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The high temperature regime of a multi-species mean field spin glass

Tim Jahn

Institut für Mathematik
Johann Wolfgang Goethe-Universität
Frankfurt am Main

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Erstprüfer: Prof. Dr. Nicola Kistler
Zweitprüfer: Prof. Dr. Anton Wakolbinger

Contents

Erklärung

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Frankfurt am Main, den 15. September 2016

(Tim Jahn)

1 Introduction

Humans naturally tend to have many different wishes. Often a situation arises where two goals are incompatible. For instance, it will be hard to be good friends with two people who hate each other.
In this spirit things become delicate when many persons are involved. Consider a population of N individuals. The feelings between two individuals i and j are quantified by a real number g_{ij}. The sign of g_{ij} determines wether i and j are friends or enemies, while the absolute value describes the intensity of these feelings. Without any additional information and to keep things simple, it is natural to assume that the g_{ij} are independent standard Gaussians. The objective is to seperate the population into two groups, such that friends are in the same group while enemies are in different ones. Obviously, for a typical realization of the g_{ij}, it will be impossible to find a perfect separation. A separation can be modeled as a vector $\boldsymbol{\sigma} \in \{-1, 1\}^N$ and a good way to measure its quality is the quantity

$$\sum_{1 \le i < j \le N} g_{ij} \sigma_i \sigma_j. \tag{1.1}$$

Maximizing this quantity can be achieved by making the summands $g_{ij}\sigma_i\sigma_j$ positive, hence taking σ_i and σ_j of the same sign if $g_{ij} > 0$ and of the opposite sign if $g_{ij} < 0$.
This optimization problem is of great difficulty. Interestingly, it can be attacked using methods of statistical physics since similiar problems arise there. The above quantity 1.1 corresponds to the Hamiltonian of the Sherrington-Kirkpatrick Model which was first investigated in [SK75]. In contrast to a crystal, the individual atoms are not aligned in a regular pattern. The competing interactions in this spinsystem yield the occurence of a new magnetic phase which is called the spin-glass phase. The physicists Mézard, Parisi and Virasoro formed a non rigorous general theory of the model which was described in [MPV87]. This theory proposes solutions at any temperature in excellent agreement with computer simulations and, moreover, is applicable to a wide range of other models. In particular, its analysis of the Sherrington-Kirkpatrick Model yields directly the solution to the above optimization problem.
In general, spin glasses show fundamentally different behaviour depending on the temperature. A universal mathematical theory, which verifies the predictions of Parisi, has not yet been found. In case of the Sherrington-Kirkpatrick Model and some others, rigorous proofs were presented comprehensively by Talagrand in [Tal10] and [Tal14]. The proofs are based on Guerra's smart path interpolation method [GT02]. However, this method does not address the central issue - even if two models show the same physical behaviour, it may work for one but not for the other. On the other hand, Talagrand's cavity method, which is restricted to the use of high temperatures, is a flexible tool describing model-independent phenomena.
In this work a multi-species version of the Sherrington-Kirkpatrick Model (see [BCMT15])

is studied. Here the interactions g_{ij} are no longer assumed to be identically distributed, rather they are grouped into a fixed number of classes. These additional correlation structures are represented by a matrix. Under the assumption that this matrix is non negative definite, results for this model (at all temperatures) were recently suggested by Panchenko in [Pan15]. Panchenko used versions of Guerra's interpolation and the Ghirlanda-Guerra identities [GG98]. This additional restriction demonstrates the weak point of the interpolation method. While it has no physical meaning, this restriction is still essential for the proofs. On the contrary, this thesis sticks only to the high temperature regime. The cavity method allows one to derive results without any additional assumptions for the matrix.

This thesis is structured as follows. First the basic definitions are given. The different versions of the Sherrington-Kirkpatrick Model are presented and the concept of the cavity method is explained.
In the main part of this work, the free energy of an important special case, the so called multi-partite Sherrington-Kirkpatrick Model, is derived rigorously. Due to the additonal notations needed for the multi-partite Sherrington-Kirkpatrick Model, some of the proofs are quite technical. To help the reader follow the important arguments of the proofs, they are prefixed by short sketches. A brief discussion about how the proofs have to be adapted to the general multi-species Sherrington-Kirkpatrick Model is set out in the appendix.
Moreover, it is shown that the multi-partite model is, in the limit of many species, an approximation of the classic model at all temperatures.
This thesis will conclude with a short summary followed by a consideration of possible future research options.

2 Basic definitions

The space considered is closely related to $\Sigma_N = \{-1, 1\}^N$, which represents an N-particle $\{\pm 1\}$ spinsystem. A point $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_N) \in \Sigma_N$ is referred to as a configuration, where the σ_i's are called sites or spins. To each $\boldsymbol{\sigma}$ a real-valued function $H_N(\boldsymbol{\sigma})$ is associated - the energy (Hamiltonian) of the configuration. Similarly to the introduction, the question arises how the maximum $\max_{\boldsymbol{\sigma} \in \Sigma_N} H_N(\boldsymbol{\sigma})$ behaves. A natural idea is to assign weights to the $H_N(\boldsymbol{\sigma})$, such that large values are given to the large elements. In this context, the function H_N induces a probability measure on Σ_N through

$$G_N(\boldsymbol{\sigma}) = \frac{\exp(\beta H_N(\boldsymbol{\sigma}))}{\sum_{\boldsymbol{\sigma}' \in \Sigma_N} \exp(\beta H_N(\boldsymbol{\sigma}'))}, \tag{2.1}$$

which is called the Gibbs measure G_N. The choice of the factors is motivated by the Boltzmann Theory of statistical physics, see [Nol01]. It suggests that $G_N(\boldsymbol{\sigma})$ can be seen as the probability of finding the spinsystem with energy levels $H_N(\boldsymbol{\sigma})$ in configuration $\boldsymbol{\sigma}$, after it has reached thermal equilibrium with an infinite heat bath at temperature $T = 1/\beta$. The term "high temperature" corresponds to β small and "low temperature" to β large. For large values of β, the Gibbs measure should concentrate on the large values of $H_N(\boldsymbol{\sigma})$.
The nominator of (2.1), in the following denoted as the partition function Z_N, is in general of great complexity. It is a large sum of terms of varying order and a priori it is not clear which of them give the main contribution. The computation of Z_N is in principle equivalent to the understanding of G_N, since derivatives of $\log(Z_N)$ with respect to various parameters arise as integrals with respect to G_N.
The function H_N usually reflects the interactions between the sites. In this thesis, the interactions will be random Gaussian and therefore the $H_N(\boldsymbol{\sigma})$ will be random Gaussians. The so called "Derrida's random energy model" [Der81], where the $H_N(\boldsymbol{\sigma})$ are assumed to be i.i.d., can be threatened with concepts from the classic probability theory. Many other models show similiar behaviour in the high temperature region. Physically, this can be compared to the breaking of chemical bonds through heat. At high temperatures "things become independent".

2.1 The classic Sherrington-Kirkpatrick Model

Things become more delicate if one introduces some correlations. As motivated in the introduction, the Hamiltonian of the Sherrington-Kirkpatirck Model is given by

$$H_N(\boldsymbol{\sigma}) = H_N(\boldsymbol{\sigma}, \beta, h) = \frac{\beta}{\sqrt{N}} \sum_{1 \le i < j \le N} g_{ij} \sigma_i \sigma_j + h \sum_{i \le N} \sigma_i, \tag{2.2}$$

where the g_{ij} are independent standard Gaussians which are called the *disorder* of the system. The Expectation with respect to the g_{ij} is denoted by E(.) . The h in the second term represents the strength of an external field, which pushes the spins in one direction (the physicists write $\beta h'$ instead of h, but here it is more comfortable to have two independent parameters). If this term is neglected, the Hamiltonian fullfills a special symmetry, namely $H_N(\boldsymbol{\sigma}) = H_N(-\boldsymbol{\sigma})$, which induces "untypical" effects, see [AST03]. The model provides two sources of randomness: the random sequence $(g_{ij})_{1\leq i<j\leq N}$, which through (2.1) and (2.2) defines a random measure G_N on Σ_N. The random interactions g_{ij} are fixed initially. They are called *quenched*. The system then is subjected to *thermal fluctuations* through Gibbs' measure in some sense. In this thesis the *static* case, where the system reached *thermal equilibrium* after a longer time, is considered. Consequently, the random variables are of the form $\langle f\rangle$, where f is a function on $\Sigma_N^n = (\Sigma_N)^n$ and $\langle . \rangle = \langle . \rangle^n$ denotes integration with respect to the (random) product Gibbs' measure $G_N^{\otimes n}$. Hence

$$\begin{aligned} \langle f\rangle &= \int f(\boldsymbol{\sigma}^1, ..., \boldsymbol{\sigma}^n) dG_N(\boldsymbol{\sigma}^1)...dG_N(\boldsymbol{\sigma}^n) \\ &= \frac{1}{Z_N^n} \sum_{\boldsymbol{\sigma}^1,...,\boldsymbol{\sigma}^n} f(\boldsymbol{\sigma}^1, ..., \boldsymbol{\sigma}^n) \exp\left(\sum_{l\leq n} H_N(\boldsymbol{\sigma}^l) \right). \end{aligned} \tag{2.3}$$

The configurations $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, ...$ belonging to the different copies of (Σ_n, G_N) are called replicas. The sequence $(\boldsymbol{\sigma}^l)_{l\leq n}$ can be seen as an i.i.d. sequence with distribution G_N. Replicas are quite useful to replace products of brackets by a single bracket which is called "linerization". For instance, a function f on Σ_N can be written as

$$\langle f(\boldsymbol{\sigma})\rangle^2 = \langle f(\boldsymbol{\sigma})\rangle \langle f(\boldsymbol{\sigma})\rangle = \left\langle f(\boldsymbol{\sigma}^1)\right\rangle \left\langle f(\boldsymbol{\sigma}^2)\right\rangle = \left\langle f(\boldsymbol{\sigma}^1) f(\boldsymbol{\sigma}^2)\right\rangle,$$

where the bracket on the left denotes expectation with respect to G_N while on the right expectation with respect to $G_N^{\otimes 2}$.
While in general $\langle \sigma_1\rangle \neq \langle \sigma_2\rangle$, the symmetry in the definition of the model implies that

$$\mathrm{E}\langle \sigma_i\rangle = \mathrm{E}\langle \sigma_j\rangle, \tag{2.4}$$

the so called "symmetry between sites". In the same fashion it holds that $\mathrm{E}\langle \sigma_1\sigma_2\rangle = \mathrm{E}\langle \sigma_3\sigma_4\rangle$, however $\mathrm{E}\langle \sigma_1\sigma_1\rangle \neq E\langle \sigma_1\sigma_2\rangle$.
The primary goal is to understand the system for a *typical* realization of the g_{ij}. In this context it means a realization, which occurs asymptotically ($N \to \infty$) with probability one. The quantity of fundamental importance is

$$p_N(\beta) = \frac{1}{N}\mathrm{E}\log Z_N(\beta), \tag{2.5}$$

which is sometimes referred to as the free energy. The expectation in (2.5) exists since the integrability of $\log(Z_N)$ can be verified separately on $\{Z_N \geq 1\}$ and $\{Z_N \leq 1\}$. On the former, one makes use of the inequality $\log x \leq x$ in combination with $\mathrm{E}Z_N < \infty$, whereas on the latter $Z_N \geq \exp(H_N(\boldsymbol{\sigma}_0))$, where $\boldsymbol{\sigma}_0$ is an arbitrary configuration, can be used. The importance of (2.5) lies in the fact that the random quantity $N^{-1}\log Z_N$ is "selfaveraging" for high temperature. Its fluctuations vanish for $N \to \infty$. Hence, as a rough approximation, one can think of the random variable $N^{-1}\log Z_N$ as being equal to

the constant p_N. A proof of this can be found in [AST03] on page 26.
In contrast to the above, $\mathrm{E}Z_N$ is not a good approximation for Z_N. The reason is, that a large part of $\mathrm{E}Z_N$ is contributed by very rare events. This can be compared to the *lottery phenomenon*: One out of 10^6 tickets offers a huge prize, whereas the expected gain might be sizeable - the typical gain is 0.

2.1.1 The analysis of the classic model at high temperatures

The Sherrington-Kirkpatrick Model is nowadays widely understood for all β. For the purpose of this thesis the analysis in the (easier) high temperature setting (called the replica symmetric phase) is of specific interest. This analysis, as presented compactly by Talagrand in [AST03] and [Tal98], is the template for this work.
It turns out, that for small β the correlations are not too strong. There will still be some kind of law of large numbers. In particular, the overlap of two configurations $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ (which are chosen according to Gibbs' measure) will be approximately constant for a typical disorder

$$R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') := \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'}{N} = \frac{1}{N} \sum_{i \leq N} \sigma_i \sigma_i' = q.$$

This is the central property, which allows the computation of p_N for the Sherrington-Kirkpatrick Model.

Theorem 2.1.1 *There is a number $\beta_0 \geq 1/3$ such that for all $\beta \leq \beta_0$,*

$$\lim_{N \to \infty} p_N(\beta) = \log 2 + \frac{\beta^2}{4}(1-q)^2 + \mathrm{E}\ \log \mathrm{ch}(h + \beta g \sqrt{q}),$$

where q is the unique solution of the equation

$$q = \mathrm{E}\ \mathrm{th}^2(h + \beta g \sqrt{q}).$$

The expectations are taken with respect to a standard Gaussian g.

