = x_2$ not holding on $(-\infty, \tau + r_+]$. Assumption of $r_- < 0$ leads to a contradiction and $r_- = 0$ implies $c = 0$ which contradicts the general assumption $c \neq 0$. The arguments for $[\tau + r_-, \infty)$ are very similar. □

This lemma enables us to reduce certain questions of uniqueness of solutions or linear independence in preimages of Λ_{c_0,x_0,ρ_0} or in $\mathfrak{K}$ or $\mathfrak{K}^*$ to finding just single points or short intervals in which the solutions coincide. We remark that with nondegenerate (i.e. nonordinary) functional differential equations, contrary to ordinary differential equations, a single coincision point is generally not enough to establish conincision of solutions on a larger scale.

Now we finally start with following through with our agenda (1.8) for the nonlinear 1d equation (2.19) and start by recalling some results on the characteristic function of a constant coefficient equation satisfying (2.23),

$$\Delta(\lambda) = -c\lambda - \sum_{i=1}^{N} A_i e^{\lambda r_i}. \tag{2.27}$$

The next lemma, Lemma 4.2 from [2] or its copy Lemma 4.2.2 from [3] together with $\sum_{i=1}^{N} A_i < 0$ from condition (2.23b) enable us to finish step (1.8a).

Lemma 2.3.2
Consider the characteristic function Δ^Λ of some linear constant coefficient operator Λ satisfying (2.23).

Then, if $a \in \mathbb{R}, b \in \mathbb{R} \setminus 0$,

$$\Delta(a) \geq 0 \Rightarrow \Delta(a + \mathrm{i}b) \neq 0.$$

Proof. Assume $\Delta(a) \geq 0$ and $\Delta(a + \mathrm{i}b) = 0$. Then

$$c(a + \mathrm{i}b) + A_0 = -\sum_{i=1}^{N} A_i e^{(a+\mathrm{i}b)r_i}$$
$$ca + A_0 \leq -\sum_{i=1}^{N} A_i e^{ar_i} < 0$$

As $|\Re\rho| \leq |\rho|$ for all complex ρ we have

$$-(ca + A_0) = |ca + A_0| \leq |c(a + \mathrm{i}b) + A_0| = |\sum_{i=1}^{N} A_i e^{r_i a} e^{\mathrm{i}r_i}| \leq |\sum_{i=1}^{N} A_i e^{r_i a}| \leq -(ca + A_0).$$

Hence all inequalities in this chain are equalities, the last two terms implying $\Delta(a) = 0$, the second and third implying

$$|ca + A_0|^2 = |c(a + \mathrm{i}b) + A_0|^2 = |ca + A_0|^2 + |\mathrm{i}cb|^2,$$

and so $b = 0$.

Thus we have

$$\Delta(a) \geq 0 \wedge \Delta(a + \mathrm{i}b) = 0 \Rightarrow \Delta(a) = 0 \wedge b = 0$$

and the proof is finished. □

An essential step here in this proof was the inequality $|a + c| \leq |(a + \mathrm{i}b) + c|$ for $a, b, c \in \mathbb{R}$, which cannot be extended to something like $|a^k + c| \leq |(a + \mathrm{i}b)^k + c|$ whenever $k > 1, c \neq 0$, foreclosing attempts to determine characteristics of the eigenvalues in a similar manner when considering equations of higher order or dimension.

Since

$$\Delta_{c_0,\pm 1,\rho_0}(0) = 0 - \sum_{i=1}^{N} A_{i,\pm 1,\rho_0} e^0 = -\sum_{i=1}^{N} A_i > 0, \tag{2.28}$$

Lemma 2.3.2 implies $\forall b \in \mathbb{R}$: $\Delta_{c_0,\pm 1,\rho_0}(0 + \mathrm{i}b) \neq 0$ so ± 1 are hyperbolic. This finishes step (1.8a).

Moreover by setting

$$A_i^\mu = (1 - \mu) A_{i,-1,\rho_0} + \mu A_{i,1,\rho_0},$$

these A_i satisfy the conditions of (2.23) so the linear equation

$$0 = \big(\Lambda^\mu x\big)(\xi) = -cx^{(1)}(\xi) - \sum_{i=1}^{N} A_i^\mu x(\xi + r_i)$$

and Λ^μ are hyperbolic. In particular Λ^μ is a Fredholm operator by (2.15a) of Theorem 2.2.1. Furthermore $\Lambda^0 = \Lambda_{c_0,-1,\rho_0}$ and $\Lambda^1 = \Lambda_{c_0,1,\rho_0}$ so the conditions of (2.16d) of Theorem (2.2.2) are met. Hence for any solution x_0 joining ± 1 we have Fredholm index 0 for Λ_{c_0,x_0,ρ_0}. Of course this argument holds not only for linearizations but any asymptotically hyperbolic linear operator as in (2.22) satisfying (2.23). This finishes step (1.8b).

The next steps need more work and will use the fact that solutions to asymptotically hyperbolic (2.1) are close to eigensolutions at the respective ends in the sense of Theorem 2.2.3. For such a study based on eigensolutions we first need to find out a little more about the eigenvalues.

The next lemma is part of Proposition 4.3 of [2] or Proposition 4.2.3 of [3].

Lemma 2.3.3

For a constant coefficient Λ as in (2.22) satisfying (2.23) there exist at most one real negative eigenvalue $\lambda_-^\Lambda \in (-\infty, 0) \cap E^\Lambda$ and at most one real positive eigenvalue $\lambda_+^\Lambda \in (0,\infty) \cap E^\Lambda$. Per convention we will set $\lambda_\pm^\Lambda = \pm\infty$ if these eigenvalues actually don't exist.

