Valuation of Performance-Dependent Options

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Abstract

Performance-dependent options are financial derivatives whose payoff depends on the performance of one asset in comparison to a set of benchmark assets. In this paper, we present a novel approach for the valuation of general performance-dependent options. To this end, we use a multidimensional Black-Scholes model to describe the temporal development of the asset prices. The martingale approach then yields the fair price of such options as a multidimensional integral whose dimension is the number of stochastic processes used in the model. The integrand is typically discontinuous which makes accurate solutions difficult to achieve by numerical approaches, though. Using tools from computational geometry, we are able to derive a pricing formula which only involves the evaluation of several smooth multivariate normal distributions. This way, performance-dependent options can efficiently be priced even for high-dimensional problems as is shown by numerical results.

Keywords: option pricing, multivariate integration, hyperplane arrangements

AMS-Classification: 65D30, 65D18, 91B28

1 Introduction

Companies make big efforts to bind their staff to them for longer periods of time in order to prevent a permanent change of executives in important positions. Besides high wages, such efforts are long-term incentive and bonus schemes. One widespread form of such schemes consists in giving the participants a conditional award of shares [24]. If the participant stays with the company for at least a prescribed time period he will receive a certain number of company shares at the end of the period. Typically, the exact amount of shares is determined by a performance criterion such as the company's gain over the period or its ranking among comparable firms (the peer group). This way, such bonus schemes induce uncertain future costs for the company.

For the corporate management and especially for the shareholders, the actual value of such bonus programs is quite interesting. One way to determine an upper bound of this value is to take the price of vanilla call options on the maximum number of possibly needed shares. This upper bound, however, often significantly overestimates the true value of the bonus program since its specific structure is not respected.

Contingent claim theory states that the accurate value of such bonus programs is given by the fair price of options which include the used performance criteria in their payoff. Such options are called performance-dependent options. Their payoff yields exactly the required shares at the end of the bonus scheme. This way, performance-dependent options minimize the amount of money the company would need to hedge the future payments arising from the bonus scheme, see, e.g. [19].

Similar performance comparison criteria are currently used in various financial products, for example many hedge funds are employing so-called portable alpha strategies. Recently, also pure performance-based derivatives have entered the market in the form of so-called alpha certificates. Here, typically the relative performance of a basket of stocks is compared to the relative performance of a stock index. Such products are either used for risk diversification or for pure performance speculation purposes.

In this paper, we define a framework for the efficient valuation of fairly general performance-dependent options. Thereby, we assume that the performance of an asset is determined by the relative increase of the asset price over the considered period of time. This performance is then compared to the performances of a set of benchmark assets. For each possible outcome of this comparison, a different payoff can be realized.

We use a multidimensional Black-Scholes model (see, e.g., [15, 18]) for the temporal development of all asset prices required for the performance ranking. The martingale approach then yields a fair price of the performance-dependent option as a multidimensional integral whose dimension is the number of stochastic processes used in the model. In the so-called full model the number of stochastic processes equals the number of assets. In the reduced model, the number of processes is smaller. Unfortunately, in either case there is no direct closed-form solution for this integral. Moreover, the integrand is typically discontinuous which makes accurate numerical solutions difficult to achieve.

The main contribution of this paper is the derivation of closed-form solutions to these integration problems. For reduced models, two novel tools from computational geometry are used, a fast enumeration method of the cells of a hyperplane arrangement and an algorithm for the determination of its orthant decomposition [13]. The resulting closed-form solutions only involve the evaluation of smooth multivariate normal distributions which can be computed quickly and robustly using numerical integration schemes. In various numerical results, we illustrate the efficiency of this approach.

The outline of this paper is as follows. In Section 2, we formally define performance-dependent options, their payoff profiles and the underlying stochastic model. A pricing formula in the full model case is presented in Section 3. The corresponding pricing formula for reduced models is then derived using tools from computational geometry in Section 4. In Section 5, we shortly discuss the numerical computation of multivariate normal distributions. Numerical results on different types of performance-dependent options are presented in Section 6. Concluding remarks are finally drawn in Section 7.

2 Performance-dependent options

In this Section, the functionality of performance-dependent options is illustrated. We formally define their payoff functions and give several example payoff profiles which correspond to possible bonus schemes. The multivariate Black-Scholes model which we use to describe the temporal development of the underlying asset prices is recalled at the end of the Section.

2.1 Framework

A performance-dependent option is a financial derivative whose payoff depends on the performance of one asset in comparison to other assets at the end of a given period. For the fair valuation of a bonus scheme, as mentioned in the introduction, the asset under consideration is the stock of our own company while the other assets are the stocks of benchmark companies.

Let us remark here that several differences between the pricing of standard derivatives and the pricing of employee stock options which are not addressed in this paper are thoroughly discussed in Hull and White [16, 17]. In their papers, only performance-independent employee stock options are considered, though.

We assume that there are n companies involved in total. Our company gets assigned label 1 and the $n-1$ benchmark companies are labeled from 2 to n. The stock price of the i-th company varying with time t is denoted by $S_i(t), 1 \leq i \leq n$. The current time is denoted by $t = 0$. All stock prices at the end of the time period $t = T$ are collected in the vector $\mathbf{S} = (S_1(T), \ldots, S_n(T))$.

2.2 Payoff profile

The character of a performance-dependent option is described by the payoff of the option at time T. To this end, we denote the relative price increase of stock i over the time interval $[0, T]$ by

$$\Delta S_i := \frac{S_i(T)}{S_i(0)}.$$

We save the performance of the first company in comparison to a given strike price K and in comparison to the benchmark assets at time T in a ranking vector $\mathbf{Rank}(\mathbf{S}) \in \{+, -\}^n$ which is defined by

$$\mathrm{Rank}_1(\mathbf{S}) := \begin{cases} + & \text{if } S_1(T) \geq K, \\ - & \text{else} \end{cases} \quad \text{and} \quad \mathrm{Rank}_i(\mathbf{S}) := \begin{cases} + & \text{if } \Delta S_1 \geq \Delta S_i, \\ - & \text{else} \end{cases}$$

for $i = 2, \ldots, n$. This means, if the first asset outperforms benchmark asset i we denote this by a plus sign in the i-th component of the ranking vector $\mathbf{Rank}(\mathbf{S})$, otherwise, there is a minus sign. For the fair valuation of a bonus scheme, the strike K is typically equal to $S_1(0)$ since this way the payoff represents the risk of the price increase of the company's own stock until time T. In the following, arbitrary strike prices K are allowed, though.

In order to define the payoff of the performance-dependent option, we require bonus factors $a_{\mathbf{R}}$ which define the bonus for each possible ranking $\mathbf{R} \in \{+, -\}^n$. It is important to distinguish here between a possible ranking denoted $\mathbf{R}$ and the realized ranking induced by $\mathbf{S}$ which is denoted by $\mathbf{Rank}(\mathbf{S})$. The payoff of the performance-dependent option at time T is then defined by

$$V((\mathbf{S}), T) := a_{\mathbf{Rank}(\mathbf{S})} (S_1(T) - K)^+ = a_{\mathbf{Rank}(\mathbf{S})} \max\{S_1(T) - K, 0\}.$$

We always define $a_{\mathbf{R}} = 0$ if $R_1 = -$, so that the payoff can be written as

$$V(\mathbf{S}, T) = a_{\mathbf{Rank}(\mathbf{S})} (S_1(T) - K). \tag{1}$$

2.3 Example payoff profiles

In the following, we illustrated some possible choices for the bonus factors $a_{\mathbf{R}}$ which are included in our framework.

