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ROBUST NORM EQUIVALENCIES AND OPTIMAL PRECONDITIONERS FOR DIFFUSION PROBLEMS

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ABSTRACT. Additive multilevel methods offer an efficient way for the fast solution of large sparse linear systems which arise from a finite element discretization of an elliptic boundary value problem. These solution methods are based on norm equivalencies for the associated bilinear form using a suitable subspace decomposition. From these, efficient preconditioners for the stiffness matrix can be derived. For the robustness of the resulting iteration schemes, it is crucial that the constants in the norm equivalence do not depend or depend only weakly on the ellipticity constants of the problem. Here we address this question for the model problem $-\nabla\omega\nabla u = f$ with a scalar weight ω.

We prove an upper bound completely independent of the weight ω, whereas our lower bound involves some information about the local variation the coefficient function. The condition on the coefficient is related to the A_2 and A_∞ Muckenhoupt classes from approximation theory. It is sufficient to require $\omega \in A_2 \subset A_\infty$, however, the results even hold for a slightly larger class of functions.

1. INTRODUCTION

The solution of large sparse linear systems arising from the discretization of a partial differential equation (PDE) is an essential ingredient in many scientific computations. The ever growing demand for efficient solvers led to the development of multigrid methods in the 1970s [4, 5, 13, 14] and multilevel preconditioning techniques in the 1980s [22]. Much research work was devoted to the question of optimal complexity; i.e., to show that the number of operations necessary to obtain the solution up to a prescribed accuracy is proportional to the number of unknowns of the linear system. However, the convergence behavior of these classical schemes is still strongly dependent on the coefficients of the considered PDE. This is the so-called robustness problem of multilevel solvers. It is one major reason which somewhat limits the applicability of classical multigrid methods and multilevel preconditioners to real world problems. Several extensions of these methods e.g. via the use of more complicated smoothing schemes or through the use of operator-dependent or matrix-dependent transfer operators [1, 10, 24, 25] in the so-called black-box multigrid method have been proposed over the years to overcome the robustness problem. Currently the most successful approach is the algebraic multigrid (AMG) method [6, 7, 8, 9, 12, 16, 19] which further generalizes the black-box multigrid idea.

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However, the design and implementation of an AMG method is rather involved and there is no satisfying theoretical foundation which proves the robustness of AMG. Hence, the development of a provably robust multilevel solver or preconditioner for general second order PDEs remains an open problem even today.

In this paper we focus on the robustness issue for diffusion problems in two space dimensions involving a scalar diffusion coefficient $\omega : \mathbb{R}^2 \to \mathbb{R}$; i.e., we consider the model problem

$$-\nabla\omega\nabla u = f \text{ in } \Omega \subset \mathbb{R}^2.$$

We discretize our model problem using linear finite elements on a sequence of uniformly refined triangulations. Furthermore, we only require the weight ω to be in the Muckenhoupt class $A_2 \subset A_\infty$; in fact, the results hold even for a slightly larger class of functions.

The main ingredients of our analysis are the use of certain weighted projection operators and auxiliary level-dependent elliptic projection operators. The tools employed in the proof are the Hardy-inequality, a hybrid Bernstein-type inequality and a generalized duality technique. We limit ourselves to the two-dimensional case using linear finite elements only, however, we believe that the results can be extended to three space dimensions and multi-linear elements.

In this paper we establish the equivalence

$$a(u,u) \leq \tilde{C} \sum_{j=0}^{J} 2^{2j} \|u_j\|_{j,\omega}^2 \leq \hat{C}\; a(u,u)$$

for a certain weighted norm where the constants $\tilde{C}$ and $\hat{C}$ depend on the initial triangulation and in general involve some information about the *local* variation of the coefficient function ω. For the standard test case of a piecewise constant coefficient function with maximal jump of height ϵ^{-1}, we obtain an optimal norm equivalence with $\hat{C} \approx \ln(\epsilon^{-1/2})$. Note that the location of the jumps, their number or their frequency is *not* restricted. Hence, unlike other articles concerned with the development of robust solvers [11, 15, 20] we do *not* require that the jumps must be resolved on any particular level.

The remainder of the paper is organized as follows: First, we introduce the notation and the employed projection operators in §2. Then, we present the main result of the paper in §3. Here, we establish two robust norm equivalencies for the considered model problem using a linear finite element discretization on a sequence of uniformly refined triangulations. We begin with the derivation of an optimal and robust upper bound for the bilinear form using a hybrid Bernstein-type estimate and a Hardy-inequality. Then, we establish a lower bound for the bilinear form under a weak assumption on the coefficient function ω using a duality technique and a Hardy-inequality. Finally, we discuss the conditions imposed on the coefficient functions and show that all weights ω in the Muckenhoupt class $A_2 \subset A_\infty$ fulfill the requirements. In §4 we present the construction of a preconditioner of BPX-type based on our robust and optimal norm equivalence before we conclude with some remarks in §5.

2. Prerequisites

Let us introduce some notation which we will use throughout this paper. We will consider a sequence of uniformly refined triangulations

$$\mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_J \tag{2.1}$$

and the associated sequence of piecewise linear finite element spaces

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_J. \tag{2.2}$$

Let $\mathcal{N}_j$ denote the set of all vertices of the triangulation $\mathcal{T}_j$ and define

$$\mathcal{R}_0 := \mathcal{N}_0, \qquad \mathcal{R}_j := \mathcal{N}_j \setminus \mathcal{N}_{j-1}, \quad \text{and } \mathcal{N} = \mathcal{N}_J. \tag{2.3}$$

Our main interest is the development of robust multilevel solvers for diffusion problems

$$-\nabla A \nabla u = f \text{ in } \Omega \subset \mathbb{R}^2, \tag{2.4}$$

with $A := (a_{\alpha\beta})$, $a_{\alpha\beta} \in L_\infty(\Omega)$ and associated bilinear form

$$a(u,v) := \int_\Omega (A\nabla u, \nabla v) = \int_\Omega \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} a_{\alpha\beta} \partial_\beta u \partial_\alpha v. \tag{2.5}$$

Let $\|\cdot\|_a$ denote the energy norm with respect to the bilinear form $a(\cdot,\cdot)$, i.e.,

$$\|u\|_a := \sqrt{a(u,u)}.$$

The fundamental prerequisite stated in Assumption 1 for our proof is related to the discretization of (2.5) on the finest level J.

Assumption 1. *The* finest *triangulation $\mathcal{T}_J$ consists of triangles T for which the bilinear form* (2.5) *satisfies the local ellipticity condition*

$$\underline{\omega}_{T,J} \sum_{\alpha=1}^{2} \xi_\alpha^2 \le \sum_{\alpha,\beta=1}^{2} a_{\alpha\beta}(x) \xi_\alpha \xi_\beta \le \overline{\omega}_{T,J} \sum_{\alpha=1}^{2} \xi_\alpha^2, \qquad \text{for all} \quad x \in T \in \mathcal{T}_J \tag{2.6}$$

with weights $\underline{\omega}_{T,J}$, $\overline{\omega}_{T,J}$ independent of $x \in T \in \mathcal{T}_J$.

Since we employ a linear finite element space $\mathcal{V}_J$, the derivatives of $v \in \mathcal{V}_J$ are constant on any element $T \in \mathcal{T}_J$, i.e., $\nabla v = \text{const}$, and Assumption 1 yields the equivalence

$$\sum_{T \in \mathcal{T}_J} \underline{\omega}_T \, \mu(T) \|\nabla v\|_T^2 \le a(v,v) \le \sum_{T \in \mathcal{T}_J} \overline{\omega}_T \, \mu(T) \|\nabla v\|_T^2 \tag{2.7}$$

for any $v \in \mathcal{V}$ where $\mu(T)$ denotes the area of T and $\|\cdot\|_T$ the Euclidean norm on $T \subset \mathbb{R}^2$.

For any coarser element $\hat{T} \in \mathcal{T}_j$ with $j < J$ let us now introduce average weights $\underline{\omega}_{\hat{T}}$ and $\overline{\omega}_{\hat{T}}$ based on the weights $\underline{\omega}_{T,J}$ and $\overline{\omega}_{T,J}$ from (2.6) which are assigned to the elements $T \in \mathcal{T}_J$ on the finest level J; i.e., we define

$$\overline{\omega}_{\hat{T}} := \frac{1}{\mu(\hat{T})} \sum_{\substack{T \in \mathcal{T}_J \\ T \subset \hat{T}}} \mu(T) \overline{\omega}_{T,J} \qquad \text{and} \qquad \underline{\omega}_{\hat{T}} := \frac{1}{\mu(\hat{T})} \sum_{\substack{T \in \mathcal{T}_J \\ T \subset \hat{T}}} \mu(T) \underline{\omega}_{T,J}. \tag{2.8}$$

Note that for elements $T \in \mathcal{T}_J$ on the finest level J we have $\underline{\omega}_T = \underline{\omega}_{T,J}$ and $\overline{\omega}_T = \overline{\omega}_{T,J}$. Let us further introduce associated weighted norms on all levels j

$$\|v\|^2_{j,\overline{\omega}} := \sum_{T\in\mathcal{T}_j} \overline{\omega}_T \int_T |v|^2 \quad \text{and} \quad \|v\|^2_{j,\underline{\omega}} := \sum_{T\in\mathcal{T}_j} \underline{\omega}_T \int_T |v|^2. \tag{2.9}$$

Now the fact that ∇v is constant on $T \in \mathcal{T}_j$ implies that (2.7) is equivalent to

$$\|\nabla v\|^2_{j,\underline{\omega}} \le a(v,v) \le \|\nabla v\|^2_{j,\overline{\omega}}, \qquad \text{for} \quad j = 0,1,\dots,J. \tag{2.10}$$

In later sections we will work only with weights $\omega_{T,J} = \underline{\omega}_{T,J} = \overline{\omega}_{T,J}$ for the ease of notation. Hence, we are essentially dealing with the weighted norms $\|\cdot\|_{j,\omega} = \|\cdot\|_{j,\overline{\omega}} = \|\cdot\|_{j,\underline{\omega}}$ from (2.9). Throughout the paper we also use the short hand notation

$$\langle u,v\rangle_\omega := \sum_{T\in\mathcal{T}_J} \omega_T \int_T uv, \qquad \|u\|_\omega := \sqrt{\langle u,u\rangle_\omega} = \|u\|_{J,\omega} \tag{2.11}$$

for the weighted norm on the finest level J.