2.1.1.1 The cavity method

The major tool used in [AST03] is Talagrand's *cavity method*. Roughly speaking, this is simply an induction over N. Usually the goal is to show convergence of $\langle f_N \rangle$ for a sequence of functions f_N on Σ_N. Often the limit will be 0.
Consider a sequence $(g_i)_{i \leq N}$ of i.i.d. standard Gaussians, independent of all g_{ij}. For $\sigma_{N+1} \in \{-1, 1\}$,

$$H_N(\boldsymbol{\sigma}, \beta, h) + \sigma_{N+1} \left(\frac{\beta}{\sqrt{N}} \sum_{i \leq N} g_i \sigma_i + h \right) = \frac{\beta}{\sqrt{N}} \sum_{1 \leq i < j \leq N+1} g_{ij} \sigma_i \sigma_j + h \sum_{i \leq N+1} \sigma_i \quad (2.6)$$

where $g_{iN+1} = g_i$. Setting $\boldsymbol{\rho} = (\boldsymbol{\sigma}, \sigma_{N+1}) \in \Sigma_{N+1}$, the right-hand side of (2.6) is $H_{N+1}(\boldsymbol{\rho}, \beta', h))$ where $\beta' = \sqrt{(N+1)/N}\beta$.
Consequently, (2.6) is the Hamiltonian of an system with $(N+1)$ spins at a slightly lower temperature. The corresponding Gibbs' measure is denoted by $G_{N+1} = G_{N+1}(\beta')$ and by

$\langle . \rangle'$ average with respect to this measure. Integrating the last spin leads to the following purely algebraic identity

$$\langle f_{N+1}(\boldsymbol{\rho}) \rangle' = \frac{1}{Z} \left\langle \operatorname*{Av}_{\sigma_{N+1}=\pm 1} f_{N+1}(\boldsymbol{\sigma}, \sigma_{N+1}) \mathcal{E} \right\rangle, \tag{2.7}$$

where

$$\mathcal{E}(\boldsymbol{\sigma}, \sigma_{N+1}) = \exp \sigma_{N+1} \left(\frac{\beta}{\sqrt{N}} \sum_{i \leq N} g_i \sigma_i + h \right),$$

and

$$\operatorname*{Av}_{\sigma_{N+1}=\pm 1} f(\boldsymbol{\sigma}, \sigma_{N+1}) \mathcal{E}(\boldsymbol{\sigma}, \sigma_{N+1}) = \frac{1}{2} f(\boldsymbol{\sigma}, 1) \mathcal{E}(\boldsymbol{\sigma}, 1) + \frac{1}{2} f(\boldsymbol{\sigma}, -1) \mathcal{E}(\boldsymbol{\sigma}, -1)$$

and

$$Z = \left\langle \operatorname*{Av}_{\sigma_{N+1}=\pm 1} \mathcal{E}(\boldsymbol{\sigma}, \sigma_{N+1}) \right\rangle.$$

After averaging, the resulting quantity depends on $\boldsymbol{\sigma}$ only and expectation with respect to G_N can be taken. This technique allows to relate $\langle f_{N+1} \rangle'$ to $\langle f_N \rangle$ and for large N there will be no big difference between $\langle . \rangle$ and $\langle . \rangle'$.

2.2 The multi-species Sherrington-Kirkpatrick Model

The classic Sherrington-Kirkpatrick Model is greatly homogenous and symmetrical. Each site interacts with all the others and these interactions underly the same law. A natural generalization is to group the sites in a finite fixed number of classes. Now the sites are just treated the same, if they belong to the same class. Particularly, consider two classes A and B and two sites $\sigma \in A$ and $\tau \in B$. The two sites can either interact or not. If they do, all sites of A will be interacting with all sites of B according to the same law. If not, there will be no interaction at all between those classes. Additionally, inside the classes there might be interaction or not.

Now a precise definition of the new model is given, which is in fact the multi-species Sherrington-Kirkpatrick Model investigated in [BCMT15] and [Pan15], with slight changes of the notations.

Consider a finite set $\mathcal{S}$ with $|\mathcal{S}| = k$ and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k) \in [0,1]^k$. The sites are grouped into classes S_m (species) indexed by elements of $\mathcal{S}$. For convenience $\mathcal{S}$ will be identified with $[k]$, hence $S_m \cong m$. The sizes of the classes are all of the same order $O(N)$ with relative sizes α_i. Precisely, given N, they are

$$N_1 = \lfloor \alpha_1 N \rfloor, \quad ... \quad , N_k = \lfloor \alpha_k N \rfloor.$$

A configuration is represented by

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_k) = (\sigma_{11}, ..., \sigma_{1N_1}, ..., \sigma_{k1}, ..., \sigma_{kN_k}) \in \{-1,1\}^{N_1} \times ... \times \{-1,1\}^{N_k} =: \Sigma_N^{\alpha}.$$

Hence σ_{mi} denotes site i of class m while $\boldsymbol{\sigma}_m$ denotes all sites of class m. The correlations are given by a symmetric matrix $\Delta^2 = (\Delta^2)_{m,n \leq k}$ with non negative entries. Write

$$L := \{m \mid m \in [k],\ \Delta^2_{mm} > 0\}$$
$$E := \{mn \mid 1 \le m < n \le k,\ \Delta^2_{mn} > 0\}$$

for the indices of the non vanishing entries of the upper triangular submatrix. The dependence structure of the classes can be characterized with an undirected graph $G = ([k], E, L)$ with vertices $[k]$, edges E and loops L. where the vertices m correspond to the classes. Two classes m and n are interacting, if and *only* if $mn \in E$. A class m is self-interacting, if and only if $m \in L$. The graph G is always assumed to be connected, which means that the system cannot be decomposed in smaller non interacting subsystems.
The Hamiltonian is defined by

$$\begin{aligned} H^{\alpha}_N(\boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_k) = & \frac{\beta}{\sqrt{N}} \sum_{m \in L} \sum_{1 \le i < j \le N_m} g^{nj}_{mi} \sigma_{mi} \sigma_{nj} \\ & + \frac{\beta}{\sqrt{N}} \sum_{mn \in E} \sum_{1 = i,j \le N_m, N_n} g^{nj}_{mi} \sigma_{mi} \sigma_{nj} \\ & + \sum_{m \in [k]} h_m \sum_{i=1}^{N_m} \sigma_{mi} \end{aligned} \tag{2.8}$$

where the g^{nj}_{mi} (the interaction between the sites σ_{mi} and σ_{nj}) are independent centered Gaussian variables, with $E((g^{nj}_{mi})^2) = \Delta^2_{mn}$. The first summand of (2.8) describes the interactions within the classes, the second the interactions in between the classes and the third the influence of the external fields. As a consequence, the *symmetry between sites* (2.4) only holds inside each class.

$$E \langle \sigma_{mi} \rangle = E \langle \sigma_{mj} \rangle .$$

This work ultimatively provides the following generalization of theorem 2.1.1 to the multi-species Sherrington-Kirkpatrick Model

Theorem 2.2.1 *There is a number $\beta_0 = \beta_0(\Delta^2)$, such that*

$$\begin{aligned} \lim_{N \to \infty} p_N(\beta, \boldsymbol{\alpha}, \boldsymbol{h}, \Delta^2) = & \log 2 \sum_{m \in [k]} \alpha_m + \frac{\beta^2}{4} \sum_{m \in L} (\alpha_m - q_m)^2 \\ & + \frac{\beta^2}{2} \sum_{mn \in E} (\alpha_m - q_m)(\alpha_n - q_n) \\ & + \sum_{m \in [k]} \alpha_m \ \mathrm{E} \ \log \mathrm{ch} \left(h_m + \beta \sum_{\substack{n \\ mn \in E}} g_{mn} \sqrt{q_n} + \beta \sum_{n \in L} \delta_{mn} g_{mn} \sqrt{q_n} \right) \end{aligned}$$

for all $\beta \le \beta_0$, where the g_{mn} are centered Gaussians with Variance Δ^2_{mn} and $\boldsymbol{q} = (q_1, ..., q_k)$ is the unique solution of

$$q_m = \alpha_m \ \mathrm{E} \ \mathrm{th}^2 \left(h_m + \beta \sum_{\substack{n \\ mn \in E}} g_{mn} \sqrt{q_n} + \beta \sum_{n \in L} \delta_{mn} g_{mn} \sqrt{q_n} \right) \qquad m = 1, ..., k.$$

The underlying dependence structure is directly reflected in the above formulas.

Remark β_0 will depend on k (the number of classes). The proofs presented in this thesis deliver $\beta_0(k) = c/\sqrt{k}$. This is (asymptotically) of optimal order, which can be seen as follows: Setting $\Delta^2_{mn} = 1$, $\alpha_m = 1$ and $h_m = h$ yields the Hamiltonian of the classic model at lower temperature $\sqrt{k}\beta$. If there was a $\beta_0(k) = w(k)$ (with $(1/\sqrt{k})/w(k) \to 0$), such that theorem 2.2.1 holds for all $\beta \leq \beta_0$, the above choices for the parameters would imply that theorem 2.1.1 holds for all $\beta \leq \sqrt{k}\beta_0(k) = \sqrt{k}w(k)$. This expression converges to ∞ for $k \to \infty$, contradicting [Tal14].

2.2.1 Adaptation of the cavity method

While the basic idea remains the same, it is a priori not clear how to implement the cavity method in the new setting. The size of the system is characterized by N and $\boldsymbol{\alpha}$, where N is the parameter which will be sent to infinity. Here two sites will be added to each class. This "affects" β and the relative sizes α_i of each class. If the starting Hamiltonian belongs to a system with parameters $(N, \boldsymbol{\alpha}, \beta)$, the resulting one is the one of a system with parameters $(N+2, \boldsymbol{\alpha}', \beta')$. Here $\alpha'_i = (\alpha_i N + 2)/(N+2)$ and $\beta' = \sqrt{(N+2)/N}\beta$. Consequently, in every following step the two new spins of each class are integrated. This provides the relation between $\langle . \rangle'$ and $\langle . \rangle$.

2.2.2 The multi-partite Sherrington-Kirkpatrick Model

Due to the amount of notations, the analysis of the general Sherrington-Kirkpatrick Model is quite unhandy. The special setup, where $\Delta^2_{mn} = \delta_{mn}$ and $\boldsymbol{\alpha} = (1, ..., 1)$ reduces this to an acceptable degree, while still keeping the new complexity. This model will be called the multi-partite Sherrington-Kirkpatrick Model.
In this case the state space is Σ_N^k with $\Sigma_N := \{-1, 1\}^N$. The Hamiltonian can be written as

$$H_N(\boldsymbol{\sigma}) = H_N(\boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_k) = \frac{\beta}{\sqrt{N}} \sum_{1 \leq m < n \leq k} \sum_{i,j \leq N} g^{nj}_{mi} \sigma_{mi} \sigma_{nj} + \sum_{m \leq k} h_m \sum_{i \leq N} \sigma_{mi}.$$

The main theorem becomes

Theorem 2.2.2 *There is a β_0, such that for $\beta \leq \beta_0$*

$$\begin{aligned} &\lim_{N \to \infty} p_N(\beta, \boldsymbol{h}) \\ &= k \log 2 + \frac{\beta^2}{2} \sum_{1 \leq m < n \leq k} (1 - q_m)(1 - q_n) + \sum_{m \leq k} \mathrm{E} \, \log \mathrm{ch} \left(h_m + \beta \sum_{n \neq m} g_{mn} \sqrt{q_n} \right), \end{aligned} \tag{2.9}$$

where $\boldsymbol{q} = (q_1, ..., q_k)$ is the unique solution of

$$q_m = \mathrm{E} \, \mathrm{th}^2 \left(h_m + \beta \sum_{n \neq m} g_{mn} \sqrt{q_n} \right) \qquad m = 1, ..., k.$$

which will be rigorously investigated in the following third chapter. After that, it will be explained what have to be taken into account in the general setting to receive theorem 2.2.1 .

3 Rigorous analysis of the multi-partite model

Similiarly to the analysis of the classic SK model, the central quantities are the overlaps. For two configurations $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_k), \boldsymbol{\sigma'} = (\boldsymbol{\sigma}'_1, ..., \boldsymbol{\sigma}'_k) \in \Sigma_N^k$, these are

$$R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) = \frac{1}{N} \sum_{i \leq N} \sigma_{mi} \sigma'_{mi} \qquad m = 1, ..., k. \tag{3.1}$$

The fundamental property is, that these overlaps are typically constant in the high temperature regime. The proof of (2.9) consists of the following steps in analogy to [AST03]:

- Given the disorder, the generic overlaps are approximately constant (first step)

$$\lim_{N \to \infty} \mathrm{E} \left\langle \left(R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) - \langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) \rangle \right)^2 \right\rangle = 0 \qquad m = 1, ..., k$$

- The integrals with respect to Gibbs' measure in the above equation, namely $\langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) \rangle$, are independent of the disorder (second step). These quantities are denoted by $\boldsymbol{q} = (q_1, ..., q_k)$, which arises as the unique fixpoint of a function $\Psi : \mathbb{R}^k \to \mathbb{R}^k$.
- Viewing the h_m's as parameters rather than free variables, the right hand side of (2.9) is a function, which depends on β only. The proof is finished by calculating the derivatives with respect to β.

3.1 Notations for the cavity method

In this section the relevant notations are introduced briefly. One starts with the Hamiltonian of an (N, β) system,

$$\frac{\beta}{\sqrt{N}} \sum_{1 \leq m < n \leq k} \sum_{i,j \leq N} g_{mi}^{nj} \sigma_{mi} \sigma_{nj} + \sum_{m \leq k} h_m \sum_{i \leq N} \sigma_{mi}.$$

Adding two sites to each class yields

$$\frac{\beta}{\sqrt{N}} \sum_{1 \leq m < n \leq k} \sum_{i,j \leq N+2} g_{mi}^{nj} \sigma_{mi} \sigma_{nj} + \sum_{m \leq k} h_m \sum_{i \leq N+2} \sigma_{mi},$$

which is the Hamiltonian of a system with parameters $(N+2, \beta')$ with $\beta' = \sqrt{(N+2)/N}\beta$. Expectation with respect to the corresponding Gibbs' measure is denoted by $\langle . \rangle'$. Integrating the sites gradually results in a sequence of systems "inbetween" the two above, where some classes contain N sites, some $N+1$ and some $N+2$.