Example 2.1 *Performance-independent option:*

$$a_{\mathbf{R}} = \begin{cases} 1 & \text{if } R_1 = + \\ 0 & \text{else.} \end{cases}$$

In this case, we recover a European call option on the stock S_1.

Example 2.2 *Linear ranking-dependent option:*

$$a_{\mathbf{R}} = \begin{cases} m/(n-1) & \text{if } R_1 = + \\ 0 & \text{else.} \end{cases}$$

Here, m denotes the number of outperformed benchmark assets. The payoff only depends on the rank of our company in the benchmark. If the company ranks first, there is a full payoff $(S_1(T) - K)^+$. If it ranks last, the payoff is zero. In between, the payoff increases linearly with the number of outperformed benchmark assets.

Example 2.3 *Outperformance option:*

$$a_{\mathbf{R}} = \begin{cases} 1 & \text{if } \mathbf{R} = (+, \ldots, +) \\ 0 & \text{else.} \end{cases}$$

A payoff only occurs if $S_1(T) \geq K$ and if all benchmark assets are outperformed.

Example 2.4 *Linear ranking-dependent option combined with an outperformance condition:*

$$a_{\mathbf{R}} = \begin{cases} m/(n-1) & \text{if } R_1 = + \text{ and } R_2 = + \\ 0 & \text{else.} \end{cases}$$

The bonus depends linearly on the number m of outperformed benchmark companies like in Example 2.2. However, the bonus is only payed if company two is outperformed. Company two could, e.g., be the main competitor of our company.

2.4 Multivariate Black-Scholes model

For the valuation of derivatives in markets with several interacting assets, the multidimensional Black-Scholes model [15, 18] has been used with great success. There, it is assumed that the stock prices are driven by $d \leq n$ stochastic processes modeled by the Black-Scholes-type system of stochastic partial differential equations (SDEs)

$$dS_i(t) = S_i(t) \left(\mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t) \right) \tag{2}$$

for $i = 1, \ldots, n$. Here, μ_i denotes the drift of the i-th stock, σ the $n \times d$ volatility matrix of the stock price movements and $W_j(t)$ the corresponding Wiener processes. The matrix $\sigma\sigma^T$ is assumed to be positive definite. If $d = n$, we call the corresponding model full. If $d < n$, the model is called reduced.

Let us remark here that for small benchmarks usually the full model with a square volatility matrix σ is used. The entries of the volatility matrix are typically estimated from historical market data. However, for larger benchmarks, the parameter estimation problem becomes more and more ill-conditioned resulting in eigenvalues of $\sigma\sigma^T$ which are close to zero. Then, reduced models with $d < n$ are often employed. If the benchmark consists of all assets in a stock index, this reduction can be achieved, for instance, by grouping assets in the same area of business. The matrix entry σ_{ij} then reflects the correlation of stock i with business area j. Such a grouping can often be obtained without much loss of information e.g. using Principal Component Analysis (PCA), as was confirmed empirically by research from Meade and Salkin [22] and Laloux *et al.* [20].

By Itô's formula, the explicit solution of the SDE is given by

$$S_i(T) = S_i(\mathbf{X}) = S_i(0) \exp \left(\mu_i T - \bar{\sigma}_i + \sqrt{T} \sum_{j=1}^{d} \sigma_{ij} X_j \right) \tag{3}$$

for $i = 1, \ldots, n$ with

$$\bar{\sigma}_i := \frac{1}{2} \sum_{j=1}^{d} \sigma_{ij}^2 T$$

and $\mathbf{X} = (X_1, \ldots, X_d)$ being a $N(\mathbf{0}, \mathbf{I})$-normally distributed random vector.

Various multivariate option pricing problems not discussed in this paper allow closed form solutions, see, e.g., Zhang [26] or Carmona and Durrleman [3]. A valuation approach for American-style performance-dependent options using a fairly general Lévy model for the underlying securities is presented in Egloff *et al.* [8]. There, a least-squares Monte Carlo scheme is used for the numerical solution of the model, but only in the case of one benchmark process. Thus, the problem of high-dimensionality does not arise which is one of the main issues in this paper.

3 Pricing formula in the full model

In this Section, we assume that the number of stochastic processes d equals the number of assets n. We derive the price of a performance-dependent option as a multivariate integral and show how this integral can be evaluated in terms of multivariate normal distributions. In the following, we nevertheless distinguish between d and n in order to be able to reuse some of the results also for the reduced model case.

3.1 Martingale approach

The multivariate Black-Scholes model induces a complete market which gives the existence of a unique equivalent martingale measure. Using the usual Black-Scholes assumptions, see, e.g., [18], the option price $V(S_1(0), 0)$ is given by the discounted expectation

$$V(S_1(0), 0) = e^{-rT} E[V(\mathbf{S}, T)] \tag{4}$$

of the payoff under the equivalent martingale measure. To this end, the drift μ_i is replaced by the riskless interest rate r for each stock i. In the case of a performance-dependent option with payoff (1) we get

$$V(S_1(0), 0) = e^{-rT} E\left[\sum_{\mathbf{R} \in \{+,-\}^n} a_{\mathbf{R}} (S_1(T) - K)\, \chi_{\mathbf{R}}(\mathbf{S})\right].$$

Thereby, the expectation runs over all possible rankings $\mathbf{R}$ and the characteristic function $\chi_{\mathbf{R}}(\mathbf{S})$ is defined by

$$\chi_{\mathbf{R}}(\mathbf{S}) = \begin{cases} 1 & \text{if } \mathbf{Rank}(\mathbf{S}) = \mathbf{R}, \\ 0 & \text{else.} \end{cases}$$

Plugging in the density function $\varphi(\mathbf{x}) := \varphi_{\mathbf{0},\mathbf{I}}(\mathbf{x})$ of the $N(\mathbf{0}, \mathbf{I})$-distributed random vector $\mathbf{X}$ (note that $\mathbf{S} = \mathbf{S}(\mathbf{X})$), we get

$$V(S_1(0), 0) = e^{-rT} \int_{R^d} \sum_{\mathbf{R} \in \{+,-\}^n} a_{\mathbf{R}} (S_1(T) - K)\, \chi_{\mathbf{R}}(\mathbf{S}) \varphi(\mathbf{x})\, d\mathbf{x} \tag{5}$$

which will be the starting point of our analysis.

3.2 Pricing formula

Looking at formula (5), we see that the fair price of a performance-dependent can be obtained by computing a d-dimensional integral. The integral can, at least at first sight, not be solved analytically and therefore requires numerical approaches for its solution. The integrand, however, is discontinuous induced by the jumps of the bonus factors $a_{\mathbf{R}}$ (see the examples in Section 2). Therefore, numerical integration methods will perform poorly and only Monte Carlo integration can be used without penalty. Thus, high accuracy solutions will be hard to obtain. In the following, we derive an analytical expression for the computation of (5) in terms of smooth functions, in our case multivariate normal distributions.

Let us first recall that the multivariate normal distribution with mean zero, limits $\mathbf{b} = (b_1, \ldots, b_d)$ and $d \times d$ covariance matrix $\mathbf{C}$ is defined by

$$\Phi(\mathbf{C}, \mathbf{b}) := \int_{-\infty}^{b_1} \ldots \int_{-\infty}^{b_d} \varphi_{\mathbf{0},\mathbf{C}}(\mathbf{x})\, dx_d \ldots dx_1$$

with the Gauss kernel

$$\varphi_{\mu,\mathbf{C}}(\mathbf{x}) := \frac{1}{(2\pi)^{d/2}(\det \mathbf{C})^{1/2}} \; e^{-\frac{1}{2}(\mathbf{x}-\mu)^T\mathbf{C}^{-1}(\mathbf{x}-\mu)}.$$

To prove our main theorem we need the following two lemmas which relate the payoff conditions to multivariate normal distributions.