Note that the considered $\omega_{T,J} = \underline{\omega}_{T,J} = \overline{\omega}_{T,J}$ case corresponds to the bilinear form $a(\cdot,\cdot)$ with coefficient functions $a_{\alpha\beta} = \omega\delta_{\alpha\beta}$, i.e.

$$a(u,v) := \int_\Omega \omega(\nabla u, \nabla v), \tag{2.12}$$

and a piecewise constant approximation

$$\omega_{T,J} := \frac{1}{\mu(T)} \int_T \omega$$

of the scalar coefficient ω with respect to the finest triangulation $\mathcal{T}_J$; i.e., we consider the bilinear form

$$a_J(u,v) = \sum_{T\in\mathcal{T}_J} \omega_T \int_T (\nabla u, \nabla v)$$

on the finest level J.

Remark 2.1. For the development of a preconditioner it is essential to employ identical operators and norms for both bounds, the lower and the upper bound. Therefore, it is necessary to consider the case $\omega_{T,J} = \underline{\omega}_{T,J} = \overline{\omega}_{T,J}$. Note, however, that respective upper and lower bounds can be obtained in terms of the weights $\underline{\omega}_T$ and $\overline{\omega}_T$ respectively.

For the weighted norm $\|\cdot\|_{J,\omega} = \|\cdot\|_\omega$ there holds a local Bernstein-type inequality under a rather weak additional condition on the weights $\omega_{T,J}$, see [17] for details.

Lemma 2.2. *Consider subsets $E \subset \mathcal{T}_J$ of elements on the finest level J which are contained in an element T on a coarser level j, i.e. $E \subset T \in \mathcal{T}_j$, such that*

$$\frac{\omega_E}{\omega_T} \le \frac{1}{2}\frac{\mu(T)}{\mu(E)} \quad \textit{and} \quad \frac{\mu(E)}{\mu(T)} \le \gamma \tag{2.13}$$

hold for some constant γ independent of E and T where the weight ω_E is defined according to (2.8)*. Then there holds the local Bernstein-type inequality*

$$\|\nabla v\|^2_{\omega,T} \le 4\gamma^{-2} C_0^2 (\operatorname{diam}(T))^{-2} \|v\|^2_{\omega,T} \tag{2.14}$$

for all $v \in \mathcal{V}_j$. *Here* $\|\cdot\|_{\omega,T}$ *denotes the restriction of the norm* $\|\cdot\|_\omega$ *to* $T \in \mathcal{T}_j$ *and the constant* C_0 *is only dependent on the initial triangulation* $\mathcal{T}_0$, *i.e.,*

$$C_0 := \max_{\hat{T}\in\mathcal{T}_0} \operatorname{diam}(\hat{T}) \|H_{\hat{T}}^{-1}\|$$

where H_T *denotes the mapping from* T *to the reference triangle* T_{ref}.

Proof. Here, we only give a sketch of the proof, see [17] for details. Let H_T denote the mapping from T to the reference triangle T_{ref} with vertices $(0,0)$, $(0,1)$, $(1,0)$. For any $v \in \mathcal{V}_j$ we have the representation

$$v(x) = q(\xi) = (g, \xi) + e \quad \text{for all } x \in T, H_T(x) = \xi \in T_{\text{ref}}$$

and by definition

$$\|v\|_{\omega,T}^2 = \sum_{\substack{\hat{T}\in\mathcal{T}_J\\ \hat{T}\subset T}} \omega_{\hat{T}}\, \mu(\hat{T}) |(g, \eta_{\hat{T}}) + e|^2 \quad \text{and} \quad |\nabla v(x)|^2 \le C_0^2 (\operatorname{diam}(T))^{-2} \|g\|^2$$

for all $x \in T$, where $\eta_{\hat{T}}$ denotes the reference point in T_{ref} for which $H_T(y_{\hat{T}}) = \eta_{\hat{T}}$ for suitable $y_{\hat{T}} \in \hat{T}$. Hence, it suffices to prove the inequality

$$\mu(T)\omega_T = \sum_{\substack{\hat{T}\in\mathcal{T}_J\\ \hat{T}\subset T}} \omega_{\hat{T}}\, \mu(\hat{T}) \le c \sum_{\substack{\hat{T}\in\mathcal{T}_J\\ \hat{T}\subset T}} \omega_{\hat{T}}\, \mu(\hat{T}) \frac{|(g, \xi_{\hat{T}})|^2}{\|g\|}. \tag{2.15}$$

We first consider the case $e = 0$ and set $b := \frac{g}{\|g\|}$. Then we can find a sub-region $S_{\text{ref},\delta} \subset T_{\text{ref}}$ for any $\delta \in (0,1)$ such that

$$S_{\text{ref},\delta} \subset S_{\text{ref},\delta}^b := \{\xi \in T_{\text{ref}} : |(b,\xi)| \ge \delta\}$$

holds for all b with $\|b\| = 1$. For the set $E_{\text{ref}} := T_{\text{ref}} \setminus S_{\text{ref},\delta}$ one can then show that $\mu(E_{\text{ref}}) \lessapprox \mu(T_{\text{ref}})$. It is clear that the most significant contribution to the validity of (2.15) is due to the points $\eta_{\hat{T}}$ which are located in $S_{\text{ref},\delta}$. With this in mind and the estimate

$$\sum_{\substack{\hat{T}\in\mathcal{T}_J\\ \hat{T}\subset T}} \omega_{\hat{T}}\, \mu(\hat{T}) |(b, \eta_{\hat{T}})|^2 \ge \delta^2 \sum_{\substack{\hat{T}\in\mathcal{T}_J\\ \eta_{\hat{T}}\in S_{\text{ref},\delta}}} \omega_{\hat{T}}\, \mu(\hat{T}) \ge \delta^2 \sum_{\substack{\hat{T}\in\mathcal{T}_J\\ \hat{T}\subset T}} \omega_{\hat{T}}\, \mu(\hat{T}) \Big(1 - \frac{\mu(E)\omega_E}{\mu(T)\omega_T}\Big)$$

we see that we can choose $E \subset T$ such that for all $\eta_{\hat{T}} \in S_{\text{ref},\delta}$ we have $\eta_{\hat{T}} \in E$ and $\mu(E) \le 4\delta\, \mu(T)$. Hence, with a choice of $\delta = \frac{\gamma}{4}$ we obtain the asserted inequality (2.15) for $e = 0$ with $c = \frac{1}{\delta^2}$. The case $e \neq 0$ then follows from this, see [17] for details. □

Remark 2.3. Note that condition (2.13) is equivalent to the following requirement on the coefficient function ω

$$F(\gamma) := \sup_{T\in\mathcal{T}_j} \sup_{E\subset T, \mu(E)\le\gamma\,\mu(T)} \frac{\int_E \omega}{\int_T \omega} \to 0$$

when $\gamma \to 0$. In fact, the proof of Lemma 2.2 (see [17]) shows that it is sufficient to require $F(\gamma) \in (0,1)$. However, in this modified form condition (2.13) is equivalent to the property A_∞ in the theory of weighted inequalities for singular integrals (see [18], p.196). It comprises the properties $A_p, 1 \le p < \infty$, which give *necessary and sufficient conditions* in order that weighted L_p-inequalities hold for a general class of singular integrals.

With the help of Lemma 2.2 we obtain the equivalence of the weighted norms $\|\cdot\|_{j,\omega}$ and $\|\cdot\|_\omega$ for $v \in \mathcal{V}_j$ for all levels j.

Theorem 2.4. *Under the condition* (2.13) *of Lemma 2.2 there holds for* $v \in \mathcal{V}_j$

$$(1+\ 2\,\gamma^{-1}\ C_0)^{-1}\ \|v\|_{j,\omega,T} \le\ \|v\|_{\omega,T} \le\ (1+\ \sqrt{\frac{3}{2}C_0})\ \|v\|_{j,\omega,T}. \tag{2.16}$$

with the constant C_0 *only depending on the initial triangulation* $\mathcal{T}_0$.

Proof. See [17]. □

This lemma is essential for our further considerations, since not all steps of the proof can be completed for both norms directly. In fact, some intermediate results can only be obtained directly either for the $\|\cdot\|_\omega$ norm or the $\|\cdot\|_{j,\omega}$ norms. Hence, the complete proof of the overall norm equivalencies makes significant use of this lemma.

Remark 2.5. The proof of Theorem 2.4 makes use of the fact that we are in two space dimension and employ linear finite elements. The extension of the result to higher dimensions and multi-linear elements is not obvious, however, we believe that a similar estimate holds true.

Finally, let us introduce the short hand notation $A \asymp B$ for two-sided inequalities $cA \le B \le CA$, and $A \succeq B$, $A \preceq B$ for one-sided inequalities $A \ge CB$, $A \le cB$ with generic constants c and C which do not depend on the arguments of A and B.

3. Robust Norm Equivalencies

The aim of this section is to establish two robust norm equivalencies which in turn can be used to derive robust multilevel preconditioners for our model problem (2.12). Namely, we are interested in the equivalence

$$a(u,u) \asymp \sum_{j=1}^{J} 2^{2j}\|u_j\|^2_{j,\omega} + a(u_0,u_0)$$

and the equivalence

$$a(u,u) \asymp \sum_{j=1}^{J} 2^{2j}\|u_j\|^2_{\omega} + a(u_0,u_0)$$

where the decomposition $u_j := P_j u - P_{j-1}u$ is based on appropriate bounded and surjective projection operators $P_j : \mathcal{V}_J \to \mathcal{V}_j$.

3.1. **Upper Bounds for the Bilinear form.** We begin with the derivation of an upper bound for the bilinearform $a(u,u)$. A straightforward computation using (2.10) shows that we have the following Bernstein-type inequality relating the bilinear form (2.5) to the weighted norms from (2.9).