Moreover, using the conventions $(-\infty,-\infty) = (\infty,\infty) = \emptyset$, we have

$$\Re\big(E^\Lambda \setminus \{\lambda_-^\Lambda, \lambda_+^\lambda\}\big) \subseteq (-\infty, \lambda_-^\Lambda) \cup (\lambda_+^\Lambda) \tag{2.29}$$

Proof. Since $f\colon \mathbb{R} \to \mathbb{R}, a \mapsto \Delta^\Lambda(a)$ is a strictly concave function as

$$f^{(2)}(a) = -\sum_{i=1}^N \underbrace{A_i}_{>0} r_i^2 e^{a r_i} < 0.$$

Together with $f(0) = -\sum_{i=1}^N A_i > 0$ the results on $\lambda_\pm^\Lambda$ follow immedeatly. Moreover it follows that $f|_{(\lambda_-^\Lambda, \lambda_+^\Lambda)} > 0$ so (2.29) follows by Lemma 2.3.2. □

For an asymptotically hyperbolic linear operator Λ as in (2.22) with (2.23) with limiting hyerpbolic constant coefficient operators $\Lambda_\pm$ at $\pm\infty$ we write

$$\lambda^{u,\Lambda} = \lambda_+^{\Lambda_-} \qquad\qquad \lambda^{s,\Lambda} = \lambda_-^{\Lambda_+} \tag{2.30}$$

For linearizations $\Lambda_{c_0,x_0,\rho}$ for solutions x_0 satisfying the boundary conditions (2.21) these are

$$\lambda^u_{c_0,x_0,\rho_0} = \lambda_+^{\Lambda_{c_0,\bar{x}_-,\rho_0}} \qquad\qquad \lambda^s_{c_0,x_0,\rho_0} = \lambda_-^{\Lambda_{c_0,\bar{x}_+,\rho_0}}. \tag{2.31}$$

The next Proposition is parts of Propositions 4.6 from [2] or 4.2.6 from [3] and gives us mentioned asymptotical approach of solutions of (2.22) to eigensolutions of the limiting operators via Theorem 2.2.3. Obviously this introduces the additional conditions of this theorem, which can later be shown to hold for Λ_{c_0,x_0,ρ_0}.

Proposition 2.3.4
Assume that the condition (2.17) on M from Theorem 2.2.3 holds for both the decompositions at $\pm\infty$ of some Λ satisfying (2.23).

Assume the remaining asymptotic conditions hold for ∞ and $x\colon [\tau, \infty)$ is a bounded solution of (2.22) on $[\tau, \infty)$ for some $\tau \in \mathbb{R}$.

1. If $\lambda^s_{c_0,x_0,\rho_0} > -\infty$ then $\exists C^+ = C^+_{c_0,x_0,\rho_0} \in \mathbb{R}, \epsilon > 0$ such that

$$x(\xi) - C^+ e^{\lambda^{s,\Lambda}\xi} \in O(\xi \mapsto e^{(\lambda^s_{c_0,x_0,\rho_0} - \epsilon)\xi}) \tag{2.32a}$$

2. If $\lambda^{s,\Lambda} > -\infty, x \geq 0, x \neq 0$ then $C^+ > 0$ (2.32b)

3. the bound obtained from formally differentiating (2.32a) hold (2.32c)

4. If $\lambda^s_{c_0,x_0,\rho_0} = -\infty, x \geq 0$ then $x = 0$ (2.32d)

The analogous result holds for $-\infty$ and $(-\infty, \tau]$.

Proof. Theorem 2.2.3 is applicable since x is a bounded and hence $O(\xi \mapsto e^{0\xi})$ solution. Both cases of (2.18) from Theorem 2.2.3 imply (2.32a), the second trivially and the first together with Lemma 2.3.3 and the fact that multiples of $\xi \mapsto e^{\xi \mapsto \lambda^{s,\Lambda}}$ are the only eigensolutions for $\lambda^{s,\Lambda}$.

For (2.32b) $C^+ \geq 0$ follows trivially from $x \geq 0$. For nonnegative solutions on $[\tau, \infty)$ it can be shown that there exist $a, b \in \mathbb{R}, R > 0$ such that

$$\forall \xi \in [\tau + R, \infty)\colon ax(\xi) \leq x^{(1)}(\xi) \leq bx(\xi) \tag{2.33}$$

Hence the second option resulting from Theorem 2.2.3 is not possible. Because x does not vanish identically it follows that (2.18a) holds for some eigenvalue $\lambda \in E^\Lambda$ with $\Re\lambda < \lambda^{s,\Lambda}$. But such an eigenvalue would have nonzero imaginary part and hence $x \geq 0$ would be impossible.

(2.32c) follows by plugging (2.32a) onto the linear equation (2.22) for $\xi \geq \tau$.

$\lambda^{s,\Lambda} = -\infty$ implies that $\Re E^\Lambda \cap (-\infty, 0) = \emptyset$ by Lemma 2.3.3 so the first option resulting from Theorem 2.2.3 is impossible. But with (2.33) the second option is only possible if $x = 0$. □

Using Proposition 2.3.4 we get a result found in Proposition 4.7, Theorem 4.1 from [2] or Proposition 4.2.7, Theorem 4.2.8 from [3].

Proposition 2.3.5
Assume that $p \geq 0$ is a nontrivial solution to the linear equation (2.22) with some Λ satisfying (2.23) and that condition (2.18) from Theorem 2.2.3 on M holds for both $\xi \to \pm\infty$.

Then

1. $\lambda^{s,\Lambda}$ and $\lambda^{u,\Lambda}$ are finite (2.34a)

2. $p > 0$ (2.34b)

3. $\mathfrak{K}^\Lambda = p\,\mathbb{R}$ (2.34c)

4. $\exists p^* > 0 \in \mathfrak{K}^{*,\Lambda}$ (2.34d)

5. $\mathfrak{K}^{*,\Lambda} = p^*\,\mathbb{R}$

Proof. (2.34a) follows from $p \neq 0$. Assuming otherwise would imply that p vanishes on some $(-\infty, \tau]$ or $[\tau, \infty)$ by (2.32d) of Proposition 2.3.4. Lemma 2.3.1 would cause p to vanish everywhere.

(2.34b) follows from Lemma 2.3.1

(2.34c) is obtained by assuming existance of some $y \in \mathfrak{K}^\Lambda$ linearily independent from p and using the asymptotic formula (2.32a) from Proposition 2.3.4 to lead to a contradiction.

(2.34c) implies $\dim \mathfrak{K}^\Lambda = 1$. Since we have already established $\operatorname{ind} \Lambda = 0$ after Lemma 2.3.2 we already know $\dim \mathfrak{K}^{*,\Lambda} = 1$. Assumption of $p^* > 0 \notin \mathfrak{K}^*$ thus implies existance of some nontrivial $p^* \in \mathfrak{K}^*$ that takes both values ± 0. p^* taking both positive and negative values implies existance of some $h \geq 0$ compactly supported such that

$$\int_{\mathbb{R}} p^* h = 0.$$

According to the Fredholm Alternative (2.15d) from Theorem 2.2.1 it would follow that $h \in \Lambda(W^{1,\infty})$. Closer examination fo the preimage of h with help from Proposition 2.3.4 yields a contradiction thus proving (2.34d).