Lemma 3.1 *Let* $\mathbf{b}, \mathbf{q} \in \mathbb{R}^d$ *and* $\mathbf{A} \in \mathbb{R}^{d\times d}$ *with full rank, then*

$$\int_{\mathbf{Ax}\geq\mathbf{b}} e^{\mathbf{q}^T\mathbf{x}}\varphi(\mathbf{x})d\mathbf{x} = e^{\frac{1}{2}\mathbf{q}^T\mathbf{q}}\Phi(\mathbf{A}\mathbf{A}^T, \mathbf{Aq}-\mathbf{b}).$$

We use $\int_{\mathbf{Ax}\geq\mathbf{b}}$ *as abbreviation for the integration over the set* $\{\mathbf{x}\in\mathbb{R}^d : \mathbf{Ax}\geq\mathbf{b}\}$.

Proof: A straightforward computation shows

$$e^{\mathbf{q}^T\mathbf{x}}\varphi(\mathbf{x}) = e^{\frac{1}{2}\mathbf{q}^T\mathbf{q}}\varphi_{\mathbf{q},\mathbf{I}}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^d$. Using the substitution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} + \mathbf{q}$ we obtain

$$\begin{aligned}\int_{\mathbf{Ax}\geq\mathbf{b}} e^{\mathbf{q}^T\mathbf{x}}\varphi(\mathbf{x})d\mathbf{x} &= e^{\frac{1}{2}\mathbf{q}^T\mathbf{q}}\int_{\mathbf{Ax}\geq\mathbf{b}} \varphi_{\mathbf{q},\mathbf{I}}(\mathbf{x})d\mathbf{x} \\ &= e^{\frac{1}{2}\mathbf{q}^T\mathbf{q}}\int_{\mathbf{y}\geq\mathbf{b}-\mathbf{Aq}} \varphi_{0,\mathbf{A}\mathbf{A}^\mathbf{T}}(\mathbf{y})\; d\mathbf{y}\end{aligned}$$

and thus the assertion. □

For the second Lemma, we first need to define a comparison relation $\geq_\mathbf{R}$ of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with respect to the ranking $\mathbf{R}$:

$$\mathbf{x} \geq_\mathbf{R} \mathbf{y} \;\; :\Leftrightarrow \;\; R_i(x_i - y_i) \geq 0 \;\; \text{for } 1 \leq i \leq n.$$

Thus, the comparison relation $\geq_\mathbf{R}$ is the usual component-wise comparison where the direction depends on the sign of the corresponding entry of the ranking vector $\mathbf{R}$.

Lemma 3.2 *We have* $\mathbf{Rank}(\mathbf{S}) = \mathbf{R}$ *exactly if* $\mathbf{AX} \geq_\mathbf{R} \mathbf{b}$ *with*

$$\mathbf{A} := \sqrt{T}\begin{pmatrix} \sigma_{11} & \dots & \sigma_{1d} \\ \sigma_{11}-\sigma_{21} & \dots & \sigma_{1d}-\sigma_{2d} \\ \vdots & & \vdots \\ \sigma_{11}-\sigma_{n1} & \dots & \sigma_{1d}-\sigma_{nd}\end{pmatrix} \quad \text{and } \mathbf{b} := \begin{pmatrix} \ln\frac{K}{S_1(0)} - rT + \bar{\sigma}_1 \\ \bar{\sigma}_1 - \bar{\sigma}_2 \\ \vdots \\ \bar{\sigma}_1 - \bar{\sigma}_n\end{pmatrix}$$

where $\mathbf{A} \in \mathbb{R}^{n\times d}$, $\mathbf{X} \in \mathbb{R}^d$ *and* $\mathbf{b} \in \mathbb{R}^n$.

Proof: Using (3) we see that $\text{Rank}_1(\mathbf{S}) = +$ is equivalent to

$$S_1(T) \geq K \quad \Longleftrightarrow \quad \sqrt{T}\sum_{j=1}^{d}\sigma_{1j}X_j \geq \ln\frac{K}{S_1(0)} - rT + \bar{\sigma}_1$$

which yields the first row of the system $\mathbf{AX} \geq_{\mathbf{R}} \mathbf{b}$. Moreover, for $i = 2, \ldots, n$ the outperformance criterion $\text{Rank}_i(\mathbf{S}) = +$ can be written as

$$\frac{S_1(T)}{S_1(0)} \geq \frac{S_i(T)}{S_i(0)} \quad \Longleftrightarrow \quad \sqrt{T} \sum_{j=1}^{d} (\sigma_{1j} - \sigma_{ij}) X_j \geq \bar{\sigma}_1 - \bar{\sigma}_i$$

which yields rows 2 to n of the system. □

Now we can state the following pricing formula which, in a slightly more special setting, is originally due to Korn [19].

Theorem 3.3 *The price of a performance-dependent option with payoff (1) is for the model (2) in the case $d = n$ given by*

$$V(S_1(0), 0) = \sum_{\mathbf{R} \in \{+,-\}^n} a_{\mathbf{R}} \left(S_1(0)\, \Phi(\mathbf{A_R A_R}^T, -\mathbf{d_R}) - e^{-rT} K \Phi(\mathbf{A_R A_R}^T, -\mathbf{b_R}) \right)$$

where the vectors $\mathbf{b_R}$, $\mathbf{d_R}$ and the matrix $\mathbf{A_R}$ are defined by $(\mathbf{b_R})_i := R_i \mathbf{b}_i$, $(\mathbf{d_R})_i := R_i \mathbf{d}_i$ and $(\mathbf{A_R})_{ij} := \mathbf{R}_i \mathbf{A}_{ij}$. Thereby, $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}^n$ are defined as in Lemma 3.2 and the vector $\mathbf{d} \in \mathbb{R}^n$ is defined by $\mathbf{d} := \mathbf{b} - \sqrt{T}\mathbf{A}\sigma_1$ with σ_1^T being the first row of the volatility matrix σ.

Proof: The characteristic function $\chi_{\mathbf{R}}(\mathbf{S})$ in the integral (5) can be eliminated using Lemma 3.2 and we get

$$V(S_1(0), 0) = e^{-rT} \sum_{\mathbf{R} \in \{+,-\}^n} a_{\mathbf{R}} \int_{\mathbf{Ax} \geq_{\mathbf{R}} \mathbf{b}} (S_1(T) - K) \varphi(\mathbf{x}) d\mathbf{x}. \tag{6}$$

By (3), the integral term can be written as

$$S_1(0) e^{rT - \bar{\sigma}_1} \int_{\mathbf{Ax} \geq_{\mathbf{R}} \mathbf{b}} e^{\sqrt{T}\sigma_1^T \mathbf{x}} \varphi(\mathbf{x}) d\mathbf{x} - K \int_{\mathbf{Ax} \geq_{\mathbf{R}} \mathbf{b}} \varphi(\mathbf{x}) d\mathbf{x}.$$

Application of Lemma 3.1 with $\mathbf{q} = \sqrt{T}\sigma_{\mathbf{1}}$ shows that the first integral equals

$$e^{\frac{1}{2}\mathbf{q}^T\mathbf{q}} \int_{\mathbf{y} \geq_{\mathbf{R}} \mathbf{b} - \mathbf{Aq}} \varphi_{0,\mathbf{AA^T}}(\mathbf{y})\, d\mathbf{y} = e^{\bar{\sigma}_1} \int_{\mathbf{y} \geq \mathbf{d_R}} \varphi_{0,\mathbf{A_R A_R^T}}(\mathbf{y})\, d\mathbf{y} = e^{\bar{\sigma}_1} \Phi(\mathbf{A_R A_R}^T, -\mathbf{d_R}).$$

By a further application of Lemma 3.1 with $\mathbf{q} = \mathbf{0}$ we obtain that the second integral equals $K\Phi(\mathbf{A_R A_R}^T, -\mathbf{b_R})$ and thus the assertion holds. □

Note that this decomposition not only provides the option price as a sum of normal distributions but can also be used to show which rankings appear with which probabilities under the model assumptions.