Lemma 3.1. *For uniformly refined triangulations* $\mathcal{T}_j$ *there holds the estimate*

$$a(v,v) \le \frac{3}{2}\ C_0^2\ 2^{2j}\ \|v\|^2_{j,\omega} \tag{3.1}$$

for all $v \in \mathcal{V}_j$, *if the bilinear form satisfies the local ellipticity condition stated in Assumption 1.*

Proof. See [17], where the result is obtained for the more general weight ϖ. □

With the help of this lemma and a strengthened Cauchy–Schwarz inequality we can derive the following *sub-optimal* but *robust* upper bound, see [17, 23] for details.

Theorem 3.2. *Let the bilinear form $a(\cdot,\cdot)$ fulfill Assumption 1 and consider a sequence of uniformly refined triangulations $\mathcal{T}_j$ and the respective sequence of nested spaces $\mathcal{V}_j$ of linear finite elements. Then, the estimate*

$$a(u,u) \leq 3\ C_0^2\ J \sum_{j=1}^{J} 2^{2j}\ \|(P_j - P_{j-1})u\|_{j,\omega}^2 + 2\ a(u_0,u_0) \tag{3.2}$$

holds true for any choice of bounded and surjective projections $P_j : \mathcal{V} = \mathcal{V}_J \to \mathcal{V}_j$.

Now we will improve this sub-optimal upper estimate of $a(u,u)$. Our improvement gives a robust and optimal estimate, i.e., it does not involve the factor J. It is achieved in three steps: First, we introduce weighted projection operators Q_j^ω based on $\langle\cdot,\cdot\rangle_\omega$. Furthermore, we need to consider a second sequence of projection operators Q_j^a based on auxiliary bilinear forms $a_j(\cdot,\cdot)$ which are defined in a level-dependent fashion. With the help of the two projections Q_j^ω and Q_j^a, we establish a hybrid Bernstein-type inequality involving both projections. Finally, we derive a robust and optimal upper bound of $a(\cdot,\cdot)$ using only the projections Q_j^ω via a Hardy inequality in Theorem 3.5.

To find appropriate projections for the bilinear form $a(u,v)$, we introduce the *weighted projections* $Q_j^\omega : \mathcal{V}_J \to \mathcal{V}_j$ by the relation

$$\langle Q_j^\omega u, v\rangle_\omega = \langle u,v\rangle_\omega \tag{3.3}$$

for all $u \in \mathcal{V}_J$ and $v \in \mathcal{V}_j$. Furthermore, we need to consider *auxiliary* projection operators $Q_j^a : \mathcal{V}_J \to \mathcal{V}_j$ defined by

$$a_j(Q_j^a u, v) = a_j(u,v) \tag{3.4}$$

for all $u \in \mathcal{V}_J$ and $v \in \mathcal{V}_j$ to obtain an optimal estimate of $a(u,u)$ by the Q_j^ω. Here, the bilinear forms $a_j(u,v)$ are defined in a *level-dependent* way, i.e.,

$$a_j(u,v) := \sum_{T\in\mathcal{T}_j} \omega_T \int_T (\nabla u, \nabla v) \tag{3.5}$$

using the average weights (2.8) for all $u \in \mathcal{V}_J$ and $v \in \mathcal{V}_j$. Due to the definition of ω_T and the use of linear finite elements, it follows that

$$\begin{aligned} a_k(u,v) &= \sum_{U\in\mathcal{T}_k} \omega_U \int_U (\nabla u,\nabla v) = \sum_{T\in\mathcal{T}_j} \sum_{U\in\mathcal{T}_k, U\subset T} \mu(U)\ \omega_U\ (\nabla u,\nabla v)|_T \\ &= \sum_{T\in\mathcal{T}_j} \omega_T \int_T (\nabla u,\nabla v) = a_j(u,v) \end{aligned}$$

for u, $v \in \mathcal{V}_j$ with $j \leq k$. This holds true, since ∇u is constant on any $T \in \mathcal{T}_k$ with $k \geq j$ for $u \in \mathcal{V}_j$. With the definition

$$v_j = v_j(u) := Q_j^a u - Q_{j-1}^a u, \tag{3.6}$$

we obtain

$$a(v_j, v_k) = a_k(v_j, Q_k^a u - Q_{k-1}^a u) = a_k(v_j,u) - a_k(v_j,u) = 0$$

for $j < k$, and since $u = Q^a_J u = \sum_{j=0}^J v_j$, where $v_0 := Q^a_0 u$, we get the equivalence

$$a(u,u) = \sum_{j,k=0}^{J} a(v_j, v_k) = \sum_{j=0}^{J} a(v_j, v_j) = \sum_{j=0}^{J} \|v_j\|_a^2 \tag{3.7}$$

for any $u \in \mathcal{V}_J$. Furthermore, we introduce the sequence

$$u_k = u_k(u) := Q^\omega_k u - Q^\omega_{k-1} u \tag{3.8}$$

based on the projections Q^ω_k.

With the definitions (3.6) and (3.8) we can now obtain an upper bound for $\|v_j\|_a$ in terms of $\|u_k\|_{k,\omega}$ for $k \geq j$; i.e., we establish a hybrid Bernstein-type estimate of $a(v_j, v_j)$ in terms of $\|v_j\|_a \|u_k\|_{k,\omega}$ for $k \geq j$.

Lemma 3.3. *Let $v_j \in \mathcal{V}_j$ and $u_k \in \mathcal{V}_k$ for $k \geq j$ be defined as in* (3.6) *and* (3.8)*, respectively. Then there holds the estimate*

$$\begin{aligned} a(v_j, v_j) &\leq 6\ C_2 \|v_j\|_a \sum_{k=j}^{J} 2^{2j-k} \|u_k\|_{k,\omega}, \\ \|v_j\|_a &\leq 6\ C_2 \sum_{k=j}^{J} 2^{2j-k} \|u_k\|_{k,\omega}, \end{aligned}$$

with a constant

$$C_2 := \max_{\hat{T} \in \mathcal{T}_0} \frac{\operatorname{diam}(\hat{T})}{\sqrt{\mu(\hat{T})}} \tag{3.9}$$

which depends on the initial triangulation $\mathcal{T}_0$ only.

Proof. Since $v_j = Q^a_j u - Q^a_{j-1} u$, we have $a(v_j, w) = 0$ for all $w \in \mathcal{V}_{j-1}$. Hence, with the choice $w = Q^\omega_{j-1} u$ we obtain

$$\begin{aligned} a(v_j, v_j) &= a_j(v_j, v_j) = a_j(v_j, u) = a_j(v_j, u - Q^\omega_{j-1} u) \\ &= \sum_{T \in \mathcal{T}_j} \omega_T \int_T (\nabla v_j, \nabla(u - Q^\omega_{j-1} u)). \end{aligned}$$

Using the identity $u - Q^\omega_{j-1} u = \sum_{k=j}^J Q^\omega_k u - Q^\omega_{k-1} u = \sum_{k=j}^J u_k$ we establish the equivalence

$$a(v_j, v_j) = \sum_{T \in \mathcal{T}_j} \omega_T \sum_{k=j}^{J} \int_T (\nabla v_j, \nabla u_k)$$

and integration by parts for each $T \in \mathcal{T}_j$ yields

$$a(v_j, v_j) = \sum_{T \in \mathcal{T}_j} \sum_{k=j}^{J} \omega_T \int_{\partial T} u_k (\nabla v_j, n_{\partial T}).$$

Now consider a fixed $T \in \mathcal{T}_j$ and $U \in \mathcal{T}_k$ with $U \subset T$. Then we obtain

$$\begin{aligned} |\omega_T \int_{\partial T} u_k(\nabla v_j, n_{\partial T})| &= |\,\mu(T)^{-1} \sum_{\substack{U\in\mathcal{T}_k\\ U\subset T}} \mu(U)\omega_U \int_{\partial U} u_k(\nabla v_j, n_{\partial U})| \\ &\le \mu(T)^{-1} \sum_{\substack{U\in\mathcal{T}_k\\ U\subset T}} \mu(U)\sqrt{\omega_U} \int_{\partial U} |u_k|\ |\nabla v_j|_{\infty,T}\sqrt{\omega_U} \\ &\le 6 \sum_{\substack{U\in\mathcal{T}_k\\ U\subset T}} \sqrt{\omega_U \int_U |u_k|^2}\ \frac{\operatorname{diam}(U)}{\mu(T)}\ \sqrt{\omega_U \int_U |\nabla v_j|^2}. \end{aligned}$$

Here, in the last line, the inequality

$$\int_{\partial U} |u_k|\ \le\ \operatorname{diam}(U)[w_1+w_2+w_3] \le 6\ \operatorname{diam}(U)\ \ \mu(U)^{-\frac{1}{2}}\ \Big(\int_U |u_k|^2\Big)^{\frac{1}{2}}$$

was used which results from the well-known formula

$$\int_U |v|^2 = \frac{\mu(U)}{12}[w_1^2+w_2^2+w_3^2+(w_1+w_2+w_3)^2]$$

for linear functions v on U with vertices w_1, w_2, and w_3. Since we consider uniformly refined triangulations in two dimensions, we have $\mu(T) = 2^{2(k-j)}\,\mu(U)$ and therefore $\operatorname{diam}(U)\ \le\ C_2\ 2^{2(j-k)}\ 2^k\,\mu(T)$ with C_2 given in (3.9) depending on the initial triangulation $\mathcal{T}_0$ only. Hence, we end up with the assertion

$$\begin{aligned} a(v_j, v_j) &\le 6\ C_2 \sum_{k=j}^{J} 2^{-k} \sum_{T\in\mathcal{T}_j} \|u_k\|_{k,\omega,T}\ 2^{2j}\ \|\nabla v_j\|_{j,\omega,T} \\ &\le 6\ C_2 \sum_{k=j}^{J} 2^{2j-k}\|u_k\|_{k,\omega}\ \|v_j\|_a. \end{aligned}$$

□

Note that this lemma is a strengthened version of the following Bernstein-type inequality of broken order for certain weighted trace norms

$$\|u\|_{\frac{1}{2},j,\omega} := \Big(\sum_{T\in\mathcal{T}_j} \omega_T \int_{\partial T} |u|^2\Big)^{\frac{1}{2}}$$

which employs only a single projection.