(5) follows directly from (2.34c) and the arguments this equation's proof. $\square$

One sees that this Proposition would serve as a good basis for all the steps (1.8c) to (1.8f) but only if the condition of exponential approach of the limiting equations (2.17) and existance of a nontrivial nonnegative solution really hold for Λ_{c_0,x_0,ρ_0}. This is established in the following subsection with a closer investigation of the original nonlinear equation (2.19).

Nonlinear

Based on the agenda (1.8) and the results from the last subsection we still need to prove existance of $0 \leq p \in \mathfrak{K}_{c_0,x_0,\rho_0}$, i.e. $x_0^{(1)} \geq 0$, and the exponential approach condition (2.17) on M for Λ_{c_0,x_0,ρ_0} to enjoy the benefits of Proposition as a basis for steps (1.8c) to (1.8f). Moreover we need to find a result that guarantees that solutions obtained from the implicit function theorem satisfy the boundary conditions (2.21).

The last requirement can be fulfilled with Lemma 6.1 from [2], Lemma 4.3.1 from [3].

Lemma 2.3.6
Let $x_0 \in W^{1,\infty}$ be a solution to (2.19) and

$$\mu_- := \inf\{x_0(\xi) | \xi \in \mathbb{R}\} \qquad \mu_+ := \sup\{x_0(\xi) | \xi \in \mathbb{R}\} \tag{2.35}$$

Then

$$\mu_- \in [-1, q(\rho)] \cup \{1\} \qquad\qquad \mu_+ \in \{-1\} \cup [q(\rho), 1] \tag{2.36}$$

The same conclusion holds for

$$\mu_{-,\infty} := \liminf_{\xi\to\infty} x_0(\xi) \qquad\qquad \mu_{+,\infty} := \limsup_{\xi\to\infty} x_0(\xi)$$
$$\mu_{-,-\infty} := \liminf_{\xi\to-\infty} x_0(\xi) \qquad\qquad \mu_{+,-\infty} := \limsup_{\xi\to-\infty} x_0(\xi)$$

Proof. The proof for $\mu_\pm$ works with a standard argument with some series $\xi \in \mathbb{R}^\mathbb{N}$ such that $x_0(\xi_j) \to \mu_\pm$ for $j \to \infty$ and continuity of F.

For the lim inf etc the proof works with some sequence $\xi \in \mathbb{R}^\mathbb{N}$ with $\xi_j \to \pm\infty$ for $j \to \infty$. Then $y_i = \tau_{-\xi_i} x$ are all solutions of (2.19) and satisfy the conditions of the Arzelà-Ascoli theorem. The resulting limits are also solutions of (2.19) and its $\inf, \sup$ coincide with x_0's $\liminf, \limsup$ etc. The already proven parts of this lemma then yield the required result. □

The next result is Corollary 6.2 from [2] or Corollary 4.3.2 from [3].

Lemma 2.3.7
Let $x_0 \in W^\infty(\mathbb{R}, \mathbb{R})$ be a solution of (2.19) with (2.20) holding and x_0 satisfying the boundary conditions (2.21). Then

$$-1 < x_0 < 1 \tag{2.37}$$

Proof. $-1 \le x_0 \le 1$ follows from the preceding Lemma 2.3.6 and the fact that x satisfies the boundary conditions.

$-1 < x_0 < 1$ follows from a usual application of Lemma 2.3.1. □

With parts of Theorem 2.2 of [2] or Theorem 4.3.3 of [3] we are able to establish trueness of the exponential approach conditions (2.17) required of M for Λ_{c_0,x_0,ρ_0} enabling us to finally use Proposition 2.3.5 for Λ_{c_0,x_0,ρ_0}.

Proposition 2.3.8
Assume that $x_0 \in W^{1,\infty}$ is a solution of (2.19) with (2.20) satisfying the boundary conditions (2.21).

Then using the asymptotic decompositions

$$\left(\Lambda_{c,x_0,\rho} x\right)(\xi) = \begin{cases} \left(\Lambda_{c,\bar{x}_-,\rho}\right)(\xi) - M_-(\xi)\tau_{-\xi} x & \xi \to -\infty \\ \left(\Lambda_{c,\bar{x}_+,\rho}\right)(\xi) - M_+(\xi)\tau_{-\xi} x & \xi \to \infty \end{cases} \tag{2.38}$$

we have

1. $\exists k > 0$ such that
$$\xi \mapsto \|M_{\pm(\xi)}\| \in O(\xi \mapsto e^{-k\xi}) \tag{2.39a}$$

2. $\exists C_{\pm} > 0$, $\exists \epsilon > 0$, such that

$$x_0(\xi) = \begin{cases} -1 + C_- e^{\lambda^u \xi} + O(\xi \mapsto e^{(\lambda^u+\epsilon)\xi}) & \xi \to -\infty \\ 1 - C_+ e^{\lambda^s \xi} + O(\xi \mapsto e^{(\lambda^s-\epsilon)\xi}) & \xi \to \infty \end{cases} \tag{2.39b}$$

and the formulas obtained from formally differentiating (2.39b) hold.

Proof. Again we only consider the case $\xi \to \infty$, as the proof(s) of the results for $\xi \to -\infty$ are similar.

Consider $y = 1 - x_0$. Let $\xi \in \mathbb{R}$. For any two $v, w \in \mathbb{R}^N$ we have

$$\begin{aligned} F(v,\rho) - F(w,\rho) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} F(tv + (1-t)w, \rho)\mathrm{d}t \\ &= \sum_{i=1}^{N} \Big(\int_0^1 \big(\frac{\partial}{\partial u_i} F\big)(tv + (1-t)w, \rho)\mathrm{d}t \Big)(v_i - w_i) \end{aligned}$$

Hence setting

$$D_i(\xi) := \int_0^1 \big(\frac{\partial}{\partial u_i} F\big)(t + (1-t)x_0(\xi + r_1), \cdots, t + (1-t)x_0(\xi + r_N))\mathrm{d}t$$

we conclude that y solves the linear equation

$$-cy^{(1)} = \sum_{i=1}^{N} D_i(\xi) y(\xi + r_i)$$

Without loss of generality we may assume $U(\rho) = \{1, \cdots, N\}$. As x_0 satisfies the boundary conditions (2.21) we have $\forall 1 \leq i \leq N$:

$$\begin{aligned} D_{i,+} &= \lim_{\xi \to \infty} A_i(\xi) \\ &= \lim_{\xi \to \infty} \int_0^1 \big(\frac{\partial}{\partial u_i} F\big)(t + (1-t)P(\xi + r_1), \cdots, t + (1-t)P(\xi + r_N), \rho)\mathrm{d}t \\ &= \int_0^1 \big(\frac{\partial}{\partial u_i} F\big)(t + (1-t)\lim_{\xi \to \infty} P(\xi + r_1), \cdots, t + (1-t)\lim_{\xi \to \infty} P(\xi + r_N), \rho)\mathrm{d}t \\ &= \int_0^1 \big(\frac{\partial}{\partial u_i} F\big)(1, \cdots, 1, \rho)\mathrm{d}t \\ &= \big(\frac{\partial}{\partial u_i} F\big)(1, \cdots, 1, \rho) \end{aligned}$$

using dominated convergence and continuity. Hence for $\xi \to \infty$ the matrix coefficients are exactly those of $\Lambda_{c,\bar{x}_+,\rho}$ obtained from a linearization around the solution $x_0 = 1$.