4 Pricing formula in the reduced model

The pricing formula in Theorem 3.3 allows a stable and efficient valuation of performance-dependent options in the case of moderate-sized benchmarks. For a large number n of benchmark assets, one is, however, confronted with the following problems:

- In total, 2^n rankings have to be considered and thus an with n exponentially growing number of cumulative normal distributions have to be computed.
- For each normal distribution, an n-dimensional integration problem has to be solved which gets increasingly more difficult with rising n.
- In larger benchmarks, stock prices are typically highly correlated. As a consequence, some of the eigenvalues of the covariance matrix σ will be very small which makes the integration problems ill-conditioned.
- There is a large number $(n(n+1)/2)$ of free model parameters in the volatility matrix which are difficult to estimate robustly for large n.

In conclusion, the pricing formula in Theorem 3.3 can only be applied to small benchmarks, although it is very useful in this case. In this Section, we aim to derive a similar pricing formula for reduced models which incorporate less processes than companies ($d < n$). This way, substantially fewer rankings have to be considered and much lower-dimensional integrals have to be computed which allows the pricing of performance-dependent options even for large benchmarks.

4.1 Geometrical view

Lemma 3.2 and thus representation (6) of the option price remains also valid in the reduced model. Note, however, that $\mathbf{A}$ is now an $(n \times d)$-matrix which prevents the direct application of Lemma 3.1. At this point, a geometrical point of view is advantageous to illustrate the effect of performance comparisons in the reduced model.

The matrix $\mathbf{A}$ and the vector $\mathbf{b}$ define a set of n hyperplanes in the space $\mathbb{R}^d$. The dissection of $\mathbb{R}^d$ into different domains or cells is called an hyperplane arrangement and denoted by $\mathcal{A} = \mathcal{A}_{n,d}$. Each cell in the hyperplane arrangement $\mathcal{A}$ is a (possibly open) polyhedron P which is uniquely represented by a ranking vector $\mathbf{R} \in \{+,-\}^n$. Each element of the ranking vector indicates on which side of the corresponding hyperplane the polyhedral cell is located. We thus have the representation of the polyhedron as the set

$$P = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \geq_{\mathbf{R}} \mathbf{b} \right\}.$$

Figure 1 illustrates two two-dimensional hyperplane arrangements, one for a full model with two assets and one for a reduced model with three assets. We see that in the reduced model fewer than the expected $2^3 = 8$ polyhedral cells arise. Indeed, it can be shown, see, e.g., [6], that the number of cells $c_{n,d}$ of the hyperplane arrangement $\mathcal{A}$ is bounded from above by

$$c_{n,d} \leq \sum_{i=0}^{d} \binom{n}{d-i}. \tag{7}$$

To illustrate this effect, note that in a full model with 30 benchmark assets, 1.1 billion cells arise while in a reduced model with 30 benchmark assets whose prices are driven by $d = 5$ underlying processes only about 170 thousand cells appear.

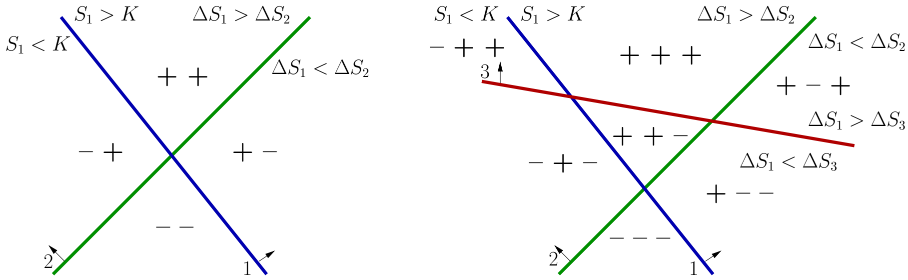

Figure 1: Polyhedral cells and ranking vectors for two hyperplane arrangements with $d = 2, n = 2$ (left) and $d = 2, n = 3$ (right).

By identifying all cells in the hyperplane arrangement, we can significantly reduce the number of integrals to be computed. This way, the representation (6) of the option price can be rewritten as

$$V(S_1(0), 0) = e^{-rT} \sum_{P \in \mathcal{A}} a_{\mathbf{R}} \int_P (S_1(T) - K)\varphi(\mathbf{x}) d\mathbf{x}. \tag{8}$$

By integrating the payoff function over each cell of the hyperplane arrangement separately, the option value can be determined as a sum over all integral values weighted with the corresponding bonus factors. Note that only smooth integrands appear in this approach.

4.2 Tools from computational geometry

Two problems remain with formula (8), however. First, it is not easy to see which ranking vectors and corresponding polyhedra appear in the hyperplane arrangement and which do not. Second, the integration region is now a general polyhedron and, therefore, involved integration rules are required. To resolve these difficulties we need some more utilities from computational geometry summarized in the following two Lemmas.

To state the first Lemma, let $\mathbf{e}_1, \ldots, \mathbf{e}_d \in \mathbb{R}^d$ denote the first unit vectors. We assume here that no row of the matrix $\mathbf{A}$ is a multiple of $\mathbf{e}_i, 1 \leq i \leq d$. Moreover, we assume that the hyperplane arrangement is non-degenerate which means that exactly d hyperplanes intersect in a unique vertex $\mathbf{v} \in \mathbb{R}^d$. In the unlikely case that these conditions are not met, they can be ensured by slightly perturbing some of entries of the volatility matrix.

The unit vectors impose an order on all vertices. A vertex $\mathbf{v}$ is said to be smaller than another vertex $\mathbf{w}$ if $v_1 < w_1$. If v_1 and w_1 happen to be equal, v_2 and w_2 are compared, and so on.

The position of each vertex can be computed by solving the corresponding $d \times d$ linear system. By computing the minimum and maximum vertex of the hyperplane arrangement in each direction, an artificial bounding box which encompasses all vertices is defined. This bounding box is only needed for the localization of the polyhedral cells in the following Lemma and does not implicate any approximation.

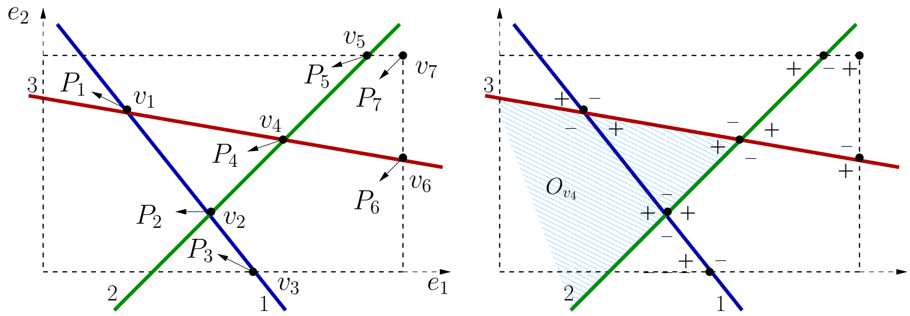

Figure 2: Illustration of the mapping between intersection points $\{\mathbf{v}_1, \dots, \mathbf{v}_7\}$ and polyhedral cells $P_j := P_{\mathbf{v}_j}$ for the right arrangement from Figure 1 (left) and corresponding reflection signs $s_{\mathbf{v},\mathbf{w}}$ as well as the orthant $O_{\mathbf{v}_4}$ (right).