Of specific interest is the ”last” step in this process, where only one class contains additional sites. For this case denote by $\langle . \rangle'_m$ the average with respect to the Gibbs' measure of the system, where only class m contains the two additional sites σ_{mN+1} and σ_{mN+2}. Sometimes it will be necessary to integrate even these two additional sites seperatley. Here the Gibbs average is denoted by eiter $\langle . \rangle'_{m^{(1)}}$ (the additional site is σ_{mN+1}) or $\langle . \rangle'_{m^{(2)}}$ (the additional site is σ_{mN+2}).

Algebraic identities are obtained similiarly to (2.7). They can be found in A.2.

3.2 The generic overlaps

In this section the overlaps are investigated in detail for small β. As already mentioned, it is firstly proven that the overlaps are approximately constant for a given disorder. From that a limit theorem for the terms, which arise in the use of the cavity method, is deduced. As a second big step it is shown, that these constants are in fact independent of the disorder. The function Ψ, which provides these values, is analyzed in detail.

3.2.1 First step

For a typical realization of the g's the function $(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \mapsto R(\boldsymbol{\sigma}, \boldsymbol{\sigma'})$ is nearly equal to its Gibb's mean $\langle R(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \rangle$.

Theorem 3.2.1 *There is a β_0 such that, if $\beta \leq \beta_0$,*

$$\lim_{N\to\infty} \mathrm{E} \left\langle \left(R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) - \langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m)\rangle\right)^2 \right\rangle = 0$$

for $m = 1, ..., k$.

In order to prove the theorem the following quantities have to be considered

$$C_N^m(\beta) := C_N^m(\beta, \boldsymbol{h}) = \mathrm{E} \left\langle \left(\frac{\boldsymbol{\sigma}_m^\sim \cdot \boldsymbol{\sigma}_m^*}{N} \right)^2 \right\rangle .$$

Remark This quantities arise naturally when one tries to prove that the correlations of two sites σ_{mi} and σ_{mj} are small

$$\lim_{N\to\infty} \mathrm{E} \left(\langle \sigma_{mi}\sigma_{mj} \rangle - \langle \sigma_{mi} \rangle \langle \sigma_{mj} \rangle \right)^2 = 0. \tag{3.2}$$

The introduction of replicas $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2$ (Note: They are independent under Gibb's measure) and using symmetrization yields

$$2 \left(\langle \sigma_{mi}\sigma_{mj} \rangle - \langle \sigma_{mi}\sigma_{mj} \rangle \right) = \left\langle \left(\sigma^1_{mi} - \sigma^2_{mi}\right) \left(\sigma^1_{mj} - \sigma^2_{mj}\right) \right\rangle .$$

To shorten the notation,

$$\boldsymbol{\sigma}^\sim = \boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2,$$

so that (3.2) is equivalent to

$$\lim_{N\to\infty} \mathrm{E} \left\langle \sigma^{\sim}_{mi}\sigma^{\sim}_{mj} \right\rangle^2 = 0.$$

The square can be removed by another two replicas $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}^3 - \boldsymbol{\sigma}^4$

$$\left\langle \sigma^{\sim}_{mi}\sigma^{\sim}_{mj} \right\rangle^2 = \left\langle \sigma^{\sim}_{mi}\sigma^{\sim}_{mj} \right\rangle \left\langle \sigma^{*}_{mi}\sigma^{*}_{mj} \right\rangle = \left\langle \sigma^{\sim}_{mi}\sigma^{\sim}_{mj}\sigma^{*}_{mi}\sigma^{*}_{mj} \right\rangle .$$

By using symmetry between sites

$$\begin{aligned} &\mathrm{E}\left\langle \left(\frac{\boldsymbol{\sigma}^{\sim}_m \cdot \boldsymbol{\sigma}^*_m}{N}\right)^2 \right\rangle \\ =&\mathrm{E}\left\langle \frac{1}{N^2}\sum_{i,j=1}^{N} \sigma^{\sim}_{mi}\sigma^{\sim}_{mj}\sigma^{*}_{mi}\sigma^{*}_{mj} \right\rangle = \frac{1}{N}\mathrm{E}\left\langle {\sigma^{\sim}_{m1}}^2 {\sigma^{*}_{m1}}^2 \right\rangle + \frac{N-1}{N}\mathrm{E}\left\langle \sigma^{\sim}_{mN-1}\sigma^{\sim}_{mN}\sigma^{*}_{mN-1}\sigma^{*}_{mN} \right\rangle . \end{aligned} \tag{3.3}$$

Since $\mathrm{E}\left\langle {\sigma^{\sim}_{m1}}^2 {\sigma^{*}_{m1}}^2 \right\rangle \le \mathrm{E}\left\langle 2^2 2^2 \right\rangle = 16$, equation (3.2) is equivalent to $\lim_{N\to\infty} C^m_N = 0$.

Proposition 3.2.2 *Set* $C_N(\beta) := \sum_{m\le k} C^m_N(\beta)$. *There is a* $\beta_0 = \beta_0(k)$*, such that, if* $\beta \le \beta_0$*,*

$$\lim_{N\to\infty} C_N(\beta) = 0.$$

In particular, $\lim_{N\to\infty} C^m_N = 0$.

Proof The technical calculations are set out in the appendix. It will be helpfull, that

$$\left| \frac{\boldsymbol{\sigma}^{\sim}_m \cdot \boldsymbol{\sigma}^*_m}{N} \right| \le \frac{1}{N}\sum_{i\le N} |\sigma^{\sim}_{mi}\sigma^{*}_{mi}| \le 4.$$

Two sites are added to all classes ($N \to N+2$ and $\beta \to \beta'$). The symmetry between sites (equation (3.3)) yields

$$\begin{aligned} C^m_{N+2}(\beta') &= \frac{1}{N+2}\mathrm{E}\left\langle {\sigma^{\sim}_{m1}}^2\ {\sigma^{*}_{m1}}^2 \right\rangle' + \frac{N+1}{N+2}\mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\ \sigma^{*}_{mN+1}\ \sigma^{\sim}_{mN+2}\ \sigma^{*}_{mN+2} \right\rangle' \\ &\le \frac{16}{N+2} + \mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\ \sigma^{*}_{mN+1}\ \sigma^{\sim}_{mN+2}\ \sigma^{*}_{mN+2} \right\rangle' . \end{aligned}$$

Since the expressions $(\boldsymbol{\sigma}^{\sim}_n \cdot \boldsymbol{\sigma}^*_n/N)$ remain bounded independently of N, the additional sites of all classes $n \ne m$ can be dropped (the detailed estimation can be found in A.3.1)

$$\mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\ \sigma^{*}_{mN+1}\ \sigma^{\sim}_{mN+2}\ \sigma^{*}_{mN+2} \right\rangle' \le A\ \mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\ \sigma^{*}_{mN+1}\ \sigma^{\sim}_{mN+2}\ \sigma^{*}_{mN+2} \right\rangle'_m . \tag{3.4}$$

Remember that $\langle . \rangle'_m$ corresponds to the system, where class m contains $N+2$ sites while the others contain N sites. The constant A is bounded for fixed k and small β. The integration of the additional sites of class m is done in A.3.2. One ends up with

$$
\begin{aligned}
&\mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\ \sigma^{*}_{mN+1}\ \sigma^{\sim}_{mN+2}\ \sigma^{*}_{mN+2}\right\rangle'_m \\
&\le \mathrm{E}\left\langle \mathrm{sh}^2\left(\beta^2 \sum_{n\neq m} \frac{\boldsymbol{\sigma}^{\sim}_n \cdot \boldsymbol{\sigma}^{*}_n}{N}\right) \exp \frac{\beta^2}{N} \sum_{n\neq m} \left(||\boldsymbol{\sigma}^{\sim}_n||^2 + ||\boldsymbol{\sigma}^{*}_n||^2\right)\right\rangle,
\end{aligned}
$$

It is crucial that, uniformly in N, the argument of the sinus hyperbolicus vanishes for $\beta \to 0$ while the argument of the exponential remains bounded. Precisely, for $|x| \le 4$ the following (Taylor-type) approximation holds

$$
\mathrm{sh}^2(\beta^2 x) \le \frac{x^2}{16}\mathrm{sh}^2(4\beta^2).
$$

This yields

$$
\begin{aligned}
\mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\ \sigma^{*}_{mN+1}\ \sigma^{\sim}_{mN+2}\ \sigma^{*}_{mN+2}\right\rangle'_m &\le A(\beta)\ \mathrm{E}\left\langle \left(\sum_{n\neq m} \frac{\boldsymbol{\sigma}^{\sim}_n \cdot \boldsymbol{\sigma}^{*}_n}{N}\right)^2\right\rangle \\
&\le A(\beta)\ (k-1) \sum_{n\neq m} \mathrm{E}\left\langle \left(\frac{\boldsymbol{\sigma}^{\sim}_n \cdot \boldsymbol{\sigma}^{*}_n}{N}\right)^2\right\rangle \qquad \text{(Cauchy-S.)} \\
&= A(\beta)\ (k-1) \sum_{n\neq m} C^n_N(\beta) \\
&\le A(\beta)\ (k-1) C_N(\beta),
\end{aligned}
\tag{3.5}
$$

where $\lim_{\beta\to 0} A(\beta) = 0$. Putting all together results in

$$
\begin{aligned}
C_{N+2}(\beta') &= \sum_{m\le k} C^m_{N+2}(\beta') \le \sum_{m\le k} \mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\ \sigma^{*}_{mN+1}\ \sigma^{\sim}_{mN+2}\ \sigma^{*}_{mN+2}\right\rangle' + o(1) \\
&\le A \sum_{m\le k} \mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\ \sigma^{*}_{mN+1}\ \sigma^{\sim}_{mN+2}\ \sigma^{*}_{mN+2}\right\rangle'_m + o(1) \\
&\le A\ A(\beta)\ (k-1) \sum_{m\le k} C_N(\beta) + o(1) = A\ A(\beta)(k-1)k C_N(\beta) + o(1) \\
&= A'(\beta)\ C_N(\beta) + o(1) \le \frac{1}{2} C_N(\beta) + o(1).
\end{aligned}
\tag{3.6}
$$

where $\beta \le \beta_0$ and β_0 is chosen small enough. A worst-case analysis finishes the proof. Set

$$
a_N = \sup_{\beta\le\beta_0} C_N(\beta)
$$

and obtain, since $\beta' \ge \beta = \sqrt{N/(N+2)}\beta'$,

$$\begin{aligned}
a_{N+2} &= \sup_{\beta' \leq \beta_0} C_{N+2}(\beta') \\
&\leq \frac{1}{2} \sup_{\beta' \leq \beta_0} C_N(\beta) + o(1) \\
&= \frac{1}{2} \sup_{\beta \leq \sqrt{N/N+2}\beta_0} C_N(\beta) + o(1) \\
&\leq \frac{1}{2} \sup_{\beta \leq \beta_0} C_N(\beta) + o(1) \\
&= \frac{1}{2} a_N + o(1).
\end{aligned}$$

Since a_1 and a_2 are bounded it follows that $a_N \to 0$.

□

Remark The estimation in A.3.1 is very rough. At this point it may be possible, that the choice of β_0 depends on $\boldsymbol{h}$ (the external field) and k (the number of classes) in a bad way. Pleasantly, the later stated results in section 3.2.2 imply that, for large N, it is $A \approx 1$ in (3.4). The Taylor-type approximation deliveres $A(\beta) \approx 1/\beta^4$ for small β in (3.5). Finally, $A'(\beta) \approx k^2/\beta^4$ in (3.6) and hence $\beta_0 \geq c/\sqrt{k}$ for some c.