We have already established hyperbolicity of $\Lambda_{c,\bar{x}_{\pm},\rho}$ so y solves an equation, that is asymptotically hyperbolic at ∞. Proposition 5.3 from [1] or Proposition from [3] or

Theorem 2.2.3 with boundedness of y yield some $a > 0$ such that $y \in O(\xi \mapsto e^{-a\xi})$ for $\xi \to \infty$.

D_vF is locally Lipschitz (2.20a) so this function is Lipschitz continuous on compact sets. By Lemma 2.3.7 we already know that $-1 < x_0(\xi) < 1$. Hence $\{t + (1-t)x_0(\xi) | \xi \in \mathbb{R}, t \in [0,1]\}^N \subseteq \mathbb{R}^N$ is bounded and hence relatively compact. Thus, if L is the Lipschitz constant of D_vF on this set's closure, then $\forall \xi \in \mathbb{R}$:

$$\begin{aligned}
|D_i(\xi) - D_{i,+}| &= |\int_0^1 (\frac{\partial}{\partial u_i}F)(t + (1-t)P(\xi), \cdots, \rho)\mathrm{d}t - (\frac{\partial}{\partial u_i}F)(1, \rho)| \\
&= |\int_0^1 (\frac{\partial}{\partial u_i}F)(t + (1-t)P(\xi), \cdots, \rho) - (\frac{\partial}{\partial u_i}F)(1, \rho)\mathrm{d}t| \\
&\leq \int_0^1 |(\frac{\partial}{\partial u_i}F)(t + (1-t)P(\xi), \cdots, \rho) - (\frac{\partial}{\partial u_i}F)(1, \rho)|\mathrm{d}t \\
&\leq \int_0^1 CN|t + (1-t)P(\xi) - 1|\mathrm{d}t \\
&= CN \int_0^1 |1-t||1 - P(\xi)|\mathrm{d}t \\
&= CN|y(\xi)| \int_0^1 1 - t\mathrm{d}t \\
&= \frac{CN}{2}|y(\xi)|
\end{aligned}$$

Hence $\xi \mapsto |D_i(\xi) - D_{i,+}| \in O(\xi \mapsto e^{-a\xi})$ for $\xi \to \infty$

Upper uniform boundedness of the functions D_i follows from continuity of F and boundedness of arguments for F in D_i's definitions. All that remains to be shown for Proposition 2.3.4 to be applicable are lower bounds for $D_1, \cdots, D_N$.

Fix $i \in \{1, \cdots, N\}$. We already know that $D_i(\xi) \to D_{i,+} > 0$ for $\xi \to \infty$. Hence $\exists \tau > 0 : \forall \xi \geq \tau : D_i(\xi) \geq \frac{D_{i,+}}{2}$. Moreover the continuous function D_i takes a minimum $\alpha_i > 0$ on the compact set $[0, \tau]$. Thus we have $\forall \xi \geq 0 : D_i(\xi) \geq \min\{\frac{A_{i,+}}{2}, \alpha_i\} > 0$.

Proposition 2.3.4 is now applicable for the case $\xi \to \infty$. As $x_0(\xi) < 1$ implies $y = 1 - x_0 > 0$ and hence y does not vanish identically on any $[\tau, \infty)$ with $\tau \in \mathbb{R}$, (2.32d) of Proposition 2.3.4 implies that $\lambda^s_{c,x_0,\rho_0} > -\infty$ and $\exists C_+ > 0, \exists \epsilon > 0$ such that

$$1 - x_0(\xi) = y(\xi) = C_+ e^{\lambda^s_{c,x_0,\rho}\xi} + O(e^{(\lambda^s_{c,x_0,\rho} - \epsilon)\xi}) \tag{2.40}$$

for $\xi \to \infty$. Moreover the formula obtained by formally differentiating the last equation holds, also by Proposition 2.3.4. Hence we have shown (2.39b).

For (2.39a) we recall the definition of L above and obtain

$$\begin{aligned}
|A_{i,x_0,\rho}(\xi) - A_{i,\bar{x}_+,\rho}| &= |\frac{\partial}{\partial v_i}F(x_0(\xi + r_1), \cdots, x_0(\xi + r_N), \rho) - \frac{\partial}{\partial v_i}F(1, \cdots, 1, \rho)| \\
&\leq L\|(x_0(\xi + r_1) - 1, \cdots, x_0(\xi + r_N) - 1)\| \\
&\leq LN \max_{1 \leq i \leq N} |y(\xi + r_i)|
\end{aligned}$$

Hence $\xi \mapsto |A_{i,x_0,\rho} - A_{i,\bar{x}_+,\rho}| \in O(\xi \mapsto e^{-a\xi})$. Finiteness of N now yields (2.39a). $\square$

Now all that is left to show for applicability of Proposition 2.3.5 is the existance of some solution $p \geq 0$. If Proposition 2.3.5 were already applicable, we would know $\dim \mathfrak{K}_{c_0,x_0,\rho_0} = 1$. Knowing $x_0^{(1)} \in \mathfrak{K}_{c_0,x_0,\rho_0}$ this would suggest $x_0^{(1)} \geq 0$ since the boundary conditions (2.21) prohibit $x_0^{(1)} \leq 0$. In fact using the formulas obtained from formally differentiating (2.39b) it is possible to show $x_0^{(1)} \geq 0$.

Originally this was covered Proposition 6.3 of [2] and reexamined in Proposition of 4.3.4 [3], i.e.

Proposition 2.3.9

Let $x_0 \in W^{1,\infty}$ be a solution of (2.19) with (2.20) holding and x_0 satisfying the boundary conditions (2.21).