Lemma 3.4. *For elements $v \in \mathcal{V}_j$ there holds*

$$\|v\|_a \le\ \sqrt{3C_2C_1}\ 2^{\frac{j}{2}}\ \|v\|_{\frac{1}{2},j,\omega},$$

where the constants $C_1 := \max_{T\in\mathcal{T}_0}\operatorname{diam}(T)$, and C_2 from (3.9) *depend on the initial triangulation $\mathcal{T}_0$ only.*

Proof. Keeping in mind that ∇v is constant on $T \in \mathcal{T}_j$, integration by parts yields

$$\begin{aligned} a(v,v) &= \sum_{T\in\mathcal{T}_j} \omega_T \int_T (\nabla v, \nabla v) = \sum_{T\in\mathcal{T}_j} \omega_T \int_{\partial T}\ v(\nabla v, n_{\partial T}) \\ &\le \Big(\sum_{T\in\mathcal{T}_j} \omega_T \int_{\partial T} |v|^2\Big)^{\frac{1}{2}}\ \Big(\sum_{T\in\mathcal{T}_j} \omega_T \int_{\partial T} \|\nabla v\|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Here, the first sum represents the semi-norm $||v||_{\frac{1}{2},\omega,j}$, whereas each term in the second sum can be bounded using the local Bernstein-type inequality

$$\int_{\partial T} \|\nabla v\|^2 \leq \; 3 \; \mathrm{diam}(T) \; \|\nabla v\|^2 \leq \; 3C_2 \; \mu(T)^{\frac{1}{2}} \; \|\nabla v\|^2 \leq \; 3C_2 C_1 \; 2^j \int_T \|\nabla v\|^2.$$

Hence, after multiplication with ω_T and summation with respect to $T \in \mathcal{T}_j$, we obtain the overall estimate

$$a(v,v) = \sum_{T\in\mathcal{T}_j} \omega_T \int_T (\nabla v, \nabla v) \leq \sqrt{3C_2C_1}\; 2^{\frac{j}{2}} \; \|v\|_{\frac{1}{2},j,\omega} \; \sqrt{a(v,v)}$$

and the assertion follows after division by $\sqrt{a(v,v)}$. □

Finally, we are in the position to prove the main result of this section, a *robust* and *optimal* upper bound for the bilinear form $a(u,u)$.

Theorem 3.5. *Let the bilinear form $a(\cdot,\cdot)$ fulfill Assumption 1 and consider a sequence of uniformly refined triangulations $\mathcal{T}_j$ and the respective sequence of nested spaces $\mathcal{V}_j$ of linear finite elements. Then, there holds the upper bound*

$$a(u,u) \; \leq \frac{32}{3} \; C_2 \sum_{j=0}^{J} \; 2^{2j} \; \|u_j\|_{j,\omega}^2 \tag{3.10}$$

for u_k from (3.8) *and the norm $\|\cdot\|_{j,\omega}$ from* (2.9)*. The constant C_2 given in* (3.9) *depends only on the initial triangulation $\mathcal{T}_0$.*

Proof. With (3.7) and Lemma 3.3 we obtain

$$\begin{aligned} a(u,u) \;\; &= \;\; \sum_{j=0}^{J} \|v_j\|_a^2 \; \leq \; 6 \; C_2 \sum_{k=0}^{J} \; 2^{-k} \|u_k\|_{k,\omega} \sum_{j=0}^{k} \; 2^{2j} \; \|v_j\|_a \\ &\leq \;\; \Big(\sum_{k=0}^{J} \; 2^{2k} \|u_k\|_{k,\omega}^2\Big)^{\frac{1}{2}} \; \Big(\sum_{k=0}^{J} \; 16^{-k} \; \big(\sum_{j=0}^{k} \; 2^{2j} \; \|v_j\|_a\big)^2\Big)^{\frac{1}{2}} \end{aligned} \tag{3.11}$$

after interchanging the sums. Using the Hardy inequality

$$\Big(\sum_{k=0}^{J} \; b^k \; s_k^2\Big)^{\frac{1}{2}} \leq \; \frac{1}{1-\sqrt{b}} \; \Big(\sum_{j=0}^{J} \; b^j \; a_j^2\Big)^{\frac{1}{2}}, \tag{3.12}$$

where

$$s_k := \sum_{j=0}^{k} \; a_j, \quad s_{-1} := 0, \quad \text{and} \quad c_k := \sum_{l=k}^{J} b^l, \quad c_{J+1} := 0,$$

we obtain

$$\Big(\sum_{k=0}^{J} \; 16^{-k} \; \big(\sum_{j=0}^{k} \; 2^{2j} \; \|v_j\|_a\big)^2\Big)^{\frac{1}{2}} \leq \; \frac{4}{3}\Big(\sum_{j=0}^{J} \; \|v_j\|_{j,\omega}^2\Big)^{\frac{1}{2}}.$$

with the choice $a_j := 2^{2j}\|v_j\|_{j,\omega}$ and $b = 1/16$. Plugging this estimate into (3.11), we establish the asserted optimal and robust upper bound (3.10). □

We obtain the respective upper bound for the weighted norm $\|\cdot\|_\omega$ on the finest level using Theorem 2.4.

Theorem 3.6. *Let the bilinear form $a(\cdot,\cdot)$ fulfill Assumption 1 and consider a sequence of uniformly refined triangulations $\mathcal{T}_j$ and the respective sequence of nested spaces $\mathcal{V}_j$ of linear finite elements. Then, there holds the upper bound*

$$a(u,u) \le \Big(1 + 2\,\gamma^{-1}\,C_0\Big)^2 \frac{32}{3} C_2 \sum_{j=0}^{J} 2^{2j}\,\|u_j\|_\omega^2 \tag{3.13}$$

for u_j given in (3.8) *and the norm $\|\cdot\|_\omega$ from* (2.11).

3.2. **Lower Bounds for the Bilinear form.** The next step in our search for robust norm equivalencies is the derivation of optimal and robust lower bounds for the bilinear form $a(\cdot,\cdot)$ in terms of the norms $\|\cdot\|_{j,\omega}$ and $\|\cdot\|_\omega$; i.e., we are looking for a Jackson-type inequality for $\|u_j\|_{j,\omega}$ and for $\|u_j\|_\omega$

$$\sum_{j=1}^{J} 2^{2j}\|u_j\|_{j,\omega}^2 + a(u_0,u_0) \preceq a(u,u) \qquad \text{and} \qquad \sum_{j=1}^{J} 2^{2j}\|u_j\|_{\omega}^2 + a(u_0,u_0) \preceq a(u,u).$$

Unlike in the Bernstein case, the constants will involve some information about the coefficient ω, yet in a very weak norm. The main reason for this slight dependence is due to the problem that we cannot obtain an optimal lower bound directly for the $\|\cdot\|_{j,\omega}$ norms. Here, we rather need to work with the $\|\cdot\|_\omega$ norm to obtain an optimal estimate. Then, we can exploit the result of Theorem 2.4 to get a similar bound for $\|\cdot\|_{j,\omega}$. Yet, it is this detour in our proof, which forces us to introduce some information about the weight ω in the estimate.

In a first step, we bound the u_j-decomposition in terms of the v_j-decomposition with respect to the $\|\cdot\|_\omega$ norm in the following lemma. The respective estimate for the $\|\cdot\|_{j,\omega}$ norms then follows with Theorem 2.4.

Lemma 3.7. *The decompositions defined in* (3.6) *and* (3.8) *allow for the estimates*

$$\sum_{j=0}^{J} 2^{2j}\|u_j\|_\omega^2 \le 4\sum_{j=0}^{J} 2^{2j}\|v_j\|_\omega^2$$

and

$$\sum_{j=0}^{J} 2^{2j}\|u_j\|_{j,\omega}^2 \le 4(1+2\gamma^{-1}C_0)^2(1+\sqrt{\frac{3}{2}C_0})^2\sum_{j=0}^{J} 2^{2j}\|v_j\|_{j,\omega}^2.$$

Proof. Observe that due to (3.3) we have

$$\begin{array}{rcccl} \|u_j\|_\omega^2 &=& \|Q_j^\omega u - Q_{j-1}^\omega u\|_\omega^2 &=& \langle Q_j^\omega u - Q_{j-1}^\omega u, Q_j^\omega u\rangle_\omega \\ &=& \langle Q_j^\omega u - Q_{j-1}^\omega u, u\rangle_\omega &=& \langle Q_j^\omega u - Q_{j-1}^\omega u, u - Q_{j-1}^a u\rangle_\omega \end{array}$$

for all j so that we obtain

$$\sum_{j=0}^{J} 2^{2j}\|u_j\|_\omega^2 \le \sum_{j=0}^{J} 2^{2j}\|u - Q_{j-1}^a u\|_\omega^2 \le \sum_{j=0}^{J} 2^{2j}\sum_{k=j}^{J}\|v_k\|_\omega^2,$$

where v_j is defined in (3.6) and u_j is defined in (3.8). Then using Hardy's inequality again, we obtain

$$\sum_{j=0}^{J} 2^{2j}\|u_j\|_\omega^2 \le 4\sum_{j=0}^{J} 2^{2j}\|v_j\|_\omega^2.$$

Now we pass back from the $\|\cdot\|_\omega$ norm to the $\|\cdot\|_{j,\omega}$ norms with the help of Theorem 2.4 and end up with the estimates

$$\sum_{j=0}^{J} 2^{2j}\|u_j\|_\omega^2 \le 4(1+\sqrt{\frac{3}{2}C_0})^2 \sum_{j=0}^{J} 2^{2j}\|v_j\|_{j,\omega}^2, \tag{3.14}$$

and

$$\sum_{j=0}^{J} 2^{2j}\|u_j\|_{j,\omega}^2 \le 4(1+2\gamma^{-1}C_0)^2(1+\sqrt{\frac{3}{2}C_0})^2 \sum_{j=0}^{J} 2^{2j}\|v_j\|_{j,\omega}^2.$$

□

Remark 3.8. Note that it is necessary for the proof to switch to the $\|\cdot\|_\omega$ norm, since the terms in the Hardy inequality must employ the *same* norm on all levels j.