The next two lemmas, which can be taken directly from [AST03], finish the proof of theorem 3.2.1. Define

$$D_N^m = D_N^m(\beta, \boldsymbol{h}) = \mathrm{E}\left\langle \left(\frac{\boldsymbol{\sigma}_m^{\sim} \cdot \boldsymbol{\sigma}_m^*}{N} \right)^2 \right\rangle.$$

This quantity is related to C_N^m through

Lemma 3.2.3

$$D_N^m \leq \sqrt{C_N^m}$$

Proof With

$$\boldsymbol{\sigma}_m^{\sim} \cdot \boldsymbol{\sigma}_m^3 = \sum_{i \leq N} \sigma_{mi}^{\sim} \sigma_{mi}^3,$$

Cauchy-Schwarz yields

$$\begin{aligned}
D_N^m &= \frac{1}{N^2} \mathrm{E} \sum_{i,j \leq N} \left\langle \sigma_{mi}^{\sim} \sigma_{mi}^3 \sigma_{mj}^{\sim} \sigma_{mj}^3 \right\rangle = \frac{1}{N^2} \sum_{i,j \leq N} \mathrm{E} \left(\left\langle \sigma_{mi}^{\sim} \sigma_{mj}^{\sim} \right\rangle \left\langle \sigma_{mi}^3 \sigma_{mj}^3 \right\rangle \right) \\
&\leq \frac{1}{N^2} \sum_{i,j \leq N} \sqrt{\mathrm{E} \left\langle \sigma_{mi}^{\sim} \sigma_{mj}^{\sim} \right\rangle^2} \sqrt{\mathrm{E} \left\langle \sigma_{mi}^3 \sigma_{mj}^3 \right\rangle^2} \\
&\leq \frac{1}{N^2} \sum_{i,j \leq N} \sqrt{\mathrm{E} \left\langle \sigma_{mi}^{\sim} \sigma_{mj}^{\sim} \right\rangle^2} \leq \sqrt{\frac{1}{N^2} \sum_{i,j \leq N} \mathrm{E} \left\langle \sigma_{mi}^{\sim} \sigma_{mj}^{\sim} \right\rangle^2} \\
&= \sqrt{\frac{1}{N^2} \sum_{i,j \leq N} \mathrm{E} \left\langle \sigma_{mi}^{\sim} \sigma_{mj}^{\sim} \sigma_{mi}^* \sigma_{mj}^* \right\rangle} = \sqrt{\mathrm{E} \left\langle \left(\frac{\sum_{i \leq N} \sigma_{mi}^{\sim} \sigma_{mi}^*}{N} \right)^2 \right\rangle} \\
&= \sqrt{C_N^m}.
\end{aligned}$$

□

Combining proposition 3.2.2 and lemma 3.2.3 one deduces that $\lim_{N\to\infty} D_N^m = 0$. Finally

Lemma 3.2.4

$$\mathrm{E}\left\langle \left(R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) - \langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m)\rangle\right)^2 \right\rangle \leq 4D_N^m$$

Proof Write

$$\boldsymbol{b}_m = \langle \boldsymbol{\sigma}_m \rangle = (\langle \sigma_{mi} \rangle)_{i \leq N}.$$

Hence, by symmetry between sites

$$\boldsymbol{b}_m \cdot \boldsymbol{b}_m = \sum_{i\leq N} \langle \sigma_{mi} \rangle^2 = \sum_{i \leq N} \left\langle \sigma_{mi}^1 \sigma_{mi}^2 \right\rangle = \left\langle \boldsymbol{\sigma}_m^1 \cdot \boldsymbol{\sigma}_m^2 \right\rangle.$$

Next, Jensens's inequality implies

$$\left\langle \left(\frac{(\boldsymbol{\sigma}_m^1 - \boldsymbol{b}_m) \cdot \boldsymbol{b}_m}{N} \right)^2 \right\rangle \leq \left\langle \left(\frac{\boldsymbol{\sigma}_m^{\sim} \cdot \boldsymbol{\sigma}_m^3}{N} \right)^2 \right\rangle,$$

when one averages $\boldsymbol{\sigma}_m^2$ for G_N inside the square rather than outside. Similarly

$$\left\langle \left(\frac{(\boldsymbol{\sigma}_m^1 - \boldsymbol{b}_m) \cdot \boldsymbol{b}_m}{N} \right)^2 \right\rangle \leq \left\langle \left(\frac{\boldsymbol{\sigma}_m^{\sim} \cdot \boldsymbol{\sigma}_m^3}{N} \right)^2 \right\rangle. \tag{3.7}$$

From that and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ it is received that

$$\begin{aligned}
\mathrm{E}\left\langle \left(R(\boldsymbol{\sigma}_m^1, \boldsymbol{\sigma}_m^2) - \left\langle R(\boldsymbol{\sigma}_m^1, \boldsymbol{\sigma}_m^2)\right\rangle\right)^2 \right\rangle &= \mathrm{E}\left\langle \left(\frac{\boldsymbol{\sigma}_m^1 - \boldsymbol{\sigma}_m^2}{N} - \left\langle \frac{\boldsymbol{\sigma}_m^1 - \boldsymbol{\sigma}_m^2}{N} \right\rangle \right)^2 \right\rangle \\
&= \mathrm{E}\left\langle \left(\frac{(\boldsymbol{\sigma}_m^1 - \boldsymbol{b}_m) \cdot \boldsymbol{\sigma}_m^2}{N} + \frac{(\boldsymbol{\sigma}_m^2 - \boldsymbol{b}_m) \cdot \boldsymbol{b}_m}{N} \right)^2 \right\rangle \\
&\leq 4D_N^m,
\end{aligned}$$

which finishes the proof of lemma 3.2.4 and thus of theorem 3.2.1. □

3.2.2 A central limit theorem

This technical section gives two important results

- As mentioned, there is no big difference between $\langle . \rangle'$ and $\langle . \rangle$

$$\mathrm{E}\left| \langle \sigma_{mN+z} \rangle' - \langle \sigma_{mN+z} \rangle'_{m^{(z)}} \right| \to 0. \tag{3.8}$$

- Moreover,

$$\mathrm{E}\left| \langle \sigma_{mN+z} \rangle'_{m^{(z)}} - \mathrm{th}\left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) \right| \to 0. \tag{3.9}$$

The usage of the cavity methods leads to quantities like

$$\left\langle \exp\left(\frac{t}{\sqrt{N}}\sum_{i\leq M} g_{ml}^{ni}\sigma_{ni}\right)\right\rangle'_{m_1,\dots,m_l}, \tag{3.10}$$

where $M = N$ or $N+2$, $l = N+1$ or $N+2$ and $t = \beta\sigma_{ml}$ (Check A.2 for the notation). In short notation

$$\sum_{i\leq M_m} g_{ml}^{ni}\sigma_{ni} = \boldsymbol{g}_n \cdot \boldsymbol{\sigma}_n$$

For the sake of readability the indices will be surpressed in the following proposition, hence $\langle.\rangle'_{m_1,\dots,m_l} = \langle.\rangle$. In fact, (3.10) is part of the moment-generating function of $(\boldsymbol{g}_n \cdot \boldsymbol{\sigma}_n)/N$ with respect to the Gibbs' measure. The next proposition shows, that for the typical choice of $\boldsymbol{g}_n$, the law of $(\boldsymbol{g}_n \cdot \boldsymbol{\sigma}_n)/N$ under the Gibbs' measure will approximately be $\mathcal{N}(\boldsymbol{g}_n \cdot \boldsymbol{b}_n/\sqrt{N}), 1-\bar{q}_n)$. So Gibb's measure resembles a product measure in some sense. The proposition has far-reaching consequences. Intuitively, the cavity method works well since the Hamiltonian is approximately invariant under the operation, which adds two spins to each class. The Hamiltonian consits mainly of large sums of i.i.d. Gaussian variables (about N^2 summands) and in each step there are only about N newcomers. This intuition will be justified. In some respect, the proposition says, that Gibbs measure is already determined by the sequences $(\langle\sigma_{mi}\rangle)_i = 1,\dots,N$. Moreover it allows to calculate $\langle\sigma_{mi}\rangle$ explicitly.

Remember that $\boldsymbol{b}_m = \langle\boldsymbol{\sigma}_m\rangle = (\langle\boldsymbol{\sigma}_{mi}\rangle)_{i\leq M}$ and set

$$\bar{q}_m = \langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m)\rangle = \frac{||\boldsymbol{b}_m||^2}{N}$$

Proposition 3.2.5 *Let $f = f_N$ be functions with $\sup_N |f_N| < \infty$ and $m_1,\dots,m_l \in [k]$ Under (3.2.1)*

$$\mathrm{E}\left|\left\langle f(\boldsymbol{\sigma})\prod_{j\leq l}\exp\left(\frac{t}{\sqrt{N}}\boldsymbol{g}_{m_j}\cdot\boldsymbol{\sigma}_{m_j}\right)\right\rangle - \langle f\rangle\prod_{j\leq l}\exp\left(\frac{t}{\sqrt{N}}\boldsymbol{g}_{m_j}\cdot\boldsymbol{b}_{m_j} + \frac{t^2}{2}(1-\bar{q}_{m_j})\right)\right| \to 0$$

Proof This follows by calculating second moments and using theorem (3.2.1) Set

$$X = \left\langle f(\boldsymbol{\sigma})\prod_{j\leq l}\exp\frac{t}{\sqrt{N}}\boldsymbol{g}_{m_j}\cdot\boldsymbol{\sigma}_{mj}\right\rangle$$
$$Y = \left\langle f(\boldsymbol{\sigma})\prod_{j\leq l}\exp\left(\frac{t}{\sqrt{N}}\boldsymbol{g}_{m_j}\cdot\boldsymbol{b}_{m_j} + \frac{t^2}{2}(1-\bar{q}_{m_j})\right)\right\rangle$$

Then it is enough to show that

$$\mathrm{E}(X-Y)^2 = \mathrm{E}\left(\mathrm{E}_g(X^2 - 2XY + Y^2)\right) \to 0 \tag{3.11}$$

where, as usual, E_g denotes integration in $(\boldsymbol{g}_{m_j})_{j\leq l}$ only. The introduction of replicas yields

$$X^2 = \left\langle f(\boldsymbol{\sigma})f(\boldsymbol{\sigma}')\prod_{j\leq l}\exp\frac{t}{\sqrt{N}}\boldsymbol{g}_{m_j}\cdot(\boldsymbol{\sigma}_{m_j}+\boldsymbol{\sigma}'_{m_j})\right\rangle,$$
$$XY = \left\langle f(\boldsymbol{\sigma})f(\boldsymbol{\sigma}')\prod_{j\leq l}\exp\left(\frac{t}{\sqrt{N}}\boldsymbol{g}_{m_j}\cdot(\boldsymbol{\sigma}_{m_j}+\boldsymbol{b}_{m_j})+\frac{t^2}{2}(1-\bar{q}_{m_j})\right)\right\rangle,$$
$$Y^2 = \langle f(\boldsymbol{\sigma})f(\boldsymbol{\sigma}')\rangle\prod_{j\leq l}\exp\left(\frac{2t^2}{\sqrt{N}}\boldsymbol{g}_{m_j}\cdot\boldsymbol{b}_{m_j}+t^2(1-\bar{q}_{m_j})\right).$$

Because of independence and (A.4.1)

$$\begin{aligned}\mathrm{E}_g X^2 &= \mathrm{E}_g\left\langle f(\boldsymbol{\sigma})f(\boldsymbol{\sigma}')\prod_{j\leq l}\prod_{i\leq M_j}\exp\frac{t}{\sqrt{N}}g_{m_j i}(\sigma_{m_j i}+\sigma'_{m_j i})\right\rangle\\ &= \left\langle f(\boldsymbol{\sigma})f(\boldsymbol{\sigma}')\prod_{j\leq l}\prod_{i\leq M_j}\exp\frac{t^2}{2N}(\sigma^2_{m_j i}+2\sigma_{m_j i}\sigma'_{m_j i}+\sigma'_{m_j i})\right\rangle\\ &= \left\langle f(\boldsymbol{\sigma})f(\boldsymbol{\sigma}')\prod_{j\leq l}\exp t^2\left(\frac{M_j}{N}+R(\boldsymbol{\sigma}_{m_j},\boldsymbol{\sigma}'_{m_j})\right)\right\rangle.\end{aligned}$$

Similarly

$$\mathrm{E}_g XY = \left\langle f(\boldsymbol{\sigma})f(\boldsymbol{\sigma}')\prod_{j\leq l}\exp t^2\left(\frac{M_j}{N}+\frac{\boldsymbol{\sigma}_{m_j}\cdot\boldsymbol{b}_{m_j}}{N}\right)\right\rangle,$$
$$\mathrm{E}_g Y^2 = \langle f(\boldsymbol{\sigma})f(\boldsymbol{\sigma}')\rangle\prod_{j\leq l}\exp t^2(1+\bar{q}_{m_j})$$

From theorem 3.2.1 it follows that (note that $M_j/N\to 1$)

$$\mathrm{E}\left\langle (R(\boldsymbol{\sigma}_m,\boldsymbol{\sigma}'_m)-\bar{q}_m)^2\right\rangle\to 0.$$

Moreover, (3.7) and

$$\boldsymbol{b}_{m_j}\cdot\boldsymbol{b}_{m_j} = \sum_i\left\langle\sigma_{m_j i}\right\rangle^2 = \sum_i\left\langle\sigma_{m_j i}\sigma'_{m_j i}\right\rangle = \left\langle\boldsymbol{\sigma}_{m_j}\boldsymbol{\sigma}'_{m_j}\right\rangle$$

imply that

$$\mathrm{E}\left\langle\left(\frac{\boldsymbol{\sigma}_m\cdot\boldsymbol{b}_m}{N}-\bar{q}_m\right)^2\right\rangle\to 0.$$

Hence

$$\mathrm{E}\left|\mathrm{E}_g X^2-\mathrm{E}_g Y^2\right|\to 0,\quad \mathrm{E}\left|\mathrm{E}_g XY-\mathrm{E}_g Y^2\right|\to 0,$$

which proves (3.11) and hence proposition 3.2.5. □

This proposition allows to deduce (3.8) and (3.9). The second expression is derived in detail in the appendix (A.6).