Then

$$x_0^{(1)} > 0 \tag{2.41}$$

Proof. By Proposition 2.3.8, i.e. from formally differentiating (2.39b), we have $\exists \tau_0 \in \mathbb{R} : \forall |\xi| \geq \tau_0 : x_0^{(1)}(\xi) > 0$.

x_0 takes a minimum $x_{0,-} \in (-1, 1)$ and a maximum $x_{0,+}$ in compact interval $[-\tau_0, \tau_0]$.

x_0 satisfies the boundary conditions (2.21) so $\exists \tau_1, \tau_2 > \tau_0$ such that $\forall \xi \geq \tau_1 \colon x_0(\xi) > x_{0,+}$ and $\forall \xi \leq -\tau_2 \colon x_0(\xi) < x_{0,-}$.

Taking $\tau := \max\{\tau_0, \tau_1, \tau_2\}$ we obtain

$$\begin{aligned} \forall |\xi| \geq \tau : \quad & P^{(1)}(\xi) > 0 \\ \forall |\xi| < \tau : \quad & P(-\tau) < P(\xi) < P(\tau) \end{aligned} \tag{2.42}$$

with $x_0^{(1)}|_{[-\tau_2,-\tau_0] \cup [\tau_0,\tau_1]} > 0$ implying $x_0(-\tau) < x_0|_{[-\tau_2,-\tau_0] \cup [\tau_0,\tau_1]} < x_0(\tau)$. Moreover it follows from the same inequality that $x_0(-\tau) < x_{0,-}$ and $x_{0,+} < x_0(\tau)$.

(2.42) also implies that $\forall k \geq 2\tau \colon \forall \xi \in \mathbb{R} \colon x_0(\xi + k) > x_0(\xi)$.

Suppose now that $x_0^{(1)}(\xi_0)) < 0$ for some $\xi_0 \in \mathbb{R}$ and set

$$k_0 := \inf\{k > 0 | \forall \xi \in \mathbb{R} : P(\xi + k) > P(\xi)\} \leq 2\tau.$$

$x_0^{(1)}(\xi_0) < 0$ implies $k_0 > 0$.

By definition $\forall \xi \in \mathbb{R} \colon x_0(\xi + k_0) \geq x_0(\xi)$ and $\forall k \in (0, k_0)$ there exists some $\xi_1 \in \mathbb{R}$ such that $x_0(\xi_1 + k) \leq x_0(\xi_1)$. From definition of τ it follows that $|\xi_1| \leq \tau$ and $|\xi_1 + k| \leq \tau$.

But using continuity of x_0 we get

$$x_0(\xi_1 + k_0) - x_0(\xi_1) = \lim_{k \nearrow k_0} x_0(\xi_1 + k) - x_0(\xi_1) \leq 0.$$

The definition of k_0 now yields $x_0(\xi_1 + k_0) = x_0(\xi_1)$.

Both x_0 and $\tau_{-k_0} x_0$ are solutions to (2.19) so Lemma 2.3.1 can be applied. Hence $x_0(\xi) = x_0(\xi + k_0)$ either $\forall \xi \leq \xi_1$ or $\forall \xi \geq \xi_1$ which are both impossible. We conclude that $x_0^{(1)}(\xi) < 0$ is impossible proving $x_0^{(1)} \geq 0$.

$x_0^{(1)} \geq 0$ together with (2.39) of Proposition 2.3.8 allow us to use Proposition 2.3.5 for $\Lambda_{c,x_0,\rho}$. (2.34a) of Propositon 2.3.5 establishes $x_0^{(1)} > 0$. $\square$

Finally we are able to finish all the steps in (1.8) and obtain Proposition 6.4 from [2] or Proposition 4.3.5 from [3].

Set

$$W_0^{1,\infty} := \{x \in W^{1,\infty} | x(0) = 0, \tag{2.43}$$

which is a closed subspace of $W^{1,\infty}$.

Any solution of x of (2.19) fulfilling the boundary conditions (2.21) has a unique representative in $\tau_{-\eta} x \in W_0^{1,\infty}$ because of Proposition 2.3.9 we can without loss assume $x \in W_0^{1,\infty}$ and get results for the original nonnormalized x witch τ_η.

Proposition 2.3.10

Let x_0 be a solution of (2.19) for some $c_0 \neq 0, \rho_0 \in V$ with F satisfying (2.20) and x_0 satisfying the boundary conditions (2.21)

Set

$$Y := \Lambda_{c_0,x_0,\rho_0}(W_0^{1,\infty}) \oplus x_0^{(1)} \mathbb{R}\,. \tag{2.44}$$

which is a closed subspace of L^∞.

Consider the restriction

$$\mathcal{G} \colon \mathbb{R} \times W^{1,\infty} \times V \to Y$$

as in (2.19) with (2.20).

Then there exists some $\epsilon > 0$ such that

$$\forall \rho \in B_\epsilon(\rho) \colon \exists! (c(\rho), x(\rho)) \colon \mathcal{G}(c(\rho), x(\rho), \rho) = 0 \tag{2.45}$$

and $\rho \mapsto (c(\rho), x(\rho))$ is C^1.

Moreover the boundary condition (2.21) hold for each $x(\rho)$.

Proof. As we have already gathered all the materials depicted in the agenda, we only need to reiterate through (1.8).

Lemma 2.1.5 ensures that $\mathcal{G}$ is C^1 and that the linearization at (c_0, x_0, ρ_0) is as in (1.4).

Now focus on Λ_{c_0,x_0,ρ_0}.

(1.8a): Lemma 2.3.2 ensures hyperbolicty of $\Lambda_{c_0,\bar{x}_\pm,\rho_0}$.

Theorem 2.2.1 yields asymptotic hyperbolicity of Λ_{c_0,x_0,ρ_0}.

(1.8b): From Lemma 2.3.2 again we get fulfillment of the conditions of (2.16d) of Theorem 2.2.2 so

$$\operatorname{ind} \Lambda_{c_0,x_0,\rho_0} = 0$$

(1.8c): From Proposition 2.3.9 we get applicability of Proposition 2.3.5, which yields $\mathfrak{K}_{c_0,x_0,\rho_0} = x_0^{(1)} \mathbb{R}$.

(1.8d): $x_0 \in W_0^{1,\infty}$ by assumption. Since $\mathfrak{K}_{c_0,x_0,\rho_0} = x_0^{(1)}$ we know that for any $k \in \mathfrak{K}_{c_0,x_0,\rho_0} \colon \exists \xi \colon k(\xi) = 0 \Rightarrow k \equiv 0$. It follows that $W_0^{1,\infty}$ is a closed subspace of the topological complement of $\mathfrak{K}_{c_0,x_0,\rho_0}$.