Lemma 4.1 *Let the set $\mathcal{V}$ consist of all vertices of the hyperplane arrangement, of the largest intersection points of the hyperplanes with the bounding box and of the largest corner point of the bounding box. Furthermore, let $P_{\mathbf{v}} \in \mathcal{A}$ be the polyhedron which is adjacent to the vertex $\mathbf{v} \in \mathcal{V}$ and which contains no other vertex which is larger than $\mathbf{v}$. Then the mapping $\mathbf{v} \mapsto P_{\mathbf{v}}$ is one-to-one and onto.*

The proof of Lemma 4.1 can be found in our companion paper [13]. For the two dimensional example with three hyperplanes in Figure 1 the mapping between intersection points and polyhedral cells is illustrated in Figure 2 (left). Each vertex from the set $\mathcal{V} := \{\mathbf{v}_1, \dots, \mathbf{v}_7\}$ is mapped to the polyhedral cell indicated by the corresponding arrow. Using Lemma 4.1, an easy to implement optimal order $O(c_{n,d})$ algorithm which enumerates all cells in an hyperplane arrangement can be constructed.

Note that by Lemma 4.1 each vertex $\mathbf{v} \in \mathcal{V}$ corresponds to a unique cell $P_{\mathbf{v}} \in \mathcal{A}$ and thus to a ranking vector $\mathbf{R}$. We can, therefore, also assign bonus factors to vertices by setting $a_{\mathbf{v}} := a_{\mathbf{R}}$.

Next, we assign each vertex $\mathbf{v}$ an associated orthant $O_{\mathbf{v}}$. An orthant is defined as an open region in $\mathbb{R}^d$ which is bounded by at most d hyperplanes. Note that each vertex is the intersection of $0 \leq k \leq d$ hyperplanes of the hyperplane arrangement with $d-k$ boundary hyperplanes of the bounding box. To find the orthant $O_{\mathbf{v}}$ associated with the vertex $\mathbf{v}$, we determine k points which are smaller than $\mathbf{v}$ and which lie on the intersection of $d-1$ of these d hyperplanes. These points are found by solving a $d \times d$ linear system where $d-1$ equations are given by the intersecting hyperplanes and the last equation is $x_1 = v_1 - \varepsilon$. The unique orthant which contains $\mathbf{v}$ and all smaller points is denoted by $O_{\mathbf{v}}$.

For illustration, the orthant $O_{\mathbf{v}_4}$ is displayed in Figure 2 (right). Note that vertices which are located on the boundary correspond to orthants with $k < d$ intersecting hyperplanes. For example, $O_{\mathbf{v}_3}$ is defined by all points which are below hyperplane one.

By definition, there exists a $(k \times d)$-submatrix $\mathbf{A}_{\mathbf{v}}$ of $\mathbf{A}$ and a k-subvector

$\mathbf{b_v}$ of $\mathbf{b}$ such that the orthant $O_\mathbf{v}$ can be characterised as the set

$$O_\mathbf{v} = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{A_v x} \geq_\mathbf{R} \mathbf{b_v} \right\}, \tag{9}$$

where $\mathbf{R}$ is the ranking vector which corresponds to $\mathbf{v}$. This way, the submatrix $\mathbf{A_v}$ and the subvector $\mathbf{b_v}$ consist of exactly those rows of $\mathbf{A}$ and $\mathbf{b}$ whose corresponding hyperplanes intersect in $\mathbf{v}$.

Furthermore, given two vertices $\mathbf{v}, \mathbf{w} \in \mathcal{V}$, we define the reflection sign $s_{\mathbf{v},\mathbf{w}} := (-1)^{r_{\mathbf{v},\mathbf{w}}}$ where $r_{\mathbf{v},\mathbf{w}}$ is the number of reflections on hyperplanes needed to map $O_\mathbf{w}$ onto $P_\mathbf{v}$. The reflection signs $s_{\mathbf{v},\mathbf{w}}$ with $\mathbf{v} \in \{\mathbf{v}_1, \ldots, \mathbf{v}_7\}$ and $\mathbf{w} \in P_\mathbf{v}$ arising in the two dimensional arrangement in Figure 1 (right) are displayed in Figure 2 (right). For instance, the three reflection signs in the cell $P_{\mathbf{v}_4}$ are given by $s_{\mathbf{v}_4,\mathbf{v}_1} = +$, $s_{\mathbf{v}_4,\mathbf{v}_2} = -$ and $s_{\mathbf{v}_4,\mathbf{v}_4} = +$. Finally, let $\mathcal{V}_\mathbf{v}$ denote the set of all vertices of the polyhedron $P_\mathbf{v}$.

Lemma 4.2 *It is possible to algebraically decompose any cell of a hyperplane arrangement into a signed sum of orthant cells by*

$$\chi(P_\mathbf{v}) = \sum_{\mathbf{w} \in \mathcal{V}_\mathbf{v}} s_{\mathbf{v},\mathbf{w}} \chi(O_\mathbf{w})$$

where χ is the characteristic function of a set. Moreover, all cells of a hyperplane arrangement can be decomposed into a signed sum of orthants using exactly one orthant per cell.

The first part of Lemma 4.2 is originally due to Lawrence [21]. The second part follows from the one-to-one correspondence between orthants $O_\mathbf{v}$ and cells $P_\mathbf{v}$. It can be found in detail in the companion paper [13].

Note that such an orthant decomposition is not unique. A different decomposition of a polyhedron into a sum of orthants is, e.g., presented in [7].

Example 4.3 *To give an example, the decomposition of all cells within the hyperplane arrangement from Figure 2 is given by*

$$\begin{aligned}
\chi(P_1) &= \chi(O_1) \\
\chi(P_2) &= \chi(O_2) - \chi(O_1) \\
\chi(P_3) &= \chi(O_3) - \chi(O_2) \\
\chi(P_4) &= \chi(O_4) - \chi(O_2) + \chi(O_1) \\
\chi(P_5) &= \chi(O_5) - \chi(O_4) - \chi(O_1) \\
\chi(P_6) &= \chi(O_6) - \chi(O_4) - \chi(O_3) + \chi(O_2) \\
\chi(P_7) &= \chi(O_7) - \chi(O_6) - \chi(O_5) + \chi(O_4)
\end{aligned}$$

where we used the abbreviations $P_j := P_{\mathbf{v}_j}$ and $O_j := O_{\mathbf{v}_j}$.

4.3 Pricing formula

Now, we are finally able to give a pricing formula for performance-dependent options also in the reduced model case.