3.2.3 The fixpoint equations

Theorem 3.2.1 says, that at high temperatures the generic overlap is nearly equal to its Gibbs' mean. In the following it will be shown, that this is even independent of the disorder in $\boldsymbol{g}$. The overlaps q_m are the unique solution of an k-dimensional fixpoint equation. This equation arises naturally as a generalization of the one established by Talagrand for the classic Sherrington-Kirkpatrick Model. The technical proposition A.5.1, known as Gaussian integration by parts, is of fundamental importance.
The equation directly reflects the dependence structure given in this model. Each q_m can be compared to the so called "quenched magnetizations" $\mathrm{E}\langle\sigma_{mi}\rangle$, which are physical quantities (this means they can be measured in an experiment). The equations arise as the fixpoint of the function

$$\Psi(x_1,...,x_k) = \begin{pmatrix} \Psi_1(x_1,...,x_k) \\ \vdots \\ \Psi_k(x_1,...,x_k) \end{pmatrix}$$

where Ψ_m is given by

$$\Psi_m(x_1,...,x_k) = \mathrm{E}\mathrm{th}^2\left(h_m + \beta\sum_{n\neq m} g_{mn}\sqrt{x_n}\right)$$

and the g_{mn} are independent standard Gaussians. Note that Ψ_m does not depend on x_m. For small β each Ψ_m is a contraction,

$$|\Psi_m(x_1,...,x_k) - \Psi_m(y_1,....,y_k)| \leq L(\beta)\sqrt{\sum_{n\neq m}(x_n - y_n)^2} \tag{3.12}$$

with arbitrary small Lipschitz constant depending on β. This follows from the fact that Ψ_m is obviously continuously differentiable in any x_n. Integration and differentiation can be interchanged. The partial derivatives are

$$\frac{\partial}{\partial x_l}\Psi_m = \mathrm{E}\left(\frac{\beta g_{ml}}{\sqrt{x_l}}\frac{\mathrm{th}}{\mathrm{ch}^2}\left(h_m + \beta\sum_{n\neq m} g_{mn}\sqrt{x_n}\right)\right) = \beta^2\mathrm{E}\frac{1-2\mathrm{sh}^2\left(h_m+\sum_{n\neq m} g_{mn}\sqrt{x_n}\right)}{\mathrm{ch}^4\left(h_m+\sum_{n\neq m} g_{mn}\sqrt{x_n}\right)}$$

where Gaussian integration is used for the second equality. Now $|\partial x_l \Psi_m| \leq 2\beta^2 = L'(\beta)$ and (3.12) results of

$$\begin{aligned} |\Psi_m(x_1,...,x_k) - \Psi_m(y_1,...,y_k)| &\leq \sum_{l=1}^{k} |\Psi_m(x_1,...,x_l,y_{l+1},...,y_k) - \Psi_m(x_1,...,x_{l-1},y_l,...,y_k)| \\ &\leq L'(\beta)\sum_{l=1}^{k}|x_l - y_l| = L'(\beta)||\boldsymbol{x}-\boldsymbol{y}||_1 \\ &\leq L(\beta)||\boldsymbol{x}-\boldsymbol{y}||_2, \end{aligned}$$

where $L = \sqrt{k}L'$. As mentioned, L will be arbitrarily small for small β. In particular, Ψ is a contraction on $\mathbb{R}^k$ and hence has a unique fixpoint $\boldsymbol{q} = (q_1,...,q_k)$ with

$$q_m = \Psi_m(q_1, ..., q_k) = \mathrm{Eth}^2 \left(h_m + \beta \sum_{n \neq m} g_{mn} \sqrt{q_n} \right) \quad m = 1, ..., k. \tag{3.13}$$

3.2.4 Second Step

Remember that $\bar{q}_m := \langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) \rangle = ||\boldsymbol{b}_m||^2/N$, where $\boldsymbol{b}_m = \langle \boldsymbol{\sigma}_m \rangle$.

Theorem 3.2.6 *Set $Q_N^m(\beta) = Q_N^m(\beta, \boldsymbol{h}) := \mathrm{E}(\bar{q}_m - q_m)^2$ and $Q_N(\beta) := \sum_{m \leq k} Q_N^m(\beta)$. There is a number $\beta_0 \geq 1/\sqrt{2k}$, such that if $\beta \leq \beta_0$,*

$$\lim_{N \to \infty} Q_N(\beta) = 0$$

In particular, for all $m = 1, ..., k$

$$\lim_{N \to \infty} Q_N^m(\beta) = 0.$$

The proof mainly reduces to the following

- Two sites will be added to each class. Equation (3.8) asserts that for large N there is no big difference between $\langle . \rangle'$ and $\langle . \rangle$. Particularly, one can replace $\langle . \rangle'$ by $\langle . \rangle'_{m^{(z)}}$ with $z = 1$ or $z = 2$. Hereby, the latter corresponds to the system in which each class $n \neq m$ contains N sites, while class m contains N sites plus σ_{mN+z}. Consequently, just the relevant site σ_{mN+z} has to be integrated.
- Using symmetry between sites and equation (3.9)) give

$$Q_{N+2}^m(\beta') \leq L(\beta) \sum_{n \neq m} Q_N^n(\beta).$$

  Summation over m yields

$$Q_{N+2}(\beta') \leq L(\beta)(k-1) Q_N(\beta),$$

  where $L(\beta)$ is the Lipschitz constant of the Ψ_m.
- The proof is finished by a worst-case analysis.

Proof Writing out the square and using symmetry between sites deliveres

$$\begin{aligned} Q_N^m(\beta) = \mathrm{E}(\bar{q}_m - q_m)^2 &= \mathrm{E} \left(\frac{1}{N} \sum_{i \leq N} \langle \sigma_{mi} \rangle^2 \right)^2 - 2 q_m \mathrm{E} \frac{1}{N} \sum_{i \leq N} \langle \sigma_{mi} \rangle^2 + q_m^2 \\ &\leq \frac{1}{N} + \mathrm{E} \langle \sigma_{mN-1} \rangle^2 \langle \sigma_{mN} \rangle^2 - 2 q_m \mathrm{E} \langle \sigma_{mN} \rangle^2 + q_m^2. \end{aligned}$$

Now one passes to $N + 2$ and β' to obtain

$$Q_{N+2}^m(\beta') \leq \frac{1}{N+2} + \mathrm{E} \langle \sigma_{mN+1} \rangle'^2 \langle \sigma_{mN+2} \rangle'^2 - 2 q'_m \mathrm{E} \langle \sigma_{mN+1} \rangle'^2 + q_m'^2,$$

where q'_m is the fixpoint solution to 3.13 for β'. Since $\beta' = \sqrt{(N+2)/N}\beta$ it is obvious that $|q_m - q'_m| = o(1)$. In the expression above, inside the angle brackets only the sites σ_{mN+1} or σ_{mN+2} of class m appear. The fact (3.8) asserts that one can pass from $\langle . \rangle'$ to $\langle . \rangle'_{m^{(z)}}$.

$$Q^m_{N+2}(\beta') \leq \mathrm{E} \left\langle \sigma_{mN+1} \right\rangle'^2_{m^{(1)}} \left\langle \sigma_{mN+2} \right\rangle'^2_{m^{(2)}} - 2q'_m \mathrm{E} \left\langle \sigma_{mN+1} \right\rangle'^2_{m^{(1)}} + q^2_m + o(1).$$

Applying 3.9 yields

$$\begin{aligned} Q^m_{N+2}(\beta') \leq & \mathrm{E}\left(\mathrm{th}^2 \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) \mathrm{th}^2 \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}'_{mn} \cdot \boldsymbol{b}_n \right) \right) \\ & - 2q\mathrm{E}\mathrm{th}^2 \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) + q^2_m + o(1), \end{aligned}$$

where $\boldsymbol{g}_{mn} = (g^{nj}_{mN+1})_{j \leq N}$ and $\boldsymbol{g}'_{mn} = (g^{nj}_{mN+2})_{j \leq N}$. The goal is to use the fixpoint equations in some way. The centred Gaussians $\boldsymbol{g}_{mn} \cdot \boldsymbol{b}_m / \sqrt{N}$ and $g_{mn}\sqrt{\bar{q}_n}$ are identical in distribution, since

$$\mathrm{E}_{\boldsymbol{g}_{mn}} \left(\frac{1}{\sqrt{N}} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right)^2 = \frac{1}{N} ||\boldsymbol{b}_n||^2 = \bar{q}_n = \mathrm{E}_g \left(g\sqrt{\bar{q}_n} \right)^2 .$$

Therefore

$$\mathrm{E}_{\boldsymbol{g}_{mn}} \mathrm{th}^2 \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) = \Psi_m(\bar{q}_1, ..., \bar{q}_k).$$

The same holds for the terms with $\boldsymbol{g}'_{mn}$. The independence allows to take the expectations in $\boldsymbol{g}_{mn}, \boldsymbol{g}'_{mn}$ separately

$$\begin{aligned} Q^m_{N+2}(\beta') &\leq \mathrm{E}(\Psi^2_m(\bar{q}_1, ..., \bar{q}_k)) - 2q_m \mathrm{E}\Psi_m(\bar{q}_1, ..., \bar{q}_k) + q^2_m + o(1) \\ &= \mathrm{E}(\Psi_m(\bar{q}_1, ..., \bar{q}_k) - q_m)^2 + o(1) \\ &= \mathrm{E}(\Psi_m(\bar{q}_1, ..., \bar{q}_m) - \Psi_m(q_1, ..., q_m))^2 + o(1) \quad \text{(fixpoint-eq.)} \\ &\leq L^2(\beta) \mathrm{E} \sum_{n \neq m} (\bar{q_n} - q_n)^2 + o(1) \\ &= L^2(\beta) \sum_{n \neq m} Q^n_N(\beta) + o(1) \leq L^2(\beta) Q_N(\beta) + o(1). \end{aligned}$$

Putting all equations together gives

$$\begin{aligned} Q_{N+2}(\beta') = \sum_{m \leq k} Q^m_{N+2}(\beta') &\leq L(\beta)^2 \sum_{m \leq k} Q_N(\beta) + o(1) \\ &= kL(\beta)^2 Q_N(\beta) + o(1) \\ &\leq \frac{1}{2} Q_N(\beta) + o(1), \end{aligned}$$

for $\beta \leq \beta_0$. The proof is finished by a worst-case analysis.

□

Theorem (3.2.6) also allows to control the joint quantities.

Corollary 3.2.7 *In the above situation, for all* $m, n = 1, ..., k$

$$\lim_{N\to\infty} \mathrm{E}\left\langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) R(\boldsymbol{\sigma}_n, \boldsymbol{\sigma}'_n) - q_m q_n \right\rangle = 0.$$

Proof

$$\begin{aligned}
&\mathrm{E}\left\langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) R(\boldsymbol{\sigma}_n, \boldsymbol{\sigma}'_n) - q_m q_n \right\rangle \\
&= \mathrm{E}\left\langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) \left(R(\boldsymbol{\sigma}_n, \boldsymbol{\sigma}'_n) - q_m\right)\right\rangle + q_n \mathrm{E}\left\langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) - q_m \right\rangle \\
&= \mathrm{E}\left\langle \left(R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) - q_m\right) \left(R(\boldsymbol{\sigma}_n, \boldsymbol{\sigma}'_n) - q_n\right)\right\rangle + q_m \mathrm{E}\left\langle R(\boldsymbol{\sigma}_n, \boldsymbol{\sigma}'_n) - q_n \right\rangle \\
&\quad + q_n \mathrm{E}\left\langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) - q_m \right\rangle .
\end{aligned}$$

First note that the second and third term converge to 0 by theorem 3.2.1. Finally applying Cauchy-Schwarz to the first term

$$\begin{aligned}
&\left|\mathrm{E}\left\langle \left(R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) - q_m\right) \left(R(\boldsymbol{\sigma}_n, \boldsymbol{\sigma}'_n) - q_n\right)\right\rangle\right| \\
&\leq \mathrm{E}\left\langle \left(R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m) - q_m\right)^2 \right\rangle \mathrm{E}\left\langle \left(R(\boldsymbol{\sigma}_n, \boldsymbol{\sigma}'_n) - q_n\right)^2 \right\rangle ,
\end{aligned}$$

finishes the proof.

□

3.3 The free energy

This section contains the calculation of the free energy based on the previous results. Since the fixpoint equations imply that the partial derivatives $\partial/\partial q_m SK(\beta, q_1, ..., q_k)$ vanish (SK is defined below) it will suffice to show that $|\partial p_N/\partial\beta - \partial SK/\partial\beta| \to 0$.

Theorem 3.3.1 *There is a* β_0 *such that for* $\beta \leq \beta_0$

$$\lim_{N\to\infty} p_N(\beta, \boldsymbol{h}) = k\log 2 + \frac{\beta^2}{2} \prod_{m<n\leq k} (1-q_m)(1-q_n) + \sum_{m\leq k} \mathrm{E}\log\mathrm{ch}\left(h_m + \beta \sum_{n\neq m} g_{mn}\sqrt{q_n}\right) \tag{3.14}$$

where $(q_1, ..., q_m)$ *is the unique fixpoint of (3.13).*

The right-hand side of (3.14) is denoted by $SK(\beta, q_1, ..., q_k)$ and $\boldsymbol{h}$ is considered as a parameter rather than a free variable. Partial differentiation deliveres

$$\frac{\partial SK}{\partial q_m} = -\frac{\beta^2}{2} \sum_{n\neq m} (1-q_n) + \sum_{n\neq m} \mathrm{E}\left(\frac{\beta g_{mn}}{2q_n}\mathrm{th}\left(h_n + \beta\sum_{l\neq n} g_{nl}\sqrt{q_l}\right)\right).$$

Consequently, one can use Gaussian integration for each term of the second sum together with the identity $\frac{1}{ch^2(x)} = 1 - \mathrm{th}^2(x)$, so that one obtains

$$\begin{aligned}
\frac{\partial SK}{\partial q_m} &= -\frac{\beta^2}{2} \sum_{n\neq m} (1-q_n) + \frac{\beta^2}{2} \sum_{n\neq m} \mathrm{E}\frac{1}{\mathrm{ch}^2\left(h_n + \beta\sum_{l\neq n} g_{nl}\sqrt{q_l}\right)} \\
&= \frac{\beta^2}{2} \sum_{n\neq m} \left(-(1-q_n) + 1 - \mathrm{E}\mathrm{th}^2\left(h_n + \beta\sum_{l\neq n} g_{nl}\sqrt{q_l}\right)\right) \\
&= 0,
\end{aligned}$$

where the fixpoint equation was used for the last equality. Thus it follows that

$$\frac{d}{d\beta}SK\left(\beta, q(\beta, \boldsymbol{h})\right) = \frac{\partial SK}{\partial \beta}\left(\beta, q(\beta, \boldsymbol{h})\right) \tag{3.15}$$

The partial derivative in β can easily be computed

$$\begin{aligned}
\frac{\partial SK}{\partial \beta} &= \beta \sum_{m<n\leq k}(1-q_m)(1-q_n) + \sum_{m\leq k}\sum_{n\neq m} \mathrm{E}\left(g_{nm}\sqrt{q_n}\mathrm{th}\left(h_m + \beta\sum_{l\neq m} g_{lm}\sqrt{q_l}\right)\right) \\
&= \beta \sum_{m<n\leq k}(1-q_m)(1-q_n) + \beta\sum_{m\leq k}\sum_{n\neq m} q_n \mathrm{E}\frac{1}{\mathrm{ch}^2\left(h_m + \beta\sum_{l\neq m} g_{ml}\sqrt{q_l}\right)} \\
&= \beta \sum_{m<n\leq k}(1-q_m)(1-q_n) + \beta\sum_{m\leq k}\sum_{n\neq m} q_n(1-q_m) \\
&= \beta \sum_{m<n\leq k}(1-q_m q_n).
\end{aligned}$$

This will be compared to p_N.