(1.8e): From Proposition 2.3.5 we have $\mathfrak{K}^*_{c_0,x_0,\rho_0} = p^*$ for some $W^{1,1} \ni p^* > 0$.

(1.8f): Hence $\forall p^* \in \mathfrak{K}^*_{c_0,x_0,\rho_0} \setminus \{0\}$ either $p^* > 0$ or $p^* < 0$. Since $x_0^{(1)} > 0$ it follows that

$$\int x_0^{(1)} p^* \neq 0$$

and therefore

$$x_0^{(1)} \notin \left(\mathfrak{K}^*_{c_0,x_0,\rho_0}\right)_{\perp L^\infty} = \Lambda_{c_0,x_0,\rho_0}(W^{1,\infty}) \supseteq \Lambda_{c_0,x_0,\rho_0}(W_0^{1,\infty})$$

(1.8g) holds already.

(1.8h): We have established that $\Lambda_{c_0,x_0,\rho_0)}$ is injective. $x_0^{(1)} \notin \Lambda_{c_0,x_0,\rho_0}(W^{1,\infty})$ implies injectivity of $D_{c,x}\mathcal{G}(c_0, x_0, \rho)$.

The definition of Y guarantees surjectivity of $D_{c,x}\mathcal{G}(x_0, x_0, \rho_0)$.

The implicit function theorem is now applicable and yields $\rho \to (c(\rho), x(\rho))$, C^1 such that $\mathcal{G}(c(\rho), x(\rho), \rho) = 0$ for some neighbourhood $B(\rho_0)$

From Proposition 2.3.8 we know that $\forall \rho \in B(\rho_0)$:

$$\liminf_{\xi\to\infty} x(\rho)(\xi) \in [-1, q(\rho)] \cup \{1\} \qquad \limsup_{\xi\to\infty} x(\rho)(\xi) \in \{-1\} \cup [q(\rho), 1]$$
$$\liminf_{\xi\to-\infty} x(\rho)(\xi) \in [-1, q(\rho)] \cup \{1\} \qquad \limsup_{\xi\to-\infty} x(\rho)(\xi) \in \{-1\} \cup [q(\rho), 1]$$

Since

$$\liminf_{\xi\to-\infty} x_0(\xi) = \limsup_{\xi\to-\infty} x_0(\xi) = -1$$
$$\liminf_{\xi\to\infty} x_0(\xi) = \limsup_{\xi\to\infty} x_0(\xi) = 1$$

it follows by continuity of $\rho \to x(\rho)$ that

$$\liminf_{\xi\to\infty} x(\rho)(\xi) = -1 \qquad \limsup_{\xi\to-\infty} x(\rho)(\xi) = 1$$

and necessarily

$$\liminf_{\xi\to-\infty} x(\rho)(\xi) \leq \limsup_{\xi\to-\infty} x(\rho)(\xi) \; \liminf_{\xi\to\infty} x(\rho)(\xi) \leq \limsup_{\xi\to\infty} x(\rho)(\xi)$$

it follows that $x(\rho)$ satisfies the boundary conditions (2.21). □

3 A 2d equation

We investigate the 2d system obtained from taking 2 components both solving the equation discussed in [2]. We already know a lot about the 1d case as presented in section 2.3 and just try to find some formulations for the 2d case, showcasing some of the differences, difficulties and problems when trying to solve equations of higher order or dimensionality. When possible and meaningful we try to base our 2d results on preceding 2d results instead of the 1d results from 2.3.

Hence the equation we investigate is

$$\begin{aligned} 0 = \mathcal{G}(c,x,\rho)(\xi) &= -cx^{(1)}(\xi) - F(x(\xi+r_1),\cdots,x(\xi+r_N),\rho) \\ &= -c\begin{pmatrix} y \\ z \end{pmatrix}^{(1)}(\xi) - \begin{pmatrix} G(y(\xi+r_1),\cdots,y(\xi+r_N),\sigma) \\ H(z(\xi+r_1),\cdots,z(\xi+r_N),\tau) \end{pmatrix} \end{aligned} \tag{3.1}$$

with $x=(y,z), \rho=(\sigma,\tau)$ and G,H both fulfilling the conditions (2.20) imposed on the previously discussed 1d equation (2.19).

For the rest of the thesis we will use supscripts $\cdot^F$ for everything relating to F and $\cdot^G$, $\cdot^H$ etc for the corresponding elements pertaining to just single components of (3.1).

For any c,ρ the stable equilibria $\bar{x}^F$ of (3.1) are

$$(-1,-1) \qquad (-1,1) \qquad (1,-1) \qquad (1,1)$$

We try to use the implicit function theorem around solutions x_0 pertaining to orbits joining these equilibria. We present results for solutions joining $\bar{x}_-^F = (-1,-1)$ and $\bar{x}_+ = (1,1)$, i.e. fulfilling the boundary conditions

$$\lim_{\xi\to\pm\infty} x(\xi) = \bar{x}_\pm \tag{3.2}$$

Equivalent results for the other pairs $\bar{z}, -\bar{z}$ follow easily and can be researched with the same steps or change of variables $\xi \to -\xi$ in both or just of the components.

Using our theory for solutions between the other pairs of the decoupled systems is trickier as solutions connecting these do not automatically have nontrivial components so the steps in the agenda around and following the steps in (1.8) might not be possible.

First some results for linear equations related to (3.1) similar to those found in section 2.3 for (2.19).

3.1 Linear

Linearizations of (3.1) around particular solutions yield

$$\big(D_{c,x}\mathcal{G}(c_0,x_0,\rho_0)\big)(d,u)(\xi) = \Big(-cx_0^{(1)}\Big)(\xi)d + \Big(\Lambda^F_{c_0,x_0,\rho_0}u\Big)(\xi) \tag{3.3}$$

with $u = (v, w)$ and

$$\begin{aligned}\left(\Lambda^F_{c_0,x_0,\rho_0} u\right)(\xi) &= -c \begin{pmatrix} v \\ w \end{pmatrix}^{(1)} (\xi) - \sum_{i=1}^{N} \begin{pmatrix} A_i^G & 0 \\ 0 & A_i^H \end{pmatrix} (\xi) \begin{pmatrix} v \\ w \end{pmatrix} (\xi + r_i) \\ &= \begin{pmatrix} \Lambda^G_{c_0,y_0,\sigma_0} \\ \Lambda^H_{c_0,y_0,\tau_0} \end{pmatrix}\end{aligned} \tag{3.4}$$