Now, we need to deal with the projection operators Q_j^a and the respective sequence v_j only. Here, we can prove a local estimate for the $\|\cdot\|_{j,\omega}$ norms using a modified version of the duality technique due to Aubin and Nitsche, see also [2], and a certain regularity result for the Neumann problem. However, we are able to obtain this local estimate on a particular level j only under the additional assumption

$$1 > C \max_{\substack{S',S\in\mathcal{T}_j\\ S',S\in U}} \sqrt{\frac{\omega_S}{\omega_{S'}}}\,\mathrm{diam}(U) \tag{3.15}$$

where C denotes an *absolute* constant, and $U = U_T$ refers to the union of the supports of all nodal basis functions $\psi_{l,j-1}$ which intersect the considered element $T \in \mathcal{T}_{j-1}$ (compare (3.16)). Note that this condition essentially intertwines the necessary refinement level j with the "smoothness" of the coefficient function ω.

Theorem 3.9. *Consider the neighborhood*

$$U := U_T = \bigcup_{\substack{\hat T\in\mathcal{T}_{j-1}\\ \hat T\subset \mathrm{supp}(\psi_{l,j-1})\\ \mathrm{supp}(\psi_{l,j-1})\cap T\neq\emptyset}} \hat T \tag{3.16}$$

of $T \in \mathcal{T}_{j-1}$ and let condition (3.15) *be fulfilled. Then there holds the estimate*

$$\|v_j\|_{j,\omega,U} \le C \left(\max_{\substack{S',S\in\mathcal{T}_j\\ S',S\subset U}} \sqrt{\frac{\omega_S}{\omega_{S'}}}\right) \mathrm{diam}(U)\ \|\nabla v_j\|_{j,\omega,U}. \tag{3.17}$$

Essential for the proof of (3.17) is the regularity result stated in the following lemma.

Lemma 3.10. *Consider the domain $\tilde U$ with a smooth boundary $\partial\tilde U$ such that $\tilde U \supset U$, all vertices $\xi \in \partial U$ are also $\xi \in \partial\tilde U$, and $\mathrm{diam}(\tilde U) < 2\ \mathrm{diam}(U)$ with U defined in* (3.16)*. Let $\tilde v_j$ denote the continuous and piecewise linear extension of v_j from U to $\tilde U$ which fulfills*

$$\|\tilde v_j\|_{0,2,\tilde U} \le 2\|v_j\|_{0,2,U}. \tag{3.18}$$

Then, the solution φ_U to the inhomogeneous Neumann problem

$$-\Delta\varphi_U = \tilde v_j \text{ in } \tilde U, \quad \frac{\partial\varphi_U}{\partial\nu} = g \text{ on } \partial\tilde U$$

is in $H^2(U)$ *and allows for the estimate*

$$|\varphi_U|_{2,2,U} \;\le\; C\; \|v_j\|_{0,2,U}. \tag{3.19}$$

Here, the boundary data for the Neumann problem are $g := \alpha h$ *with a piecewise linear* $h \in L_2(\partial\tilde{U})$ *such that* $h \ge 0$ *on the boundary* $\partial\tilde{U}$ *and* $h(\xi) = 0$ *for all vertices* $\xi \in \partial U \cap \partial\tilde{U}$*. Furthermore,* $\alpha \in \mathbb{R}$ *is such that the compatibility condition*

$$\int_{\tilde{U}} \tilde{v}_j = -\int_{\partial\tilde{U}} g \tag{3.20}$$

is fulfilled.

Proof. Consider the scaled domain $\hat{\Omega} := R\tilde{U}$ and the spaces

$$\begin{aligned} V(\hat{\Omega}) \;&:=\; \{\varphi \in W_2^2(\hat{\Omega}) : \int_{\hat{\Omega}} \varphi\, dx = 0\} \qquad \text{and} \\ W(\hat{\Omega}) \;&:=\; \{\langle f, g\rangle \in L_2(\hat{\Omega}) \times W_2^{1/2}(\partial\hat{\Omega}) : \int_{\hat{\Omega}} f\, dx + \int_{\partial\hat{\Omega}} g\, ds = 0\} \end{aligned}$$

and the mapping $T : \varphi \in V \mapsto T\,\varphi \in W$ defined by

$$T\,\varphi := \begin{cases} f = \Delta\varphi & \in L_2(\hat{\Omega}), \\ g = \partial\varphi/\partial\nu & \in W_2^{1/2}(\partial\hat{\Omega}). \end{cases}$$

This mapping is linear and continuous; i.e., there exists a constant $M_{\hat{\Omega}}$ such that

$$\|T\,\varphi\|_W \equiv \|f\|_{0,2,\hat{\Omega}} + \|g\|_{1/2,2,\partial\hat{\Omega}} \;\le\; M_{\hat{\Omega}} \|\varphi\|_{2,2,\hat{\Omega}}$$

holds. Furthermore, T is also bijective from V onto W, see [21], p. 336-339. Hence by the open mapping theorem its inverse $T^{-1} :\; \langle f, g\rangle \mapsto \varphi$ is also continuous and satisfies

$$\|\varphi\|_V \equiv \|\varphi\|_{2,2,\hat{\Omega}} \;\le\; L_{\hat{\Omega}}\Big(\|\Delta\varphi\|_{0,2,\hat{\Omega}} + \|\nabla\varphi/\partial\nu\|_{1/2,2,\partial\hat{\Omega}}\Big), \qquad \varphi \in W(\hat{\Omega}),$$

with a constant $L_{\hat{\Omega}}$. However, it is still necessary to determine the dependence of $L_{\hat{\Omega}}$ on the size of $\hat{\Omega} = R\tilde{U}$, i.e., on the scaling R, since by definition (3.16) $U = U_T$ depends on the level j. To this end, let us consider the scaling R such that $R^{-1} := \operatorname{diam}(\tilde{U}) \le 2\; \operatorname{diam}(U)$ so that

$$\|\psi\|_{2,2,\hat{\Omega}} \;\le\; C^*\Big(\|\Delta\psi\|_{0,2,\hat{\Omega}} + \|\partial\psi/\partial\nu\|_{1/2,2,\partial\hat{\Omega}}\Big), \qquad \psi \in W(\hat{\Omega}), \tag{3.21}$$

with C^* depending only on the *shape* of U but *not* on the size. Hence, C^* depends only on the initial triangulation $\mathcal{T}_0$. The connection between φ and ψ is given by $\psi(t) = \varphi(t/R) := \varphi(x)$. Therefore, there hold the equivalencies

$$\|\psi\|_{0,2,\hat{\Omega}} = R\;\|\varphi\|_{0,2,\tilde{U}}, \qquad |\psi|_{1,2,\hat{\Omega}} = |\varphi|_{1,2,\tilde{U}}, \quad \text{and} \quad |\psi|_{2,2,\hat{\Omega}} = R^{-1}\;|\varphi|_{2,2,\tilde{U}}.$$

Using the explicit form of the trace norm $\|\cdot\|_{1/2,2,\partial\hat{\Omega}}$ given by (see e.g. [21], p. 94)

$$\begin{aligned} \|\psi\|^2_{1/2,2,\partial\hat{\Omega}} \;&:=\; \sum_i \|\psi_i\|^2_{0,2,\partial\hat{\Omega}} + \sum_i |\psi_i|^2_{1/2,2,\partial\hat{\Omega}} \\ |\psi_i|^2_{1/2,2,\partial\hat{\Omega}} \;&:=\; \int_{\partial\hat{\Omega}}\int_{\partial\hat{\Omega}} \frac{|\psi_i(t) - \psi_i(s)|^2}{|t-s|^2} d\sigma\; d\sigma, \end{aligned} \tag{3.22}$$

where $\psi = \sum_i \psi_i$ is a partition of ψ with respect to the representation by charts of the curve $\partial\hat{\Omega}$ with curve element $d\sigma$, we conclude

$$\|\partial\psi/\partial\nu\|_{1/2,2,\partial\hat{\Omega}} = R^{-1/2}\;\|\partial\varphi/\partial\nu\|_{0,2,\partial\tilde{U}} + R^{-1}\;|\partial\varphi/\partial\nu|_{1/2,2,\partial\tilde{U}}.$$