Lemma 3.3.2 *In this setting,*

$$\frac{\partial p_N}{\partial \beta} = \beta \sum_{m<n\leq k}\left(1 - \mathrm{E}\left\langle R(\boldsymbol{\sigma}_m, \boldsymbol{\sigma}'_m)R(\boldsymbol{\sigma}_n, \boldsymbol{\sigma}'_n)\right\rangle\right). \tag{3.16}$$

Proof Integration and differentiation may be interchanged by A.2 in [Tal10]. Once more, Gaussian integration by parts will play the keyrole. Define

$$B(\boldsymbol{\sigma}) = B_N(\beta, \boldsymbol{h}, \boldsymbol{\sigma}) = \exp\left(\frac{\beta}{\sqrt{N}}\sum_{m<n\leq k}\sum_{i,j\leq N} g_{mi}^{jn}\sigma_{mi}\sigma_{nj} + \sum_{m\leq k} h_m \sum_{i\leq N}\sigma_{mi}\right)$$

such that

$$p_N(\beta, \boldsymbol{h}) = \frac{1}{N}\mathrm{E}\log Z_N = \frac{1}{N}\mathrm{E}\log\sum_{\boldsymbol{\sigma}} B(\boldsymbol{\sigma}). \tag{3.17}$$

Then

$$\frac{\partial B(\boldsymbol{\sigma})}{\partial \beta} = \frac{1}{\sqrt{N}}\left(\sum_{m<n\leq k}\sum_{i,j\leq N} g_{mi}^{nj}\sigma_{mi}\sigma_{nj}\right) B(\boldsymbol{\sigma}) \tag{3.18}$$

$$\frac{\partial B(\boldsymbol{\sigma})}{\partial g_{mi}^{nj}} = \frac{\beta}{\sqrt{N}}\sigma_{mi}\sigma_{nj}B(\boldsymbol{\sigma}). \tag{3.19}$$

Differentiation of (3.17) yields

$$\begin{aligned}
\frac{\partial}{\partial \beta}p_N(\beta, \boldsymbol{h}) &= \frac{1}{N}\mathrm{E}\left(\frac{1}{Z_N}\sum_{\boldsymbol{\sigma}}\frac{\partial B(\boldsymbol{\sigma})}{\partial \beta}\right) \\
&= \frac{1}{N^{3/2}}\mathrm{E}\left(\frac{1}{Z_N}\sum_{\boldsymbol{\sigma}}\sum_{m<n\leq k}\sum_{i,j\leq N} g_{mi}^{nj}\sigma_{mi}\sigma_{nj}B(\boldsymbol{\sigma})\right) \\
&= \frac{1}{N^{3/2}}\sum_{\boldsymbol{\sigma}}\sum_{m<n\leq k}\sum_{i,j\leq N}\sigma_{mi}\sigma_{nj}\mathrm{E}\left(\frac{g_{mi}^{nj}B(\boldsymbol{\sigma})}{Z_N}\right).
\end{aligned}$$

The independence of the g_{mi}^{nj} allows to use Gaussian integration conditionally upon $(g_{i',j'})$, with $(i',j') \neq (i,j)$

$$\begin{aligned}\mathrm{E}\left(g_{mi}^{nj}\frac{B(\boldsymbol{\sigma})}{Z_N}\right) &= \mathrm{E}\left(\frac{1}{Z_N}\frac{\partial B(\boldsymbol{\sigma})}{\boldsymbol{g}_{mi}^{nj}} - \frac{B(\boldsymbol{\sigma})}{Z_N^2}\frac{\partial Z_N}{\partial g_{mi}^{nj}}\right)\\ &= \frac{\beta}{\sqrt{N}}\left(\sigma_{mi}\sigma_{nj}\mathrm{E}\left(\frac{B(\boldsymbol{\sigma})}{Z_N}\right) - \mathrm{E}\frac{B(\boldsymbol{\sigma})\sum_{\boldsymbol{\sigma}'}\sigma'_{mi}\sigma'_{nj}B(\boldsymbol{\sigma}')}{Z_N^2}\right).\end{aligned}$$

Since $\sigma_{mi}^2 = 1$,

$$\begin{aligned}&\frac{\partial p_N}{\partial\beta}\\ =&\frac{1}{N^{3/2}}\sum_{\boldsymbol{\sigma}}\sum_{m<n\le k}\sum_{i,j\le N}\sigma_{mi}\sigma_{nj}\mathrm{E}\left(\frac{g_{mi}^{nj}B(\boldsymbol{\sigma})}{Z_N}\right)\\ =&\frac{\beta}{N^2}\mathrm{E}\left(\sum_{\boldsymbol{\sigma}}\sum_{m<n\le k}\sum_{i,j\le N}\sigma_{mi}^2\sigma_{nj}^2\frac{B(\boldsymbol{\sigma})}{Z_N} - \frac{1}{Z_N^2}\sum_{\boldsymbol{\sigma}}\sum_{m<n\le k}\sum_{i,j\le N}\sigma_{mi}\sigma_{nj}B(\boldsymbol{\sigma})\sum_{\boldsymbol{\sigma}'}\sigma'_{mi}\sigma'_{nj}B(\boldsymbol{\sigma}')\right)\\ =&\frac{\beta}{N^2}\mathrm{E}\left(\sum_{m<n\le k}\sum_{i,j\le N}\frac{\sum_{\boldsymbol{\sigma}}B(\boldsymbol{\sigma})}{Z_N} - \sum_{m<n\le k}\sum_{i,j\le N}\frac{\sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}\sigma_{mi}\sigma_{nj}\sigma'_{mi}\sigma'_{nj}B(\boldsymbol{\sigma})B(\boldsymbol{\sigma}')}{Z_N^2}\right)\\ =&\frac{\beta}{N^2}\sum_{m<n\le k}\left(N^2 - \sum_{i,j\le N}\mathrm{E}\left\langle\sigma_{mi}\sigma_{nj}\sigma'_{mi}\sigma'_{nj}\right\rangle\right)\\ =&\beta\sum_{m<n\le k}\left(1-\mathrm{E}\left\langle\frac{\sum_{i\le N}\sigma_{mi}\sigma'_{mi}}{N}\frac{\sum_{j\le N}\sigma_{nj}\sigma'_{nj}}{N}\right\rangle\right)\\ =&\beta\sum_{m<n\le k}\left(1-\mathrm{E}\left\langle R(\boldsymbol{\sigma}_m,\boldsymbol{\sigma}'_m)R(\boldsymbol{\sigma}_n,\boldsymbol{\sigma}'_n)\right\rangle\right).\end{aligned}$$

□

Finally, corollary (3.2.7) implies that

$$\lim_{N\to\infty}\left|\frac{\partial}{\partial\beta}p_N(\beta,\boldsymbol{h}) - \beta\sum_{m<n\le k}(1-q_mq_n)\right| = 0.$$

Note that (3.16) is bounded uniformly in N for $0\le\beta\le 1$. Moreover, β and $\boldsymbol{h}$ are independent variables so

$$\begin{aligned}SK(\beta,q(\beta,\boldsymbol{h})) - SK(0,q(0,\boldsymbol{h})) &= \int_0^\beta\frac{d}{d\beta'}SK(\beta',q(\beta',\boldsymbol{h}))d\beta'\\ &= \int_0^\beta\frac{\partial}{\partial\beta'}SK(\beta',q(\beta',\boldsymbol{h}))d\beta'\\ &= \int_0^\beta\lim_{N\to\infty}\frac{\partial}{\partial\beta'}p_N(\beta',\boldsymbol{h})d\beta'\\ &= \lim_{N\to\infty}\int_0^{\beta'}\frac{\partial}{\partial\beta'}p_N(\beta',\boldsymbol{h})d\beta' \quad \text{(dom. convergence)}\\ &= \lim_{N\to\infty}p_N(\beta,\boldsymbol{h}) - \lim_{N\to\infty}p_N(0,\boldsymbol{h}),\end{aligned}$$

Finally

$$\begin{aligned}
p_N(0,\boldsymbol{h}) &= \frac{1}{N}\mathrm{E}\log\left(\sum_{\boldsymbol{\sigma}}\exp\left(\sum_{m\le N} h_m \sum_{i\le N}\sigma_{mi}\right)\right)\\
&= \frac{1}{N}\left(\sum_{m\le k}\log\left(\sum_{\boldsymbol{\sigma}_m}\exp\left(h_m\sum_{i\le N}\sigma_{mi}\right)\right)\right)\\
&= \frac{1}{N}\sum_{m\le k}\log\left(\sum_{l\le N}\binom{N}{l}\exp(h_m(N-2l))\right)\\
&= \frac{1}{N}\sum_{m\le k}\log\left(\exp(h_m N)\sum_{l\le N}\binom{N}{l}(\exp(-2h_m))^l\right)\\
&= \frac{1}{N}\sum_{m\le k}\log\left(\exp(h_m N)(1+\exp(-2h_m))^l\right)\\
&= \frac{1}{N}\sum_{m\le k}\log\left(2^N\mathrm{ch}(h_m)\right) = \sum_{m\le k}\left(\log(2)+\mathrm{E}\log(h_m)\right)\\
&= SK(0,q(0,\boldsymbol{h})).
\end{aligned}$$

3.4 Approximation of the classic model

Here it is shown that the multi-partite Sherrington-Kirkpatrick model is, in the limit of many classes, an approximation to the classic model at any temperature. The proof of that is based on the following lemma, an application of Jensen's inequality, which is taken from [ASS03].

Lemma 3.4.1 *Let $Z(H)$ denote the partition function for a system with the Hamiltonian $H(\boldsymbol{\sigma})$, and let $U(\boldsymbol{\sigma})$ be, for each $\boldsymbol{\sigma}$, a centered Gaussian variable which is independent of H. Then*

$$0 \le \mathrm{E}\left(\log\frac{Z(H+U)}{Z(H)}\right) \le \frac{1}{2}\mathrm{E}\left(U^2\right). \tag{3.20}$$

For $k|N$ the Hamiltonian of the multi-partite model with k classes containing N/k sites is denoted by

$$H := H_N^{(k)}(\boldsymbol{\sigma}) := \frac{\beta}{\sqrt{N}}\sum_{1\le m<n\le k}\sum_{i,j\le N/k} g_{mi}^{nj}\sigma_{mi}\sigma_{mj} + \sum_{m\le k} h\sum_{i\le N/k}\sigma_{mi}.$$

Therefore, with

$$U := U_N^{(k)}(\boldsymbol{\sigma}) := \frac{\beta}{\sqrt{N}}\sum_{m\le k}\sum_{1\le i<j\le N/k} g_{mi}^{mj}\sigma_{mi}\sigma_{mj},$$

the Hamiltonian of the classic model on N sites can be written as $H+U$. Set

$$\begin{aligned}
p_N &:= \frac{1}{N}\mathrm{E}\ \log Z(H+U)\\
p_N^{(k)} &:= \frac{1}{N}\mathrm{E}\ \log Z(H).
\end{aligned}$$

Theorem 3.4.2 *For all $\epsilon > 0$ there is a $k_0 \in \mathbb{N}$ such that*

$$|p_N - p_N^{(k)}| \leq \epsilon$$

for all N, k with $N \geq k \geq k_0$.