For constant coefficients this gives us the characteristic equation

$$\begin{aligned} 0 &= \det \Delta^F_{c_0,x_0,\rho_0}(\lambda) \\ &= \det \begin{pmatrix} \Delta^G_{c_0,y_0,\sigma_0} & 0 \\ 0 & \Delta^H_{c_0,z_0,\tau_0} \end{pmatrix} (\lambda) \\ &= \Delta^G(\lambda)\Delta_H(\lambda) \end{aligned}$$

for $\lambda \in \mathbb{C}$. This implies that for decoupled systems hyperbolicity of each component of a constant coefficient system equates hyperbolicity of the whole system and asymptotic hyperbolicity of each component of a linear system as (3.4) warrants asymptotically hyperbolicity of the whole system, as

$$\Delta^F(\lambda) = 0 \qquad \Leftrightarrow \Delta^G(\lambda) = 0 \vee \Delta^H(\lambda) = 0. \tag{3.5}$$

So, denoting the eigenvalues of the G, H components of the a 2d constant coefficient equation as obtained from (3.4) around equilibria with E^G, E^H and the eigenvalues of the whole system with just E^F, we get $E^F = E^G \cup E^H$.

However this is not of significant interest, as we do not need another eigenvalue based asymptotic analysis of solutions but can simply use the 1d results. However we do point out here, that there are difficulties with finding properties of these eigenvalues as implied by Lemma (2.3.2) and the comments thereafter.

Steps (1.8a) and (1.8b) on calculation of the Fredholm index can be summarized as follows with the help of our 1d results.

Lemma 3.1.1

Consider linearizations of (3.1). Let $\bar{x}_1^F, \bar{x}_2^F$ be any two of the 4 stable equilibria and x_0 a solution of (3.1) joining $\bar{x}_1$ at $-\infty$ and $\bar{x}_2$ at ∞ for paramterers $c = c_0, \rho = \rho_0$.

Then

1. $\Lambda^F_{c_0,\bar{x}_1,\rho_0}$ is hyperbolic (3.6a)
2. $\Lambda^F_{c_0,x_0,\rho_0}$ is asymptotically hyperbolic (3.6b)
3. $\Lambda^F_{c_0,x_0,\rho_0}$ is Fredholm (3.6c)
4. $\operatorname{ind} \Lambda^F_{c_0,x_0,\rho_0} = \imath(\Lambda^F_{c_0,\bar{x}_1^F,\rho_0}, \Lambda^F_{c_0,\bar{x}_2^F,\rho_0,}) = 0$ (3.6d)

Proof. From Lemma 2.3.2 we have already obtained hyperbolicity of ± 1 as equlibria of the 1d equations 2.19 for G, H. From (3.5) it follows that all equlibria $(\pm 1, \pm 1)$ of the 2d linear equation (3.4) are hyperbolic. This proves (3.6a).

As in the 1d case continuity of F, i.e continuity of G, H imply asymptotic hyperbolicity of $\Lambda^F_{c_0,x_0,\rho_0}$ if x_0 joins any of the equlibria $\bar{x}^F$ finishing the proof of (3.6b).

As in the 1d case Fredholmness and hence validity of (3.6c) follows from Theorem 2.2.1.

As in the 1d case the conditions imposed on F, G suggest that a homotopy as in the condition of (2.16d) of Theorem 2.2.2 preserves hyperbolicity of the equlibria. Thus it preserves asymptotic hyperbolicity and hence includedness in $F(W^{1,\infty}(\mathbb{R},\mathbb{R}^2), L^\infty(\mathbb{R},\mathbb{R}^2))$. Theorem 2.2.2 is applicable now and (2.16d) yields (3.6d). □

For x_0 fulfilling the boundary condition (3.2) we can copy even more of the 1d results giving us an equivalent to Proposition 2.3.5.

Proposition 3.1.2

Consider (3.4) with the preceding assumptions and additionally existance of some solution (v, w) to (3.4) with nontrivial components $v \geq 0$ and $w \geq 0$. Morever, using $L(\xi) = L_\pm + M_\pm(\xi)$ for the decompositions with regards to the limiting operators/equations at $\pm\infty$ we assume that $\exists k > 0$:

$$\xi \mapsto \|M_\pm(\xi)\| \in O\big(\xi \mapsto e^{-k\xi}\big) \tag{3.7}$$

Then, with $\Lambda\colon W^{1,\infty}(\mathbb{R},\mathbb{R}^2) \to L^\infty(\mathbb{R},\mathbb{R}^2)$, we have

1. Λ is asymptotically hyperbolic (3.8a)
2. Λ is Fredholm (3.8b)
3. $\operatorname{ind}\Lambda = 0$ (3.8c)
4. $v > 0, w > 0$ (3.8d)
5. $\exists v^* > 0, w^* > 0 \in L^\infty\colon (v^*, w^*) \in \mathfrak{K}^{F,*}$ (3.8e)
6. Finally for the kernels of Λ^F and $\Lambda^{F,*}$ we have

$$\mathfrak{K}^F = (v,0)\,\mathbb{R} \oplus (0,w)\,\mathbb{R}\; \mathfrak{K}^{F,*} \quad = (v^*,0)\,\mathbb{R} \oplus (0,w^*)\,\mathbb{R}\; \dim\mathfrak{K} = \dim\mathfrak{K}^* = 2 \tag{3.8f}$$

Proof. (3.8a), (3.8b) and (3.8c) follow are part of the preceding Lemma 3.1.1.

Since the components solutions of (3.4) solve the 1d equation (2.22) and we have (3.7) we can apply Proposition 2.3.5.

(3.8d) follows from (2.34b) of Proposition 2.3.5.

(3.8e) follows from (2.34d) of Proposition 2.3.5.

(3.8f) follows from (3.8d) and (3.8e) since, by definition of (3.1) based on (2.19), we have (3.4) which implies that

$$\Lambda^F_{c_0,(y_0,z_0),(\rho_0,\sigma_0)}(v,w) = 0 \Leftrightarrow \Lambda^G_{c_0,y_0,\rho_0} v = 0 \wedge \Lambda^H_{c_0,z_0,\rho_0} w = 0$$

□

3.2 Nonlinear

Now we can formulate results equivalent to those of section 2.3 of the treatise on (2.19).

For the remainder of this part $x_0 = (y_0, z_0)$ shall always be a solution of (3.1) fulfilling the boundary conditions (3.2) for some $c_0 \neq 0$ and $\rho_0 \in V$.