Consequently, by (3.21) we obtain the estimate

$$|\varphi_U|_{1,2,\tilde U} \ + \ \mathrm{diam}(U)\ |\varphi_U|_{2,2,\tilde U} \le C^* \,\mathrm{diam}(U) \left(\|\tilde v_j\|_{0,2,\tilde U} + |g|_{1/2,2,\partial\tilde U} + \frac{\|g\|_{0,2,\partial\tilde U}}{\sqrt{\mathrm{diam}(U)}}\right) \tag{3.23}$$

for the data of the Neumann problem above, i.e., $\varphi = \varphi_U \in W_2^2(\tilde U)$, $-\Delta\varphi_U = \tilde v_j$ in $\tilde U$ and $\partial\varphi/\partial\nu = g$ on $\partial\tilde U$. Now we use the fact that g is piecewise linear on $\partial\tilde U$ by construction. Therefore,

$$|g|_{1/2,2,\partial\tilde U} \ \le \ C(\mathrm{diam}(U))^{-1}\ \|g\|_{0,1,\partial\tilde U} \tag{3.24}$$

$$\|g\|_{0,2,\partial\tilde U} \ \le \ C\ (\mathrm{diam}(U))^{-1/2}\ \|g\|_{0,1,\partial\tilde U} \tag{3.25}$$

holds with a constant C independent of $\tilde U$ and g. For completeness, we give the proof of the inequalities (3.24) and (3.25) here. To this end, let $\partial\tilde U_i$ be one of the two pieces of the segment of $\partial\tilde U$ between two vertices of $\mathcal{T}_{j-1}$. Here g is a linear function. Then by (3.22) with $\psi_i = g$

$$|g|_{1/2,2,\partial\tilde U_i} = \left(\int_{\partial\tilde U_i}\int_{\partial\tilde U_i} \frac{|g(t)-g(s)|^2}{|t-s|^2} d\sigma\ d\sigma\right)^{1/2} \ \le\ C\ \ \mathrm{diam}(\partial\tilde U_i)\ |\nabla g|_{\infty,\partial\tilde U_i}$$

follows. Furthermore, we can replace g by $\tilde g := g - \bar g$ where $\bar g := \min_{\partial\tilde U_i}|g|$. Then $\tilde g$ has a zero on $\partial\tilde U_i$ and we have the equivalence

$$\int_{\partial\tilde U_i} |\tilde g|\ d\sigma \asymp (\mathrm{diam}(\partial\tilde U_i))^2\ |\nabla g|_{\infty,\partial\tilde U_i},$$

with constants independent of $\tilde U_i$ and g. Since

$$\int_{\partial\tilde U_i} |\tilde g|\ d\sigma \le \int_{\partial\tilde U_i} |g|\ d\sigma \ + \ C\ \ \mathrm{diam}(\partial\tilde U_i)\ \bar g \le\ C\ \int_{\partial U_i} |g|\ d\sigma = C\|g\|_{0,1,\partial\tilde U_i},$$

the comparison with the previous inequality yields (3.24). The second inequality (3.25) follows from

$$\begin{aligned}
\int_{\partial\tilde U_i} |g|^2\ d\sigma \ &\le\ C\left(\int_{\partial\tilde U_i} |\tilde g|^2\ d\sigma + \int_{\partial\tilde U_i}|\bar g|^2\ d\sigma\right)\\
&\le\ C\ (\mathrm{diam}(\partial\tilde U_i))^3\ |\nabla g|^2_{\infty,\partial\tilde U_i} + C(\mathrm{diam}(\partial\tilde U_i))^{-1}\left(\int_{\partial\tilde U_i}|g|\ d\sigma\right)^2\\
&\le\ C(\mathrm{diam}(\partial\tilde U_i))^{-1}\left(\int_{\partial\tilde U_i}|g|\ d\sigma\right)^2.
\end{aligned}$$

This completes the proof of (3.24) and (3.25), since we can choose $\tilde U$ such that $\mathrm{diam}(\tilde U) \asymp \mathrm{diam}(\partial\tilde U)$.

Inserting (3.24) and (3.25) into inequality (3.23) and taking (3.18) and (3.20) into account, the desired inequality (3.19) directly follows:

$$\begin{aligned}
|\varphi_U|_{2,2,U} \ &\le\ C\left(\|\tilde v_j\|_{0,2,\tilde U} \ + (\mathrm{diam}(U))^{-1}\|g\|_{0,1,\partial\tilde U}\right)\\
&=\ C\Big(\|\tilde v_j\|_{0,2,\tilde U} \ + (\mathrm{diam}(U))^{-1}\|\tilde v_j\|_{0,1,\tilde U}\Big) \le C\ \|v_j\|_{0,2,U}.
\end{aligned}$$

□

With this lemma, we are in the position to prove Theorem 3.9.

Proof (Theorem 3.9). At first we note that we can assume $v_j(x) \geq 0$ in U since otherwise v_j changes sign in U and we can obtain the desired inequality (3.17) directly. Therefore we need to consider in the following only U which do not intersect boundary elements $T \in \mathcal{T}_{j-1}$.

With the help of (2.9) and Lemma 3.10 we obtain, after integration by parts on *each* $S \subset U$,

$$\|v_j\|_{j,\omega,U}^2 = -\int_U v_j \; \Delta\varphi_U \; = \sum_{S\subset U} \omega_S \; \int_S (\nabla v_j, \nabla\varphi_U) \; - \sum_{S\subset U} \omega_S \; \int_{\partial S} v_j (\nabla\varphi_U, n_{\partial S}). \tag{3.26}$$

Concerning the first sum, observe that by definition of v_j, see (3.6),

$$0 = a(v_j, w) = \sum_{S\in\mathcal{T}_j} \omega_S \int_S (\nabla v_j, \nabla w)$$

holds for all $w \in \mathcal{V}_{j-1}$. Now, we choose w to be the function in $\mathcal{V}_{j-1}$ which interpolates φ_U at the nodes in T and has support in $U = U_T$. Then, we can estimate the first sum via the Bramble–Hilbert Lemma on U

$$\begin{aligned} I_1(U) := \Big|\sum_{S\subset U} \omega_S \int_S (\nabla\varphi_U, \nabla v_j)\Big| \;&=\; \Big|\sum_{S\subset U} \omega_S \int_S (\nabla(\varphi_U - w), \nabla v_j)\Big| \\ &\leq\; \Big(\sum_{S\subset U} \omega_S \int_S |\nabla(\varphi_U - w)|^2\Big)^{1/2} \|\nabla v_j\|_{j,\omega,U} \\ &\leq\; C \operatorname{diam}(U) |\varphi_U|_{2,2,U} \Big(\max_{S\subset U} \sqrt{\omega_S}\Big) \|\nabla v_j\|_{j,\omega,U}, \end{aligned} \tag{3.27}$$

where the constant C depends only on the shape of U, i.e., by assumption only on the initial triangulation $\mathcal{T}_0$.

The second sum in (3.26) can be estimated by

$$\Big|\sum_{S\subset U} \omega_S \; \int_{\partial S} v_j (\nabla\varphi_U, n_{\partial S})\Big| \leq \sum_{S\subset U} \omega_S \|v_j\|_{\infty,\partial S} \; \int_{\partial S} |\nabla\varphi_U| := I_2(U).$$

Note that due to the choice of g in Lemma 3.10, the normal derivative $\partial\varphi_U/\partial\nu_\xi = g$ vanishes at each vertex $\xi \in \partial U$. And since any normal vector n_K of an arbitrary edge $K \subset U$ can be represented as a linear combination of three such normal vectors at $\xi_i, i = 1, 2, 3$, we can bound the normal derivative at $x \in K$ by

$$\begin{aligned} |(\nabla\varphi_U, n_K)(x)| \;&=\; \Big|\sum_{i=1}^3 \beta_i \frac{\partial\varphi_U}{\partial\nu_{\xi_i}}(x)\Big| \\ &=\; \Big|\sum_{i=1}^3 \beta_i \Big(\frac{\partial\varphi_U}{\partial\nu_{\xi_i}}(x) - \frac{\partial\varphi_U}{\partial\nu_{\xi_i}}(\xi_i)\Big)\Big| \\ &\leq\; \sum_{i=1}^3 |\beta_i| \int_0^1 \Big|(\nabla \frac{\partial\varphi_U}{\partial\nu})(\xi_i + t(x - \xi_i), x - \xi_i) dt\Big| \end{aligned}$$

with $\sum_{i=1}^3 |\beta_i|^2 = 1$. Hence after integration over K we obtain

$$\int_K |(\nabla\varphi_U, n_K)(x)| dx \leq \; C \; \operatorname{diam}(U) \sum_{|\alpha|=2} \int_U |D^\alpha\varphi_U| dx \leq \; C \; (\operatorname{diam}(U))^2 |\varphi_U|_{2,2,U}.$$

Altogether, we can now establish the estimate

$$
\begin{aligned}
I_2(U) &\leq C \sum_{S\subset U} \omega_S \|v_j\|_{\infty,\partial S} \; (\operatorname{diam}(U))^2 |\varphi_U|_{2,2,U} \\
&\leq C \; \|v_j\|_{j,\omega,U} \Big(\max_{S\subset U} \sqrt{\omega_S}\Big) \; \operatorname{diam}(U) \; |\varphi_U|_{2,2,U},
\end{aligned}
\tag{3.28}
$$

due to $\sum_{S\subset U} \sqrt{\omega_S} \; \operatorname{diam}(U) \; \|v_j\|_{\infty,\partial S} \leq C \; \|v_j\|_{j,\omega,U}$. Now the assertion (3.17) follows easily with the aid of (3.19): Insert (3.19) into (3.27) and (3.28) and obtain

$$
\begin{aligned}
\|v_j\|^2_{j,\omega,U} &\leq I_1(U) + I_2(U) \\
&\leq C \; \operatorname{diam}(U) \Big(\|v_j\|_{j,\omega,U} + \|\nabla v_j\|_{j,\omega,U} \Big) (\max_{S\subset U} \sqrt{\omega_S}) \; \|v_j\|_{0,2,U}.
\end{aligned}
$$

With the estimate $(\max_{S\subset U} \sqrt{\omega_S}) \; \|v_j\|_{0,2,U} \leq \Big(\max_{S',S\subset U} \sqrt{\omega_S/\omega_{S'}}\Big) \; \|v_j\|_{j,\omega,U}$, this yields the inequality

$$
\|v_j\|_{j,\omega,U} \leq C \; \operatorname{diam}(U) \Big(\max_{S',S\subset U} \sqrt{\frac{\omega_S}{\omega_{S'}}} \Big) \Big(\|v_j\|_{j,\omega,U} + \|\nabla v_j\|_{j,\omega,U} \Big)
$$

after division by $\|v_j\|_{j,\omega,U}$. Hence, we end up with

$$
\Big(1 - C \; \operatorname{diam}(U) \Big(\max_{S',S\subset U} \sqrt{\frac{\omega_S}{\omega_{S'}}} \Big)\Big) \|v_j\|_{j,\omega,U} \leq C \; \operatorname{diam}(U) \Big(\max_{S',S\subset U} \sqrt{\frac{\omega_S}{\omega_{S'}}} \Big) \|\nabla v_j\|_{j,\omega,U}.
$$

Thus, the assertion follows with assumption (3.15)

$$
\|v_j\|_{j,\omega,U} \leq \frac{C \; \operatorname{diam}(U) \Big(\max_{S',S\subset U} \sqrt{\frac{\omega_S}{\omega_{S'}}} \Big)}{1 - C \; \operatorname{diam}(U) \Big(\max_{S',S\subset U} \sqrt{\frac{\omega_S}{\omega_{S'}}} \Big)} \|\nabla v_j\|_{j,\omega,U}.
$$

□

A direct consequence of Theorem 3.9 is the following lower bound.