Proof The same notation as above will be used for the case, where k does not divide N. Here the sizes of the classes are either $\lfloor N/k \rfloor$ or $\lceil N/k \rceil$.
Note first, that

$$p_N - p_N^{(k)} = \frac{1}{N}\mathrm{E}\ \log Z(H+U) - \frac{1}{N}\mathrm{E} \log Z(H) = \frac{1}{N}\mathrm{E}\left(\log \frac{Z(H+U)}{Z(H)}\right).$$

Hence, according to lemma 3.4.1,

$$0 \leq p_N - p_N^{(k)} \leq \frac{1}{2N}\mathrm{E}\left(U^2\right).$$

Finally, due to independence and $k \leq N$,

$$\begin{aligned}
\frac{1}{2N}\mathrm{E}\left(U^2\right) &= \frac{\beta^2}{2N^2} \sum_{m,m' \leq k} \sum_{1 \leq i,i' < j,j' \leq \lfloor N/k \rfloor} E\left(g_{mi}^{mj} g_{m'i'}^{m'j'}\right) \sigma_{mi}\sigma_{mj}\sigma_{m'i'}\sigma_{m'j'} + O(k/N^2) \\
&= \frac{\beta^2}{2N^2} \sum_{m,m' \leq k} \sum_{1 \leq i,i' < j,j' \leq \lfloor N/k \rfloor} \delta_{mm'}\delta_{ii'}\delta_{jj'} + o(1/N) \\
&= \frac{\beta^2}{2N^2} k \frac{N}{k}\left(\frac{N}{k} - 1\right)\frac{1}{2} + o(1/N) \\
&= \frac{\beta^2}{4}\left(\frac{1}{k} - \frac{1}{N}\right) + o(1/N) \\
&\leq \frac{\beta^2}{4}\frac{1}{k} + o(1/N) \to 0 \qquad \text{for } k \to \infty.
\end{aligned} \tag{3.21}$$

4 Summary

In this work, Talagrand's cavity method demonstrates once more its wide applicability in the analysis of mean-field spin-glasses at high temperatures.
The analogies between the classic model and the multi-species model investigated here become visible. All statements, which were proven for the classic model by Talagrand in [AST03], generalize naturally. In both cases the central property is that the generic overlaps are approximately constant in the high temperature regime.
The major challenge of this thesis was to find the right implementation of the cavity method. The state space of the generalized model is $\{-1,1\}^{N_1} \times ... \times \{-1,1\}^{N_k}$. A priori it is not clear, if one should view the system as determined by $N_1, ..., N_k$, or by $N, \alpha_1, ..., \alpha_k$, where $N_i = \lfloor \alpha_i N \rfloor$. In the first case, the sizes of the classes are considered independently. In the second, however, if one thinks of $(\alpha_1, ..., \alpha_k)$ being fixed in the beginning, the size of the system is dictated by the main variable N. Here the second approach is chosen. Consequently, in every step of the cavity method, the sites are added to all classes simultaneously. Often the studied quantities were of "squared nature", e.g. $(\boldsymbol{\sigma}_m^{\sim} \cdot \boldsymbol{\sigma}_m^{*}/N)^2$ or $(\bar{q}_m - q)^2$. Adding two sites to each class guarantees that one obtains squared quantities after such additional sites are integrated.
Another big problem was to find a suitable notation to make the proofs readable. Therefore, only the multi-partite case was investigated in full detail. On the one hand the complexity of having various classes is present here, while on the other hand, due to the symmetry of this case, some technical details need not be considered.
The bounds given for β_0 in the proofs are probably far from the optimum. Even in the case of the classic Sherrington-Kirkpatrick Model, the maximum region in the (β, h)-space, where theorem 2.1.1 holds (called the high temperature regime or the replica symmetric regime), is not yet known. Nevertheless, it is believed to be determined by the so called Almeida-Thouless line.
The results obtained should be viewed only as a first step. They are meant to be a demonstration of the flexibility of the cavity method. As a subsequent step, it would be interesting to take a look at the TAP-equations of the multi-species model at low temperatures. These are self-consistent equations for the quenched magnetizations of the sites. In the case of the classic model at low temperatures, the number of solutions of these equations is growing rapidly with N, which is problematic. The hope is that the situation is simpler for the multi-species model, and that restults can be transferred to the classic model through the approximation theorem 3.4.2.

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Appendix

A.1 The general case

The cavity method will be used in a similiar way - again two sites are added to each class. If one starts with an $(N, \boldsymbol{\alpha}, \beta)$ system, this results in a $(N+2, \boldsymbol{\alpha}', \beta')$ system, where $\alpha_i' = (\alpha_i N + 2)/(N+2)$ and $\beta' = \sqrt{(N+2)/N}\beta$.
The following points have to be taken into consideration:

- The restriction $\alpha_i \leq 1$ implies that $\alpha_i \leq \alpha_i'$. Moreover, since $\beta \leq \beta'$, this allows to use worst-case arguments in β and $\boldsymbol{\alpha}$.
- Without loss of generality, assume that $\Delta_{mn}^2 \leq 1$ for all $m, n \in [k]$. Larger values would just yield a smaller β_0. With this choice, for a Gaussian $g = g_{mi}^{nj}$ with $\mathrm{E} g^2 = \Delta_{mn}^2 \leq 1$, it holds that
$$\mathrm{E} \exp a g = \exp \frac{a^2 \Delta_{mn}^2}{2} \leq \exp \frac{a^2}{2}.$$
Therefore, expectation in additional Gaussians can be taken in the same manner.
- In case of m being a self-interacting class (which additionally interacts with the classes $n \in L \subset [k] \setminus m$), the calculation in section 3.2.1 changes to
$$\begin{aligned} &\mathrm{E}\, \langle \sigma_{mN+1}^{\sim}\ \sigma_{mN+1}^{*}\ \sigma_{mN+2}^{\sim}\ \sigma_{mN+2}^{*} \rangle_m' \\ &\leq C\ \mathrm{E} \left\langle \mathrm{sh} \left(\beta^2 \left(\frac{\dot{\boldsymbol{\sigma}}_m^{\sim} \cdot \dot{\boldsymbol{\sigma}}_m^{*}}{N} + \sum_{n \in L} \frac{\boldsymbol{\sigma}_n^{\sim} \cdot \boldsymbol{\sigma}_n^{*}}{N} \right) \right) \mathrm{sh} \left(\beta^2 \sum_{n \in L} \frac{\boldsymbol{\sigma}_n^{\sim} \cdot \boldsymbol{\sigma}_n^{*}}{N} \right) \right\rangle \\ &\approx C\ \mathrm{E} \left\langle \mathrm{sh}^2 \left(\beta^2 \sum_{n \in L \cup \{m\}} \frac{\boldsymbol{\sigma}_n^{\sim} \cdot \boldsymbol{\sigma}_n^{*}}{N} \right) \right\rangle, \end{aligned}$$
where $\dot{\sigma}_n = (\sigma_{n1}, ..., \sigma_{nN}, \sigma_{nN+1})$.

A.1.1 The general fixpoint equation

The function Ψ changes due to the new dependence structure
$$\Psi_m(x_1, ..., x_k) = \alpha_m \mathrm{E}\ \mathrm{th}^2 \left(h_m + \beta \sum_{\substack{n \\ mn \in E}} g_{mn} \sqrt{x_n} + \beta \sum_{n \in L} \delta_{mn} g_{mn} \sqrt{x_n} \right).$$
Similarly, it follows that Ψ_m is a contraction. The unique fixpoint is given by
$$q_m = \alpha_m \mathrm{E}\ \mathrm{th}^2 \left(h_m + \beta \sum_{\substack{n \\ mn \in E}} g_{mn} \sqrt{q_n} + \beta \sum_{n \in L} \delta_{mn} g_{mn} \sqrt{q_n} \right) \qquad m = 1, ..., k.$$

A.2 Algebraic identities for the cavity method

Integrating the spins gradually results in a sequence of systems "inbetween" the $N+2$-system and the N-system, where some classes contains N sites and some $N+1$ or $N+2$. Specifically, consider $\boldsymbol{m} = (m_1, ..., m_k)$ with $m_i \in [k]$ and $m_i \neq m_j$ for $i \neq j$. The sequence $\boldsymbol{m}$ determines the "integration order". First the sites of the class S_{m_1} are integrated, then the one of S_{m_2} and so forth. With the notation $M_i = [k] \setminus \{m_1, ..., m_i\}$ and $M = [k]$, the Hamiltonian after the i-th step is given by

$$\frac{\beta}{\sqrt{N}} \sum_{\substack{s<t \\ m_s, m_t \in M \setminus M_i}} \sum_{i,j \leq N} g_{m_s i}^{m_t j} \sigma_{m_s i} \sigma_{m_t j} + \frac{\beta}{\sqrt{N}} \sum_{\substack{s<t \\ m_s \in M \setminus M_i, m_t \in M_i}} \sum_{\substack{i \leq N \\ j \leq N+2}} g_{m_s i}^{m_t j} \sigma_{m_s i} \sigma_{m_t j}$$
$$+ \frac{\beta}{\sqrt{N}} \sum_{\substack{s<t \\ m_s, m_t \in M_i}} \sum_{i,j \leq N+2} g_{m_s i}^{m_t j} \sigma_{m_s i} \sigma_{m_t j}$$
$$+ \sum_{m_s \in M \setminus M_i} h_{m_s} \sum_{i \leq N} \sigma_{m_s i} + \sum_{m_s \in M_i} h_{m_s} \sum_{i \leq N+2} \sigma_{m_s i}.$$

Integration with respect to the corresponding Gibbs' measure is denoted by $\langle . \rangle'_{M_i}$. Algebraic identities are obtained similiarly to (2.7). For a function f on $\Sigma_N^{i-1} \times \Sigma_{N+z}^{k-(i-1)}$ it holds that

$$\langle f \rangle'_{M_{i-1}} = \frac{1}{Z_{M_i}} \left\langle \underset{\sigma_{m_i N+1}, ..., \sigma_{m_i N+z}}{\text{Av}} f \mathcal{E}_{M_i} \right\rangle'_{M_i},$$

where

$$\mathcal{E}_{M_i}$$
$$= \prod_{n=N+1}^{N+2} \exp \sigma_{m_i n} \left(\frac{\beta}{\sqrt{N}} \left(\sum_{m_s \in M \setminus M_{i-1}} \sum_{j \leq N} g_{m_i n}^{m_s j} \sigma_{m_s j} + \sum_{m_s \in M_i} \sum_{j \leq N+2} g_{m_i n}^{m_s j} \sigma_{m_s j} \right) + h_{m_i} \right),$$

and

$$Z_{M_i} = \left\langle \underset{\sigma_{m_j N+1}, ..., \sigma_{m_j N+z}}{\text{Av}} \mathcal{E}_{M_i} \right\rangle'_{M_i}.$$

The formulas can be generalized to the case when multiple replicas are involoved. Let f be a function on $\left(\Sigma_N^{i-1} \times \Sigma_{N+z}^{k-(i-1)} \right)^n$. Then

$$\langle f \rangle'_{M_{i-1}} = \frac{1}{(Z_{M_i})^n} \left\langle \underset{\sigma^1_{m_i N+1}, ..., \sigma^n_{m_i N+z}}{\text{Av}} f \mathcal{E}^s_{M_i} \right\rangle'_{M_i},$$

where

$$\mathcal{E}^n_{M_i}$$
$$= \prod_{n=N+1}^{N+2} \exp \sum_{l \leq n} \sigma^l_{m_i n} \left(\frac{\beta}{\sqrt{N}} \left(\sum_{m_s \in M \setminus M_{i-1}} \sum_{j \leq N} g_{m_i n}^{m_s j} \sigma^l_{m_s j} + \sum_{m_s \in M_i} \sum_{j \leq N+2} g_{m_i n}^{m_s j} \sigma^l_{m_s j} \right) + h_{m_i} \right)$$

A.3 Supplements to Proposition 3.2.2

A.3.1 Dropping uninvolved sites

It holds that

$$\mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\sigma^{*}_{mN+1}\sigma^{\sim}_{mN+2}\sigma^{*}_{mN+2}\right\rangle' \leq A\ \mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\sigma^{*}_{mN+1}\sigma^{\sim}_{mN+2}\sigma^{*}_{mN+2}\right\rangle'_{m},$$

where

$$A = A(k, \boldsymbol{h}, \beta) = \exp(8k^2\beta^2)\prod_{m\leq k}\mathrm{ch}^8(h_m).$$

Hence, A remains bounded for fixed k and $\boldsymbol{h}$ (uniformly for small β).

Proof These are simple calculations. Choose $m_1 = n \neq m$ and integrate the two sites of class m_1 first. Similar arugments as in the next subsection (A.3.2) are used.
Remember that $\langle . \rangle_{M_1}$ denotes the Gibbs' average with respect to the system, where class m_1 contains N sites and the others $N+2$. By setting $f = \sigma^{\sim}_{mN+1}\sigma^{*}_{mN+1}\sigma^{\sim}_{mN+2}\sigma^{*}_{mN+2}$ and $\sigma^{(i)}_{m_1N+1} = \epsilon_i$, $\sigma^{(i)}_{m_1N+2} = \iota_i$, one obtains

$$\begin{aligned}
&\mathrm{E}\left\langle f\right\rangle' \\
=&\frac{1}{Z_{M_1}}\mathrm{E}\left\langle f \operatorname*{Av}_{\epsilon_1,\ldots,\epsilon_8}\mathcal{E}_{M_1}\right\rangle'_{M_1} \\
\leq&\mathrm{E}\left\langle f \operatorname*{Av}_{\epsilon_1,\ldots,\iota_4}\mathcal{E}_{M_1}\right\rangle'_{M_1} \\
=&\mathrm{E}\langle f\ \mathrm{Av}\exp\sum_{l\leq 4}\epsilon_l\left(\frac{\beta}{\sqrt{N}}\sum_{n\neq m_1}\sum_{j\leq N+2}g^{nj}_{m_1N+1}\sigma^{l}_{nj} + h_{m_1}\right) \\
&\qquad + \iota_l\left(\frac{\beta}{\sqrt{N}}\sum_{n\neq m_1}\sum_{j\leq N+2}g^{nj}_{m_1N+2}\sigma^{l}_{nj} + h_{m_1}\right)\rangle'_{M_1} \\
=&\mathrm{E}\left\langle f\prod_{l\leq 4}\operatorname*{Av}_{\epsilon_l,\iota_l}\exp\epsilon_l h_{m_1}\exp\iota_l h_{m_1}\prod_{n\neq m_1}\prod_{j\leq N+2}\exp\left(g^{nj}_{m_1N+1}\frac{\epsilon_l\sigma^l_{nj}\beta}{\sqrt{N}}\right)\exp\left(g^{nj}_{m_1N+2}\frac{\iota_l\sigma^l_{nj}\beta}{\sqrt{N}}\right)\right\rangle_{M_1} \\
=&\mathrm{E}\left\langle f\prod_{l\leq 4}\operatorname*{Av}_{\epsilon_l,\iota_l}\exp\epsilon_l h_{m_1}\exp\iota_l h_{m_1}\prod_{n\neq m_1}\prod_{j\leq N+2}\exp\frac{\beta^2}{2N}\exp\frac{\beta^2}{2N}\right\rangle_{M_1} \\
\leq&\mathrm{E}\left\langle f\prod_{l\leq 4}\exp(2(k-1)\beta^2)\operatorname*{Av}_{\epsilon_l,\iota_l}\exp\epsilon_l h_{m_1}\exp\iota_l h_{m_1}\right\rangle'_{M_1} \\
=&\mathrm{ch}^8 h_m\exp(8(k-1)\beta^2)\mathrm{E}\left\langle f\right\rangle'_{M_1}.
\end{aligned}$$

Iterating this process for all $m_i \neq m$ finishes the proof.