Lemma 3.2.1
If preceding assumptions are met then

$$y_0^{(1)} > 0 \qquad\qquad z_0^{(1)} > 0 \tag{3.9}$$

Proof. Again, since y_0, z_0 are both solutions of the 1d equations (2.19) satifying the boundary conditions (2.21), Proposition 2.3.9 can be applied which gives us (3.9). □

Trying to find X as in step (1.8d) proves to be more of a hassle for the 2d case. In the 1d case we could use $W_0^{1,\infty}$. However in the 2d case (3.1), we may not find a unique translate in $\{f \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^2) | f(0) = 0\}$ for all solutions joining two stable equilibria $\pm\bar{x}, \bar{y}$ since the components need not match up their zero so the construction of the spaces for use in the implicit function theorem is more x_0 centric. So, while the 1d case enabled us to find continuations via implicit function theorem in spaces independent of the points around which we try find solutions, our way of handling X from (1.8d) does not permit this.

Lemma 3.2.2
There exists a closed subspace $X^F \subseteq W^{1,\infty}(\mathbb{R}, \mathbb{R}^2)$:

$$W^{1,\infty} = X^F \oplus \mathfrak{K}^F_{c_0,x_0,\rho_0} \tag{3.10a}$$

$$L^\infty = \Lambda^F_{c_0,x_0,\rho_0}(W^{1,\infty}) \oplus \mathfrak{K}^F_{c_0,x_0,\rho_0} = \Lambda^F_{c_0,x_0,\rho_0}(X^F) \oplus \mathfrak{K}^F_{c_0,x_0,\rho_0} \tag{3.10b}$$

Proof. Existance of X^F as complement of $\mathfrak{K}^F_{c_0,x_0,\rho_0}$ follows from finite dimensionality of $\mathfrak{K}^F_{c_0,x_0,\rho_0}$ since $\Lambda^F_{c_0,x_0,\rho_0}$ is Fredholm by Lemma 3.1.1.

$\Lambda^F_{c_0,x_0,\rho_0}(X^F) = \Lambda^F_{c_0,x_0,\rho_0}(W^{1,\infty})$ follows from construction of X^F as complement of the kernel.

Lemma 3.2.1 enables us to use Proposition 3.1.2. Theorem 2.2.1 implies that $\forall (r, s) \in \Lambda(X)\colon \forall a, b \in \mathbb{R}$:

$$\int_R \langle (s, t), av^*, bw^* \rangle) = a \int_{\mathbb{R}} sv^* + b \int_{\mathbb{R}} tw^*.$$

Since $y_0^{(1)} > 0, z_0^{(1)} > 0$ and $v^* > 0, w^* > 0$ it follows that $\mathfrak{K}^F_{c_0,x_0,\rho_0} \cap \Lambda^F_{c_0,x_0,\rho_0}(X^F) = \{0\}$. $\operatorname{codim} \Lambda^F_{c_0,x_0,\rho_0}(X^F) = 2$, also from 3.1.2 implies $L^\infty = \Lambda^F_{c_0,x_0,\rho_0}(X^F) \oplus \mathfrak{K}^F_{c_0,x_0,\rho_0}$. □

Proposition 3.2.3
Set

$$Y := \Lambda^F_{c_0,x_0,y_0}(X^F) \oplus x_0\,\mathbb{R} \tag{3.11}$$

and

$$B := \{x \in X | x \text{ fulfills (3.2)}\}$$

Consider $\mathcal{G}^F$ from (3.1) as a map $\mathbb{R} \times X^F \times V \to Y$.

Then there exists open neighbourhood $U(\rho_0) \subseteq V$ and C^1 maps $c\colon U(\rho) \to \mathbb{R}, x\colon U(\rho) \to S$ such that $\forall \rho \in U(\rho_0)$

$$\mathcal{G}^F(c(\rho), x(\rho), \rho) = 0$$

Proof. Differentiability of $\mathcal{G}^F$ follows from Lemma 2.1.5.

$\Lambda^F_{c_0,x_0,\rho_0}$ fulfills all requirements of Proposition 3.1.2 and together with construction of X in Lemma 3.2.2 shows injectivity of $\Lambda^F_{c_0,x_0,\rho_0}$. As $x_0^{(1)} \notin \Lambda^F_{c_0,x_0,\rho_0}(X)$ it follows that $\big(D_{c,x}G\big)(c_0, x_0, \rho_0)$ is injective.

Construction of Y yields surjectivity of $\big(D_{c,x}\mathcal{G}^F\big)(c_0, x_0, \rho_0)$.

An application of the implicit function theorem yields U, c, x as required, but with $x\colon U(\rho) \to X$ instead of $x\colon U(\rho) \to B$. We hence have to check fulfillment of the boundary conditions for $x(\rho)$. From Lemma 2.3.6 we get, with inf, sup taken componentwise

$$\mu_{-,\pm\infty}(\tilde{x}) := \liminf_{\xi\to\pm\infty} \tilde{x}(\xi) \in \big([-1, q^F(\sigma)] \cup \{1\}\big) \times \big([-1, q^G(\tau)] \cup \{1\}\big)$$
$$\mu_{+,\pm\infty}(\tilde{x}) := \limsup_{\xi\to\pm\infty} \tilde{x}(\xi) \in \big(\{-1\} \cup [q^F(\sigma), 1]\big) \times \big(\{-1\} \cup [q^G(\tau), 1]\big)$$

for any solutions $\tilde{x}$ of our nonlinear equation 3.1 with parameters $c, \rho = (\sigma, \tau)$ including those not fulfilling the boundary conditions. x_0 fulfilling the boundary conditions gives us $\mu_{\pm,-\infty}(x_0) = (-1, -1)$, $\mu_{\pm,\infty}(x_0) = (1, 1)$. Thus continuity of x yields $\forall \rho \in U(\rho_0)$:

$$\mu_{-,\infty}(x(\rho)) = 1 \qquad\qquad \mu_{+,-\infty}(x(\rho)) = -1$$

Since $\mu_{-,\pm\infty} \leq \mu_{+,\pm\infty}$ and $x \in L^\infty$ it follows that the boundary conditions (3.2) are fulfilled along x so $x\colon U(\rho_0) \to S$. □

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Bibliography

[1] Mallet-Paret, J. The fredholm alternative for functional differential equations of mixed type. *Journal of Dynamics and Differential Equations 11*, 1 (1999).

[2] Mallet-Paret, J. The global structure of traveling waves in spatially discrete dynamical systems. *Journal of Dynamics and Differential Equations 11*, 1 (1999).

[3] Wu, S. A fredholm alternative for linear functional differential equations of mixed type. Bachelor's thesis, Technische Universität München, 2009.

[4] Zeidler, E. *Applied Functional Analysis : Main Principles and their Applications.* Applied mathematical sciences ; v. 109. Springer, 1995.

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