Theorem 4.4 *The price of a performance-dependent option with payoff (1) is for the model (2) in the case $d \leq n$ given by*

$$V(S_1(0), 0) = \sum_{\mathbf{v} \in \mathcal{V}} c_\mathbf{v} \left(S_1(0) \Phi(\mathbf{A_v A_v}^T, -\mathbf{d_v}) - e^{-rT} K \Phi(\mathbf{A_v A_v}^T, -\mathbf{b_v}) \right)$$

with $\mathbf{A_v}, \mathbf{b_v}$ *as in (9) and* $\mathbf{d_v}$ *being the corresponding subvector of* $\mathbf{d}$*. The weights* $c_\mathbf{v}$ *are given by*

$$c_\mathbf{v} := \sum_{\mathbf{w}\in\mathcal{V}:\ \mathbf{v}\in P_\mathbf{w}} s_{\mathbf{v},\mathbf{w}} a_\mathbf{w}. \tag{10}$$

Proof: By Lemma 4.1 we see that the integral representation (8) is equivalent to a summation over all vertices $\mathbf{v} \in \mathcal{V}$, i.e.

$$V(S_1(0), 0) = e^{-rT} \sum_{\mathbf{v}\in\mathcal{V}} a_\mathbf{v} \int_{P_\mathbf{v}} (S_1(T) - K)\varphi(\mathbf{x})d\mathbf{x}.$$

By Lemma 4.2 we can decompose the polyhedron $P_\mathbf{v}$ into a signed sum of orthants and obtain

$$V(S_1(0), 0) = e^{-rT} \sum_{\mathbf{v}\in\mathcal{V}} a_\mathbf{v} \sum_{\mathbf{w}\in\mathcal{V}_\mathbf{v}} s_{\mathbf{v},\mathbf{w}} \int_{O_\mathbf{w}} (S_1(T) - K)\varphi(\mathbf{x})d\mathbf{x}.$$

By the second part of Lemma 4.2 we know that only $c_{n,d}$ different integrals appear in the above sum. Rearranging the terms leads to

$$V(S_1(0), 0) = e^{-rT} \sum_{\mathbf{v}\in\mathcal{V}} c_\mathbf{v} \int_{O_\mathbf{v}} (S_1(T) - K)\varphi(\mathbf{x})d\mathbf{x}.$$

Since now the integration domains $O_\mathbf{v}$ are orthants, Lemma 3.1 can be applied exactly as in the proof of Theorem 3.3 which finally implies the Theorem. □

To compute the weights $c_\mathbf{v}$, all cells $P_\mathbf{w}$ incident in $\mathbf{v}$ have to be traversed and their ranking vectors have to be be determined. This can be done symbolically by flipping the signs in the ranking vector of $P_\mathbf{v}$ which correspond to the hyperplanes intersecting in $\mathbf{v}$. By the non-degeneracy condition there are at most 2^d cells adjacent to each vertex which bounds the number of terms in the definition of $c_\mathbf{v}$. Moreover, the number of vertices in $\mathcal{V}$ equals $c_{n,d}$ which yields the number of integrals which have to be computed in the worst case. The structure of the valuation algorithm is summarized in Figure 3.

Example 4.5 *Consider the bonus scheme from Example 2.2 with* $n = 3$*,* $d = 2$ *and the hyperplane arrangement from Figure 2. Then, the bonus factors* $a_j := a_{\mathbf{v}_j}$ *are given by*

$$a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = \frac{1}{2},\ a_5 = 1,\ a_6 = 0,\ a_7 = \frac{1}{2}.$$

Following the steps in the proof of Theorem 4.4 and employing the decomposition from Example 4.3 we see that the price of this option satisfies

$$\begin{aligned} V(S_1(0), 0) &= e^{-rT} \left(\frac{1}{2}\, I(P_4) + I(P_5) + \frac{1}{2}\, I(P_7)\right) \\ &= e^{-rT} \left(-\frac{1}{2}\, I(O_1) - \frac{1}{2}\, I(O_2) + \frac{1}{2}\, I(O_5) - \frac{1}{2}\, I(O_6) + \frac{1}{2}\, I(O_7)\right) \end{aligned}$$

where we define

$$I(B) := \int_B (S_1(T) - K)\varphi(\mathbf{x})d\mathbf{x}.$$

1) to compute the set of all intersection points $\mathcal{V}$:
   a) compute the set of vertices of the hyperplane arrangement
   b) compute the bounding box of these vertices
   c) compute the set of boundary intersection points
2) for each intersection point $\mathbf{v} \in \mathcal{V}$:
   a) determine the submatrix $\mathbf{A_v}$ and the subvectors $\mathbf{d_v}$ and $\mathbf{b_v}$
   b) evaluate the cumulative normal distributions $\Phi(\mathbf{A_v}\mathbf{A_v}^T, -\mathbf{d_v})$ and $\Phi(\mathbf{A_v}\mathbf{A_v}^T, -\mathbf{b_v})$
   c) for all vertices $\mathbf{w} \in \mathcal{V}$ whose polyhedra $P_\mathbf{w}$ contain $\mathbf{v}$: determine the reflection signs $s_{\mathbf{v},\mathbf{w}}$ and bonus factors $a_\mathbf{w}$
   d) compute the weight $c_\mathbf{v}$ using formula (10)
3) compute the price of the option as the weighted sum over all normal distributions according to Theorem 4.4.

Figure 3: Valuation algorithm for performance-dependent options in the reduced model case.

4.4 Special cases

Let us first remark that, if the payoff function has a special structure, many weights $c_\mathbf{v}$ are zero in the formula from Theorem 4.4. This way, the corresponding normal distributions do not have to be computed. This is, for example, true for the outperformance option of Example 2.3.

In addition, if the vertex $\mathbf{v}$ is located on the artificial boundary, see for example vertex $\mathbf{v}_3$ in Figure 2, the corresponding orthant is defined by $k < d$ intersecting hyperplanes. As a consequence, only a k-dimensional normal distribution instead of a d-dimensional one has to be computed. Consider, for example, a bonus scheme which is defined by the bonus factors

$$a_\mathbf{R} = \begin{cases} \sum\limits_{\{i:R_i=+\}} \bar{a}_i & \text{if } \mathbf{R}_1 = + \\ 0 & else \end{cases} \tag{11}$$

for some given $\bar{a}_i \in \mathbb{R}$, where the sum goes over all $i \in \{2,\ldots,n\}$ where $R_i = +$. Example 2.2 is a special case of such a scheme with $\bar{a}_i \equiv 1/(n-1)$. The pricing formula for such a scheme only contains vertices which are located on at least $d-2$ boundary hyperplanes. Thus, independently of d and n, at most two-dimensional normal distributions have to be evaluated. Moreover, the number of two-dimensional normal distributions is bounded by $n-1$. This behaviour is most easily understood if the payoff function of the bonus scheme (11) is rewritten in the equivalent form

$$V(\mathbf{S},T) = \sum_{i=2}^{n} \bar{a}_i\,(S_1(T)-K)^+\,\chi_{\Delta S_1(T)\geq \Delta S_i(T)}$$

which shows that only the two-dimensional joint distributions of the random variables $S_1(T)$ and $S_i(T)$ are required for $i = 2,\ldots,n$.

Note that these special cases are automatically recognized by our algorithm and only the minimum number of integrals with the corresponding minimal dimensions are computed.

5 Quadrature

The efficient application of the formulas from Theorem 3.3 and 4.4 crucially depends on the availability of accurate and fast numerical methods for the evaluation of multivariate normal probabilities. For small dimensions $d \leq 3$ there is reliable and efficient software, see e.g. [5, 10, 25]. For larger dimensions, standard multivariate numerical integration software, like ADAPT [11] or DCUHRE [2], can be applied but their accuracy usually suffers from the fact that the infinite integration limits need to be transformed or cut off. Moreover, they do not take advantage of the special form of the integrand.

Instead, Genz [9] proposed a simple sequence of transformations of the multivariate normal distribution function which reduces the dimension by one and places the problem to the unit square. One obtains

$$\Phi(\mathbf{A}, \mathbf{b}) = e_1 \int_{[0,1]^{d-1}} \prod_{i=2}^{d} e_i(w_1, \ldots, w_{i-1}) \, d\mathbf{w} \tag{12}$$

with

$$e_i(w_1, \ldots, w_{i-1}) = \Phi((b_i - \sum_{j=1}^{i-1} c_{ij} \Phi^{-1}(w_j \, e_j))/c_{ii})$$

where $\Phi(x)$ denotes the standard univariate normal distribution function and c_{ij} the entries of the Cholesky decomposition $\mathbf{CC}^T$ of the matrix $\mathbf{A}$.