□

A.3.2 Integrating the last sites

It holds that

$$\begin{aligned}
&\mathrm{E}\left\langle \sigma^{\sim}_{mN+1}\ \sigma^{*}_{mN+1}\ \sigma^{\sim}_{mN+2}\ \sigma^{*}_{mN+2}\right\rangle'_m \\
\leq &\mathrm{E}\left\langle \mathrm{sh}^2\left(\beta^2\sum_{n\neq m}\frac{\boldsymbol{\sigma}^{\sim}_n\cdot\boldsymbol{\sigma}^{*}_n}{N}\right)\exp\frac{\beta^2}{N}\sum_{n\neq m}\left(||\boldsymbol{\sigma}^{\sim}_n||^2+||\boldsymbol{\sigma}^{*}_n||^2\right)\right\rangle,
\end{aligned}$$

Proof The notation is lightened by setting $\epsilon_l=\sigma^l_{N+1}$ and $\iota_l=\sigma^l_{N+2}$. One obtains

$$\mathrm{E}\left\langle \epsilon^{\sim}\epsilon^{*}\iota^{\sim}\iota^{*}\right\rangle'_m=\frac{1}{Z^4_m}\mathrm{E}\left\langle \mathrm{Av}\ \epsilon^{\sim}\epsilon^{*}\iota^{\sim}\iota^{*}\mathcal{E}^4_m\right\rangle,$$

where the average is taken over $\epsilon^1,...,\iota^4$ and

$$\mathcal{E}^4_m=\prod_{z=N+1}^{N+2}\exp\sum_{l\leq 4}\left(\sigma_{mN+z}\frac{\beta}{\sqrt{N}}\sum_{n\neq m}g^{nj}_{mN+z}\sigma^l_{nj}+h_m\right)$$

Since $Z_m\geq 1$ and

$$\mathrm{E}\left\langle \mathrm{Av}\ \epsilon^{\sim}\epsilon^{*}\iota^{\sim}\iota^{*}\mathcal{E}^4_m\right\rangle'_m=\mathrm{E}\left\langle \mathrm{Av}\ \epsilon^{\sim}\epsilon^{*}\mathcal{E}^2_m\right\rangle'^2_m\geq 0$$

one deduces

$$\mathrm{E}\left\langle \epsilon^{\sim}\epsilon^{*}\iota^{\sim}\iota^{*}\right\rangle'_m\leq\mathrm{E}\left\langle \mathrm{Av}\ \epsilon^{\sim}\epsilon^{*}\iota^{\sim}\iota^{*}\mathcal{E}^4_m\right\rangle.$$

The integrand is different from 0, if $\epsilon^{\sim},\epsilon^{*},\iota^{\sim},\iota^{*}\neq 0$. This is equivalent to $\epsilon^2=-\epsilon^1,\ \epsilon^4=-\epsilon^3,\ \iota^2=-\iota^1,\ \iota^4=-\iota^3$. It follows that

$$\epsilon^{\sim}\epsilon^{*}\iota^{\sim}\iota^{*}\mathcal{E}^4_m=\epsilon^{\sim}\epsilon^{*}\mathcal{E}'_{m,1}\iota^{\sim}\iota^{*}\mathcal{E}'_{m,2},$$

where

$$\begin{aligned}
\mathcal{E}'_{m,1}&=\exp\frac{\beta}{\sqrt{N}}\sum_{n\neq m}\sum_{j\leq N}g^{nj}_{mN+1}(\epsilon^1\sigma^{\sim}_{nj}+\epsilon^3\sigma^{*}_{nj}),\\
\mathcal{E}'_{m,2}&=\exp\frac{\beta}{\sqrt{N}}\sum_{n\neq m}\sum_{j\leq N}g^{nj}_{mN+2}(\iota^1\sigma^{\sim}_{nj}+\iota^3\sigma^{*}_{nj}).
\end{aligned}$$

Since

$$\underset{\epsilon^2}{\mathrm{Av}}\epsilon^{\sim}=\epsilon^1\quad\underset{\epsilon^3}{\mathrm{Av}}\epsilon^{*}=\epsilon^3\quad\underset{\iota^2}{\mathrm{Av}}\epsilon^{\sim}=\iota^1\quad\underset{\iota^4}{\mathrm{Av}}\iota^{*}=\iota^3$$

it follows that

$$\mathrm{E}\left\langle \mathrm{Av}\ \epsilon^{\sim}\epsilon^{*}\iota^{\sim}\iota^{*}\mathcal{E}^4_m\right\rangle=\mathrm{E}\left\langle \mathrm{Av}\ \epsilon^1\epsilon^3\mathcal{E}'_{m,1}\iota^1\iota^3\mathcal{E}'_{m,2}\right\rangle,$$

where on the right hand side $\epsilon^2,\ \epsilon^4,\ \iota^2,\ \iota^4$ are averaged out. Expectation in $(g^{nj}_{mN+1})_{j\leq N,n\neq m}$ or $(g^{nj}_{mN+2})_{j\leq N,n\neq m}$ only is denoted by $\mathrm{E}_{g,1}$ and $\mathrm{E}_{g,2}$ respectively. Thus

$$\begin{aligned}\mathcal{E}''_{m,1} = \mathrm{E}_{g,1}\mathcal{E}'_{m,1} &= \exp\frac{\beta^2}{2N}\sum_{n\neq m}\sum_{j\leq N}((\sigma^{\sim}_{nj})^2 + (\sigma^{*}_{nj})^2 + 2\epsilon^1\epsilon^3\sigma^{\sim}_{nj}\sigma^{*}_{nj}) \\ &= \exp\frac{\beta^2}{2N}\sum_{n\neq m}\left(||\boldsymbol{\sigma}_n{}^{\sim}||^2 + ||\boldsymbol{\sigma}_n{}^{*}||^2 + 2\epsilon^1\epsilon^3\boldsymbol{\sigma}_n{}^{\sim}\cdot\boldsymbol{\sigma}_n{}^{*}\right)\end{aligned}$$

where $||\boldsymbol{\sigma}^{\sim}_n||^2 := \sum_{i\leq N}(\sigma_{ni})^2$. Now

$$\begin{aligned}\mathcal{E}''_{m,2} = \mathrm{E}_{g,2}\mathcal{E}'_{m,2} &= \exp\frac{\beta^2}{2N}\sum_{n\neq m}\sum_{j\leq N}((\sigma^{\sim}_{nj})^2 + (\sigma^{*}_{nj})^2 + 2\iota^1\iota^3\sigma^{\sim}_{nj}\sigma^{*}_{nj}) \\ &= \exp\frac{\beta^2}{2N}\left(\sum_{n\neq m}||\boldsymbol{\sigma}^{\sim}_n||^2 + ||\boldsymbol{\sigma}^{*}_n||^2 + 2\iota^1\iota^3\boldsymbol{\sigma}_n{}^{\sim}\cdot\boldsymbol{\sigma}_n{}^{*}\right).\end{aligned}$$

Since the disorder in $\langle . \rangle$ is independent of the g^{nj}_{mN+1}'s and g^{nj}_{mN+2}'s,

$$\begin{aligned}&\mathrm{E}\left\langle \mathrm{Av}\ \epsilon^1\epsilon^3\mathcal{E}'_{m,1}\iota^1\iota^3\mathcal{E}'_{m,2}\right\rangle \\ =&\mathrm{E}\left\langle \underset{\epsilon^1,\epsilon^3}{\mathrm{Av}}\ \epsilon^1\epsilon^3\mathcal{E}''_{m,1}\underset{\iota^1,\iota^3}{\mathrm{Av}}\iota^1\iota^3\mathcal{E}''_{m,2}\right\rangle \\ =&\mathrm{E}\left\langle \mathrm{sh}^2\left(\frac{\beta^2}{N}\sum_{n\neq m}\boldsymbol{\sigma}^{\sim}_n\cdot\boldsymbol{\sigma}^{*}_n\right)\exp\frac{\beta^2}{N}\sum_{n\neq m}\left(||\boldsymbol{\sigma}^{\sim}_n||^2 + ||\dot{\boldsymbol{\sigma}}^{*}_n||^2\right)\right\rangle.\end{aligned}$$

□

A.3.3 Taylor Expansion

For $|x| \leq 4$ it holds that

$$\mathrm{sh}^2(\beta^2 x) \leq x^2/16\ \mathrm{sh}^2(4\beta^2)$$

Proof Note that sh(x) is monotonically increasing. For $0 \leq x \leq 4$, it holds that

$$x\mathrm{sh}\beta^2 x \leq x^2/4\ \mathrm{sh}4\beta^2.$$

Respectively, for $0 \geq x \geq -4$, the antisymmetry of sh(x) implies that

$$x\mathrm{sh}\beta^2 x \leq x\mathrm{sh}(-4\beta^2) = -x\mathrm{sh}4\beta^2 \leq x^2/4\mathrm{sh}4\beta^2.$$

Thus for $|x| \leq 4$,

$$x\mathrm{sh}\beta^2 x \leq \frac{x^2}{4}\mathrm{sh}4\beta^2.$$

Taking squares first and then dividing both sides by x^2 finishes the proof. □

A.4 Moment-generating function of a Gaussian

Proposition A.4.1 *Let g be a centered Gaussian with $E[g^2] = \tau^2$. Then*

$$\mathrm{E}\ \exp ag = \exp \frac{a^2\tau^2}{2}$$

Proof

$$\begin{aligned}
\mathrm{E}\ \exp ag &= \frac{1}{\sqrt{2\pi\tau^2}} \int \exp at \exp -\frac{t^2}{2\tau^2}\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi\tau^2}} \int \exp -\left(\frac{t}{\sqrt{2\tau^2}} - \frac{a\tau}{\sqrt{2}}\right)^2 \exp \frac{a^2\tau^2}{2}\mathrm{d}t \\
&= \exp \frac{a^2\tau^2}{2} \frac{1}{\sqrt{2\pi\tau^2}} \int \exp -\xi^2 \mathrm{d}\xi \sqrt{2\tau^2} \\
&= \exp \frac{a^2\tau^2}{2}
\end{aligned}$$

□

A.5 Gaussian integration by parts

Proposition A.5.1 *Let f be a smooth function, such that $(1+x^2)^{-k} f(x)$ is bounded for some k. Additionally, g is a centered Gaussian with $E(g^2) = \tau^2$. Then*

$$E(gf(g)) = E(g^2)E(f'(g))$$

Proof This is a direct consequence of the integration by parts formula for integrals

$$\frac{1}{\sqrt{2\pi\tau^2}} \int tf(t) \exp\left(-\frac{t^2}{2\tau^2}\right) dt = \tau^2 \frac{1}{\sqrt{2\pi\tau^2}} \int f'(t) \exp\left(-\frac{t^2}{2\tau^2}\right) dt$$

□

A.6 Proof of (3.9)

Proof

$$\langle \sigma_{mN+1} \rangle'_{M\backslash m} = \frac{1}{Z} \left\langle \underset{\sigma_{mN+1}}{\mathrm{Av}} \sigma_{mN+1} \mathcal{E} \right\rangle,$$

where

$$\mathcal{E} := \exp \sigma_{mN+1} \left(\frac{\beta}{\sqrt{N}} \sum_{n \neq m} g^{nj}_{mN+1} \sigma_{nj} + h_m \right)$$

$$Z = \langle \mathrm{Av}\mathcal{E} \rangle .$$

So

$$
\begin{aligned}
Z =& \left\langle \operatorname{Av} \exp \sigma_{mN+1} \left(\frac{\beta}{\sqrt{N}} \sum_{n \neq m} g_{mN+1}^{nj} \sigma_{nj} + h_m \right) \right\rangle \\
=& \frac{1}{2} \left\langle \exp h_m \prod_{n \neq m} \exp \frac{\beta}{\sqrt{N}} \boldsymbol{g}_{mn} \boldsymbol{b}_n \right\rangle + \frac{1}{2} \left\langle \exp -h_m \prod_{n \neq m} \exp \frac{-\beta}{\sqrt{N}} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right\rangle \\
.
\end{aligned}
$$

Now proposition 3.2.5 yields

$$
\mathrm{E} \left| Z - \exp \frac{\beta^2}{2} \sum_{n \neq m} (1 - \bar{q}_n) \operatorname{ch} \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) \right|
$$

and similarly

$$
\mathrm{E} \left| \langle \sigma_{mN+1} \rangle'_{M \setminus m} - \exp \frac{\beta^2}{2} \sum_{n \neq m} (1 - \bar{q}_n) \operatorname{sh} \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) \right| .
$$

Finally, since $Z \geq 1$,

$$
\begin{aligned}
& \mathrm{E} \left| \langle \sigma_{mN+1} \rangle'_{M \setminus m} - \operatorname{th} \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) \right| \\
=& \frac{1}{Z} \mathrm{E} \left| \langle \operatorname{Av} \sigma_{mN+1} \mathcal{E} \rangle - Z \ \operatorname{th} \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) \right| \\
\leq& \mathrm{E} \left| \langle \operatorname{Av} \sigma_{mN+1} \mathcal{E} \rangle - \exp \frac{\beta^2}{2} \sum_{n \neq m} (1 - \bar{q}_n) \operatorname{sh} \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) \right| \\
& + \mathrm{E} \left| \exp \frac{\beta^2}{2} \sum_{n \neq m} (1 - \bar{q}_n) \operatorname{sh} \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) - Z \ \operatorname{th} \left(h_m + \frac{\beta}{\sqrt{N}} \sum_{n \neq m} \boldsymbol{g}_{mn} \cdot \boldsymbol{b}_n \right) \right| \\
& \to 0.
\end{aligned}
$$

Note that, by subsequent integration, (3.8) can be obtained similiarly.