Theorem 3.11. *Let the bilinear form $a(\cdot,\cdot)$ fulfill Assumption 1 and consider a sequence of uniformly refined triangulations $\mathcal{T}_j$ and the respective sequence of nested spaces $\mathcal{V}_j$ of linear finite elements. Consider the sets $U = U_T$ and S, $S' \in \mathcal{T}_j$ defined in Theorem 3.9 for every $T \in \mathcal{T}_{j-1}$. Now define*

$$
C_{j,\omega} := \max_{T\in\mathcal{T}_{j-1}} \max_{\substack{S',S\in\mathcal{T}_j \\ S',S\in U_T}} \frac{\omega_S}{\omega_{S'}} \quad \textit{and} \quad C_{J,\max,\omega} := \max_{0\leq j\leq J} C_{j,\omega}
\tag{3.29}
$$

and let ω be such that we can find a minimal refinement level $j_0 < J$ so that

$$
2^j > C \; \sqrt{C_{J,\max,\omega}}
\tag{3.30}
$$

holds for all $j \geq j_0$ and Theorem 3.9 is applicable for all $j \geq j_0$. Then there holds the estimate

$$
\sum_{j=j_0}^{J} 2^{2j} \|u_j\|^2_\omega \leq 4 \; (1 + \sqrt{\frac{3}{2} C_0})^2 C \; C_{J,\max,\omega} \; a(u,u)
\tag{3.31}
$$

for u_j defined in (3.8).

Proof. Using Lemma 3.7, compare (3.14), we establish the estimate

$$\begin{aligned}\sum_{j=j_0}^{J} 2^{2j}\|u_j\|_\omega^2 &\leq 4\,(1+\sqrt{\frac{3}{2}C_0})^2 \sum_{j=j_0}^{J} 2^{2j}\|v_j\|_{j,\omega}^2 \\ &\leq 4\,(1+\sqrt{\frac{3}{2}C_0})^2\; C\; C_{J,\max,\omega} \sum_{j=j_0}^{J} a(v_j,v_j)\end{aligned}$$

after squaring (3.17) and summation with respect to U_T, i.e., over all $T \in \mathcal{T}_{j-1}$, where the constant C_0 depends on the shape of the triangles of the initial triangulation only. Then, with (3.7) the assertion (3.31) follows. □

Note that this estimate employs the minimal conditions on the coefficient function ω in the sense, that assuming a fixed a refinement level J we may find such a j_0 for a rather large set of functions ω. However, from a more practical point of view it is rather important to ensure the existence of j_0 *independent* of J involving only information obtainable from ω directly. Hence, let us now focus on the formulation of conditions on the coefficient function ω such that the results of Theorem 3.11 still hold, but for j_0 independent of J.

In essence, we are looking for coefficient functions ω such that

$$C_{j,\omega} = \max_{T\in\mathcal{T}_{j-1}} \max_{\substack{S',S\in\mathcal{T}_j \\ S',S\in U_T}} \frac{\omega_S}{\omega_{S'}} < \operatorname{diam}^2(U_T), \tag{3.32}$$

starting from a certain level j_0. Hence, if we consider a non-negative function ω such that for all x and y with $\|x-y\| < 2^{j_0}$ and $\omega(x) > \omega(y)$ the estimate

$$\omega(x) < 2^{2j_0}\; w(y)$$

holds, then condition (3.32) is fulfilled for all $j \geq j_0$. A trivial sufficient condition is of course

$$C_{j,\omega} \leq C_\omega < \infty \quad \text{for all } j.$$

Also, it is obvious that continuous weights ω are admissible. Another simple condition which suffices is

$$E_{j,\omega} := \max_{T\in\mathcal{T}_j} \max_{x,y\in T} \left\| \frac{\omega(x)-\omega(y)}{\omega(y)} \right\| \to 0 \quad \text{for } j\to\infty,$$

since

$$\frac{\omega_S}{\omega_{S'}} = \frac{\int_S \omega}{\int_{S'}\omega}\frac{\mu(S')}{\mu(S)} = \frac{\int_S \frac{\omega}{\omega(y)}}{\int_{S'}\frac{\omega}{\omega(y)}}\frac{\mu(S')}{\mu(S)} \leq \frac{(1+E_{j,\omega})\int_S 1}{(1-E_{j,\omega})\int_{S'}1}\frac{\mu(S')}{\mu(S)} = \frac{(1+E_{j,\omega})}{(1-E_{j,\omega})}.$$

Furthermore, *any* piecewise constant coefficient function allows for the determination of j_0 independent of the number of jumps, their frequency and their location. Note that the estimate does *not* rely on any specific knowledge about the "resolution" of the jumps on a particular level or grid. Only the height of the maximal jump is relevant for j_0. Let ϵ^{-1} denote the height of the maximal jump, then choosing j_0 such that $\ln(\epsilon^{-1/2}) < j_0$ is sufficient.

Let us now consider $\omega \in A_\infty$, since we already required ω to be in the Muckenhoupt class in Lemma 2.2 to establish the norm equivalence between $\|\cdot\|_\omega$ and $\|\cdot\|_{j,\omega}$ in Theorem 2.4.

A function ω is in $A_\infty(\Omega)$ if and only if

$$\frac{1}{\mu(B)}\int_B \omega dx \exp\Big(\frac{1}{\mu(B)}\int_B \ln(\frac{1}{\omega})dx\Big) \le A < \infty$$

holds for all balls $B \subset \Omega$. Another characterization is given by: A weight ω is in A_∞ if and only if for all balls B and all $F \subset B$ with $\mu(F) \ge \alpha\,\mu(B)$ there exists a constant $\beta \in (0,1)$ such that

$$\int_F \omega \ge \beta \int_B \omega \tag{3.33}$$

holds. Using this property, we obtain that any such weight fulfills our limited growth condition automatically. Consider S, $S' \subset U$, since we employ uniformly refined triangulations, we have $\mu(S') \approx \mu(S) \ge \mu(U)/16$. Now, let B denote the smallest ball which completely contains U, i.e. $U \subset B$, then the estimate

$$\mu(S') \approx \mu(S) \ge C_\infty\,\mu(B)$$

with C_∞ depends on the initial triangulation $\mathcal{T}_0$ only. Hence, with the characterization (3.33) given above we obtain

$$\int_{S'} \omega \ge \beta \int_B \omega$$

and therefore

$$\frac{\omega_S}{\omega_{S'}} \le \frac{\int_B \omega}{\beta \int_B \omega}\frac{\mu(S')}{\mu(S)} \le \frac{C_2}{\beta} := \mathcal{C}_{\infty,\omega}. \tag{3.34}$$

Therefore, all quotients $\frac{\omega_S}{\omega_{S'}}$ are bounded independent of the level j by $\mathcal{C}_{\infty,\omega}$ for $\omega \in A_\infty$ and the results of Theorem 3.9 hold.

Remark 3.12. Note that there is a close connection between the Muckenhoupt class A_∞ with the space BMO via the implications

$$\omega \in A_p \subset A_\infty \qquad \Rightarrow \qquad \ln(\omega) \in \mathrm{BMO}$$

and

$$f \in \mathrm{BMO} \qquad \Rightarrow \qquad f = c\ln\omega \qquad \text{with } \omega \in A_\infty.$$

Let us summarize our results in the following theorem.

Theorem 3.13. *Let the bilinear form $a(\cdot,\cdot)$ fulfill Assumption 1 and consider a sequence of uniformly refined triangulations $\mathcal{T}_j$ and the respective sequence of nested spaces $\mathcal{V}_j$ of linear finite elements. Furthermore, let us consider $\omega \in A_\infty$. Then $C_{j,\omega} < \mathcal{C}_{\infty,\omega}$ holds for all j. Hence, there exists a minimal refinement level j_0 independent of J such that the limited growth condition*

$$2^j > C\ \sqrt{\mathcal{C}_{\infty,\omega}} \tag{3.35}$$

is fulfilled for all $j \ge j_0$. Therefore, the equivalencies

$$a(u,u) \le \sum_{j=0}^{j_0-1} a(v_j,v_j) + C_3 \sum_{j=j_0}^{J} 2^{2j}\|u_j\|^2_{j,\omega} \le C_4\ a(u,u) \tag{3.36}$$

and

$$a(u,u) \le \sum_{j=0}^{j_0-1} a(v_j,v_j) + C_5 \sum_{j=j_0}^{J} 2^{2j}\|u_j\|^2_{\omega} \le C_6 a(u,u) \tag{3.37}$$

hold with the constants

$$C_3 := \frac{32}{3} C_2, \qquad C_4 := C_3 \; 4 \; (1 + 2\gamma^{-1} C_0)^2 (1 + \sqrt{\frac{3}{2} C_0})^2 C \; \mathcal{C}_{\infty,\omega} \tag{3.38}$$

and

$$C_5 := C_3 \; (1 + s\gamma^{-1} C_0)^2, \qquad C_6 := C_5 \; C \; \mathcal{C}_{\infty,\omega}, \tag{3.39}$$

where C_0, C_1, and C_2 depend on the initial triangulation $\mathcal{T}_0$ only and γ is a constant determined by the A_∞ property of the coefficient function ω.