This way, the convergence of standard numerical integration software can be significantly accelerated. Usually, the computation time can be further reduced if the variables are reordered such that the variables associated with the largest integration intervals are the innermost variables. The standard univariate normal distribution function and its inverse can efficiently and up to high accuracy be computed by a Moro [23] scheme.

For the computation of the integral in (12), standard deterministic integration methods such as quasi-Monte Carlo methods, product or sparse grid integration can be used. Quasi-Monte Carlo methods in the context of problems from mathematical finance are discussed in detail in Acworth *et al.* [1] and Glasserman [14]. More information about product integration can be found in Davis and Rabinowitz [4]. The sparse grid approach is based on Gerstner and Griebel [12].

6 Numerical Results

In this Section we present numerical examples to illustrate the performance of our approach to price performance-dependent options using Theorem 4.4. In particular, we compare the efficiency of our algorithm to the direct pricing approach of (quasi-)Monte Carlo simulation of the expected payoff (4).

We consider a reduced Black-Scholes market with $n = 30$ assets and $d = 5$ processes. This setting corresponds, e.g., to the case of a performance-dependent option which includes the performance of all companies of the German stock index DAX in its payoff profile. We investigate the four different choices according to the Examples 2.1 – 2.4 from Section 2 for the bonus factors $a_{\mathbf{R}}$ in the payoff function (1). Throughout this Section we use the following model parameters:

Example	$V(S_1(0), 0)$	Discount	$\sharp$ Int	Dim	STD	QMC	P	SG
2.1	14.4995	-	1	1	1.1	-	-	-
2.2	12.9115	10.95%	41	2	0.58	0.88	1.45	1.55
2.3	1.8774	87.05%	31	5	0.6	1.1	0.27	1.87
2.4	8.6024	40.67%	38	3	0.52	1.3	0.89	1.54

Table 1: Option prices, discounts compared to the corresponding plain vanilla option, intrinsic dimensions and convergence rates of the different numerical approaches for the considered examples.

$K = 100$, $S_1(0) = 100$, $T = 1$, $r = 5\%$ and σ being a 30×5 volatility matrix whose entries are uniformly distributed in $[-1/d, 1/d]$. The computations were performed on a dual Intel(R) Xeon(TM) CPU 3.06GHz workstation.

In the performance-independent case of Example 2.1, an analytical solution is readily obtained by the Black-Scholes formula. In all other cases, we computed reference values for the option price on a very fine integration grid as a benchmark value to compare the efficiency of the different pricing approaches.

The prices of the performance-dependent options from the Examples 2.1–2.4 are displayed in the second column of Table 1. In principle, all bonus schemes described above could be hedged by the plain vanilla option in Example 2.1. The differences of the prices of the performance-dependent options (yielding the accurate value) and the corresponding plain vanilla options are displayed in the third column of Table 1. We see that the usage of plain vanilla options substantially (up to 87 %) overestimates the fair values of the bonus schemes. As explained in Section 4.4, the complexity and dimensionality of our formula is often substantially reduced depending on the choice of the bonus factors. The number ($\sharp$ Int) and the maximum dimension (Dim) of normal distributions which have to be computed in the Examples 2.1–2.4 are displayed in the fourth and fifth column of Table 1. One can see that the number of required normal distributions is substantially lower than the theoretical bound (7) which is $174,437$ for these examples. The maximum dimension varies from one to the nominal dimension five depending on the specific example.

In the last four columns of Table 1, the estimated asymptotic convergence rates are listed for four different schemes. In the standard approach denoted by STD, we used quasi-Monte Carlo integration to simulate the expected payoff (4). In the other three cases, the option prices were computed with the formula from Theorem 4.4. For the approximation of the normal distributions, we used the following integration schemes from Section 5:

- Quasi-Monte Carlo integration based on Sobol point sets (QMC),
- Product integration based on the Clenshaw-Curtis rule (P),
- Sparse grid integration based on the Clenshaw-Curtis rule (SG).

The convergence behaviour of the four different approaches STD, QMC, P and SG to price the performance-dependent options from the Examples 2.2 – 2.4 are displayed in Figure 4. There, the time is displayed which is needed to obtain a given accuracy. In the special case of Example 2.1, the application of Theorem 4.4 combined with the transformation (12) automatically reduces to the analytical solution given by the Black-Scholes formula with variance $\bar{\sigma}_1$. The exact solution up to machine precision is obtained in about 4.7 seconds by all integration schemes (QMC, P, SG). This is the time which is needed in the setup step of our algorithm to compute all vertices $\mathbf{v}$ and all weights $c_{\mathbf{v}}$. In the same time, the STD approach approximates the solution up to an error of $1e-03$. One can see that a simulation of the expected payoff (STD) performs similarly in all examples. Low accuracies are quickly achieved, the convergence rate is slow, though. The rate is about 0.6 in all examples and thus lower than one, as may be expected. The integration scheme suffers under the irregularity of the integrand which is highly discontinuous and not of bounded variation. The QMC scheme clearly outperforms the STD approach in all examples. It exhibits a convergence rate of about one and leads to much smaller errors after the setup time of 4.7 seconds. In contrast to the two previous approaches, the product integration approach (P) exhibits a high dependency on the specific example. While it performs very well in the Examples 2.1 and 2.2 it only converges with a rate of 0.27 in Example 2.3. Here, the curse of dimension, under which the product approach suffers, is clearly visible. While the intrinsic dimensions of Examples 2.1 and 2.2 are only one and two, respectively, the intrinsic dimension of Example 2.3 is five and, thus, equal to the nominal dimension. The combination of sparse grid integration with our pricing formula (SG) leads to the best convergence rates. The curse of dimension can be broken to some extent, while the favorable accuracy of the product approach is maintained. It is the most efficient scheme for the Examples 2.1, 2.2 and 2.4. However, for higher dimensional problems as Example 2.3, this advantage is only visible if very accurate solutions are required. In the preasymptotic regime, the QMC scheme leads to smaller errors.

7 Conclusions

In this paper, we presented several approaches for the valuation of performance-dependent options in a Black-Scholes framework. The price of a such an option depends on the joint distribution of all stock prices in the benchmark. Thus, its valuation must be regarded as a high-dimensional integration problem.

As an alternative to a direct integration of the payoff we presented two analytical pricing formulas which involve the evaluation of several cumulative normal distribution functions. The pricing formula for the full model is useful in case of small benchmarks. It suffers, however, under a very high complexity and dimensionality if a larger number of benchmark companies are considered. Using novel tools from computational geometry we derived a more general formula for reduced models which incorporate less stochastic processes than companies and can be used for larger benchmarks as well.

In numerical examples we demonstrated for different typical bonus schemes that our pricing approach outperforms standard methods even for large benchmarks which may include as much as $n = 30$ companies and $d = 5$ stochastic processes. Thereby several deterministic integration methods were compared regarding their efficiency. Furthermore, for specific bonus schemes (11) we showed that, independently of n and d, the pricing problem can be analytically reduced to a sum of two-dimensional normal distributions.

An additional advantage of our approach compared to standard Monte Carlo pricing is given by the fact that option price sensitivities can be obtained by analytical differentiation of the pricing formulas. The computation of the Greek letters can thus be integrated in the valuation algorithm without much additional effort.

Let us finally remark that we restricted ourselves to payoff profiles which depend on relative performance comparisons to a specific benchmark. Payoff profiles which include absolute performance criteria, e.g., performance comparisons with different strike prices, can also be included. The corresponding valuation formulas then include weighted sums of gap option prices.

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References

[1] P. Acworth, M. Broadie, and P. Glasserman. A comparison of some Monte Carlo and Quasi-Monte Carlo methods for option pricing. In *Monte Carlo and Quasi-Monte Carlo Methods, H. Niederreiter et al., (eds.)*, pages 1–18. Springer, 1998.

[2] J. Bernsten, T. Espelid, and A. Genz. Algorithm 698: DCUHRE – an adaptive multidimensional integration routine for a vector of integrals. *ACM Transactions on Mathematical Software*, 17(452-456), 1991.

[3] R. Carmona and V. Durrleman. Generalizing the Black-Scholes formula to multivariate contingent claims. *J. Computational Finance*, 9(2), 2006.

[4] P. Davis and P. Rabinowitz. *Methods of Numerical Integration*. Academic Press, 1975.

[5] Z. Drezner and G. O. Wesolowsky. On the computation of the bivariate normal integral. *Journal of Statistical Computation and Simulation*, 35:101–107, 1990.

[6] H. Edelsbrunner. *Algorithms in Combinatorial Geometry*. Springer, 1987.

[7] H. Edelsbrunner. Algebraic decomposition of non-convex polyhedra. In *Proc. 36th Ann. IEEE Sympos. Found. Comput. Sci.*, pages 248–257. 1995.

[8] D. Egloff, W. Farkas, and M. Leippold. American options with stopping time constraints. *submitted to Mathematical Finance, available at SSRN: http://ssrn.com/abstract=798124*, 2005.

[9] A. Genz. Numerical computation of multivariate normal probabilities. *J. Comput. Graph. Statist.*, 1:141–150, 1992.

[10] A. Genz. Numerical computation of rectangular bivariate and trivariate normal and t probabilities. *Statistics and Computing*, 14:151–160, 2004.

[11] A. Genz and A. Malik. An adaptive algorithm for numerical integration over an n-dimensional rectangular region. *J. Comput. Appl. Math.*, 6:295–302, 1980.

[12] T. Gerstner and M. Griebel. Numerical integration using sparse grids. *Numerical Algorithms*, 18:209–232, 1998.

[13] T. Gerstner and M. Holtz. Algorithms for the cell enumeration and orthant decomposition of hyperplane arrangements. Working paper, Institute for Numerical Simulation, University of Bonn, 2006.

[14] P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer, 2003.

[15] J. Harrison and S. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.*, 11:215–260, 1981.

[16] J. Hull and A. White. Accounting for employee stock options: A practical approach to handling the valuation issues. *Journal of Derivatives Accounting*, 1(1):3–9, 2004.

[17] J. Hull and A. White. How to value employee stock options. *Financial Analysts Journal*, 60(1):114–119, 2004.

[18] I. Karatzas. *Lectures on the Mathematics of Finance*, volume 8 of *CRM Monograph Series*. American Mathematical Society, Providence, R.I., 1997.

[19] R. Korn. A valuation approach for tailored options. Working paper, Department of Mathematics, University of Kaiserslautern, 1996.

[20] L. Laloux, P. Cizeau, J. Bouchaud, and M. Potters. Noise dressing of financial correlation matrices. *Phys. Rev. Lett.*, 83(7):1467–1470, 1999.

[21] J. Lawrence. Polytope volume computation. *Math. Comp.*, 57(195):259–271, 1991.

[22] N. Meade and G. Salkin. Index funds – construction and performance measurement. *J. Operational Research Society*, 40(10):871–879, 1989.

[23] B. Moro. The full monte. *RISK*, 8(2), 1995.

[24] S. Rodrick (ed.). *Incentive Compensation and Employee Ownership*. National Center for Employee Ownership, 5th edition, 2004.

[25] M. Schervish. Multivariate normal probabilities with error bound. *Applied Statistics*, 33:81–87, 1984.

[26] P. Zhang. *Exotic Options*. World Scientific Publishing, 1998.

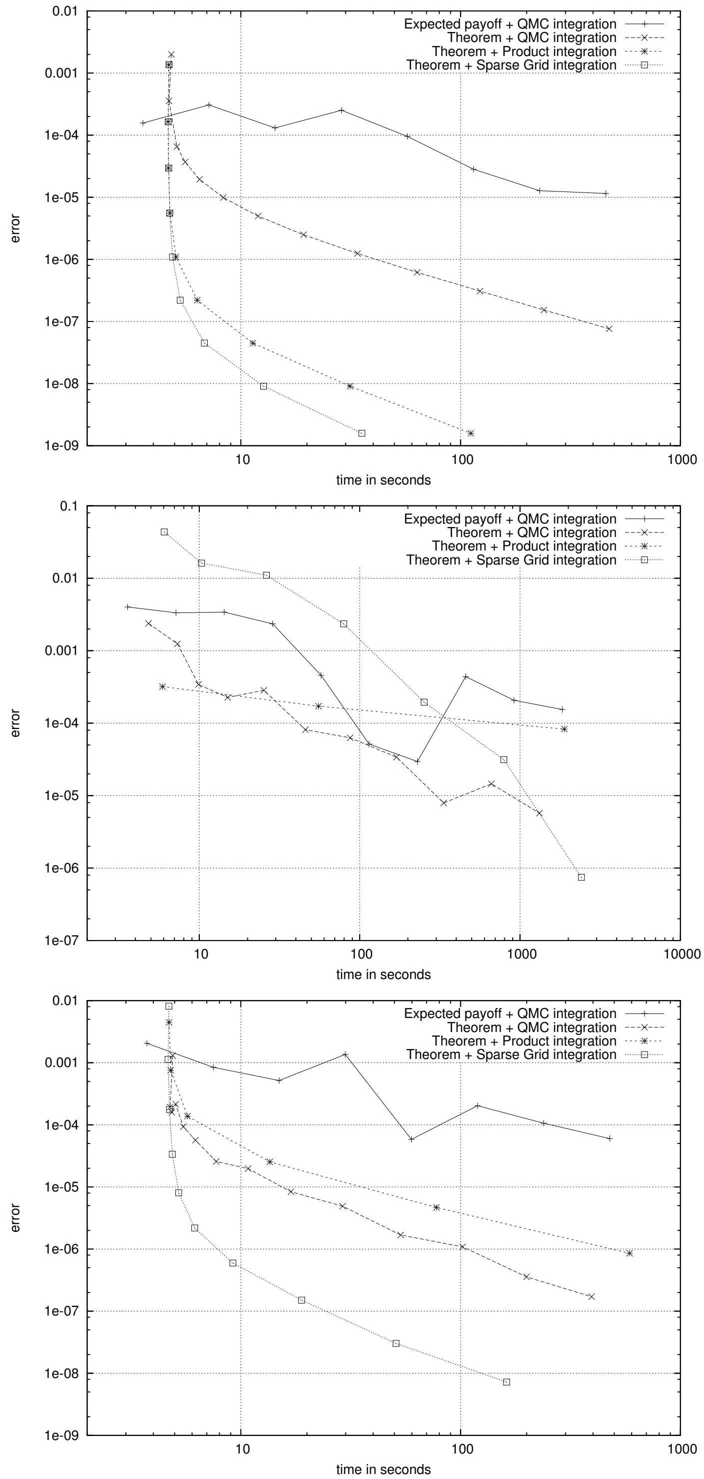

Figure 4: Errors and timings of the different numerical approaches to price the performance-dependent options of Examples 2.2 (top), 2.3 (middle) and 2.4 (bottom).