Let us now consider the remainder terms for the coarser levels $j = 0, \ldots, j_0 - 1$. Since j_0 is independent of J it is sufficient to establish an equivalence which explicitly involves j_0 (compare Theorem 3.2). Hence, we can employ a similar approach as in [23]. To this end, let us introduce the averaging operators M_j similar to [23]. However, our averaging operators M_j are based on the weighted scalar product $\langle \cdot, \cdot \rangle_\omega$, namely we use

$$M_j^\omega u := \sum_{i=1}^{n_j} \frac{\langle u, \psi_i^j \rangle_\omega}{\langle 1, \psi_i^j \rangle_\omega} \; \psi_i^j \tag{3.40}$$

where ψ_i^j denotes the piecewise linear finite element basis function at vertex i on level j and $n_j = \#\mathcal{N}_j$ denotes the number of vertices in the triangulation $\mathcal{T}_j$. Similarly as in the proof of Lemma 3.7 we obtain

$$||u_j||_\omega \leq ||u - M_{j-1}^\omega v||_\omega. \tag{3.41}$$

Omitting the level superscript j for the basis functions, we have on each triangle $T \in \mathcal{T}_j$ the equivalence

$$M_j^\omega u\Big|_T = \sum_{\nu=1}^{3} \frac{\langle u, \psi_{i(T,\nu)} \rangle_\omega}{\langle 1, \psi_{i(T,\nu)} \rangle_\omega} \; \psi_{i(T,\nu)}$$

where $i(T,\nu)$ indicates the respective vertex of T. For the ease of notation we use ψ_ν for $\psi_{i(T,\nu)}$ in the following. With this local formulation we directly obtain the estimate

$$||M_j^\omega u||_{\omega,T}^2 \leq \Big(\sum_{\nu=1}^{3} \frac{|\langle u, \psi_\nu \rangle_\omega|}{\langle 1, \psi_\nu \rangle_\omega} \; ||\psi_\nu||_{\omega,T} \Big)^2 \leq 3 \; \sum_{\nu=1}^{3} |\langle u, \psi_\nu \rangle_\omega|^2 \frac{||\psi_\nu||_{\omega,T}^2}{\langle 1, \psi_\nu \rangle_\omega^2}.$$

With $(\psi(x))^2 \leq \psi(x)$ for all basis functions ψ we obtain

$$\frac{||\psi_\nu||_{\omega,T}^2}{\langle 1, \psi_\nu \rangle_\omega^2} = \frac{\int_T \omega(x) |\psi_\nu(x)|^2 \; dx}{\Big(\int_{\mathrm{supp}(\psi_\nu)} \omega(x) |\psi_\nu(x)| \; dx \Big)^2} \leq \Big(\int_{\mathrm{supp}(\psi_\nu)} \omega(x) |\psi_\nu(x)| \; dx \Big)^{-1}.$$

Hence, we have the local boundedness of the averaging operators M_j^ω with the estimate

$$\begin{aligned} |\langle u, \psi_\nu \rangle_\omega|^2 &= \Big| \int_{\mathrm{supp}(\psi_\nu)} u(x) \omega(x) \psi_\nu(x) \; dx \Big|^2 \\ &= \Big| \int_{\mathrm{supp}(\psi_\nu)} \big(u(x) \sqrt{\omega(x) \psi_\nu(x)} \big) \sqrt{\omega(x) \psi_\nu(x)} \; dx \Big|^2 \\ &\leq \int_{\mathrm{supp}(\psi_\nu)} \omega(x) \psi_\nu(x) \; dx \int_{\mathrm{supp}(\psi_\nu)} |u(x)|^2 \omega(x) \psi_\nu(x) \; dx. \end{aligned}$$

Lemma 3.14. *For any $T \in \mathcal{T}_j$ let $U = \bigcup_{\nu=1}^{3} \mathrm{supp}(\psi_\nu)$ be the union of the supports of the three nodal basis functions $\psi_\nu = \psi_{i(T,\nu)}$ with $T \subset \mathrm{supp}(\psi_\nu)$. Then the estimate*

$$\|M_j^\omega u\|_{\omega,T}^2 \leq 3 \sum_{\nu=1}^{3} \int_{\mathrm{supp}(\psi_\nu)} \omega(x)|u(x)|^2 \psi_\nu(x)dx \leq 3 \int_U \omega(x)|u(x)|^2 dx.$$

holds for any $u \in \mathcal{V}$ so that

$$\|M_j^\omega u\|_{\omega,T}^2 \leq 3 \ \|u\|_{\omega,U}^2. \tag{3.42}$$

Since M_j^ω preserves constants we obtain the estimate

$$||M_j^\omega u - u||_{\omega,T} \leq ||M_j^\omega (u - a_T)||_{\omega,T} + ||u - a_T||_{\omega,T} \leq (1+\sqrt{3}) \ ||(u - a_T)||_{\omega,U} \tag{3.43}$$

with the help of this lemma. Now we use the Sobolev inequality in $\mathbb{R}^2$, i.e., the Taylor-remainder formula in integral form, which reads

$$u(x) = \int_U \varphi(y)u(y)dy \ + \ R^1 u(x) = \int_U \varphi(y)u(y)dy \ + \sum_{|\alpha|=1} \int_U K_\alpha(x,y)u^{(\alpha)}(y)dy$$

where φ is a smooth function with support in a ball $B \subset U$ and $\int \varphi(x)dx = 1$, and the kernels K_α are given by

$$K_\alpha(x,y) = (x-y)^\alpha \int_0^1 s^{-3} \varphi(x + s^{-1}(y-x))ds.$$

One can show that the estimate

$$\left| \int_U K_\alpha(x,y)u^{(\alpha)}(y)dy \right| \leq C \frac{(\mathrm{diam}(U))^2}{\delta^2} \int_U |u^{(\alpha)}(z)|(x-z)^\alpha| \ |x-z|^{-2} dz$$

holds, with $C_K = C\frac{(\mathrm{diam}(U))^2}{\delta^2}$ depending only on the initial triangulation $\mathcal{T}_0$, where C is an absolute constant independent of u, and δ is the radius of the ball B. Thus, choosing $a_T = \int_U \varphi(y)u(y)dy$ we obtain the estimate

$$\|u(x) - a_T\|_{\omega,U}^2 \ \leq C_K (\mathrm{diam}(U))^2 \int_U \omega(x) \left| \sum_{|\alpha|=1} \int_U |u^{(\alpha)}(z)||x-z|^{-2} dz \right|^2 dx.$$

Now we introduce the kernel $G(x) := |x|^{-2}$ and rewrite the right-hand side as a singular integral with the notation $(u^{(\alpha)} *_U G)(x) = \int_U u^{(\alpha)}(z)G(x-z) \ dz$

$$\|u(x) - a_T\|_{\omega,U}^2 \ \leq C_K (\mathrm{diam}(U))^2 \sum_{|\alpha|=1} \int_U \omega(x) \big|(u^{(\alpha)} *_U G)(x)\big|^2 dx.$$

Then, using a result from harmonic analysis [18] for weights $\omega \in A_2$ (a sub-class of the Muckenhoupt class A_∞) we obtain

$$\int_U \omega(x) \big|(u^{(\alpha)} *_U G)(x)\big|^2 dx \ \leq \ \mathcal{C}_{2,\omega} \int_U \omega(x)|u^{(\alpha)}|^2 dx$$

with $\mathcal{C}_{2,\omega}$ depending on ω only. Hence, with $\alpha = 1$ and (3.43) we end up with

$$\|M_j^\omega u\|_{\omega,T} \leq (1+\sqrt{3}) \ ||(u - a_T)||_{\omega,U} \leq \mathcal{C}_{2,\omega} \ C_K (\mathrm{diam}(U))^2 \|\nabla u\|_{\omega,U}^2. \tag{3.44}$$

Finally, we choose $v = M_j^\omega u$ in (3.41) and we obtain

$$\|u_j\|_{j,\omega}^2 \leq 4 \ \|M_j^\omega u - u\|_{j,\omega}^2 \leq \mathcal{C}_{2,\omega} \ 2^{-2j} a(u,u)$$

where we have used $\|\nabla u\|_\omega^2 = a(u,u)$ after summation with respect to U, i.e. $T \in \mathcal{T}_j$.

Recall that we can determine j_0 for $\omega \in A_\infty$ due to the limited growth condition

$$2^j > C \sqrt{\mathcal{C}_{\infty,\omega}}$$

for all $j \geq j_0$ and we obtain

$$j_0 := \ln(C \sqrt{\mathcal{C}_{\infty,\omega}}).$$

Altogether, we can now establish our robust and optimal norm equivalencies in the following theorem.

Theorem 3.15. *Let the bilinear form $a(\cdot,\cdot)$ fulfill Assumption 1 and consider a sequence of uniformly refined triangulations $\mathcal{T}_j$ and the respective sequence of nested spaces $\mathcal{V}_j$ of linear finite elements. Furthermore, let us consider $\omega \in A_2 \subset A_\infty$. Then the equivalence*

$$a(u,u) \leq C_3 \sum_{j=0}^{J} 2^{2j} \|u_j\|_{j,\omega}^2 \leq C_7 \; a(u,u) \tag{3.45}$$

where

$$C_7 := C_3 \; (\ln(C \sqrt{\mathcal{C}_{\infty,\omega}})\mathcal{C}_{2,\omega} + 4 \; (1 + 2\gamma^{-1}C_0)^2 (1 + \sqrt{\frac{3}{2}C_0})^2 \; C \; \mathcal{C}_{\infty,\omega})$$

γ, $\mathcal{C}_{2,\omega}$ and $\mathcal{C}_{\infty,\omega}$ depend on ω only, C_0 and C_3 depend on the initial triangulation $\mathcal{T}_0$ only. With respect to the $\|\cdot\|_\omega$ norm we have the equivalence

$$a(u,u) \leq C_3 \; (1 + 2\gamma^{-1}C_0)^2 \sum_{j=0}^{J} 2^{2j} \|u_j\|_\omega^2 \leq C_8 \; a(u,u) \tag{3.46}$$

where

$$C_8 := C_3 \; (1 + 2\gamma^{-1}C_0)^2 (\ln(C \sqrt{\mathcal{C}_{\infty,\omega}})\mathcal{C}_{2,\omega}(1 + \sqrt{\frac{3}{2}C_0})^2 + C \; \mathcal{C}_{\infty,\omega}).$$

4. Robust and Optimal Preconditioners

Let us now focus on the design of appropriate preconditioners based on the established norm equivalencies. Here, however, we can derive a simple preconditioner based on the $\|\cdot\|_\omega$ norm only. The following standard construction for BPX-type preconditioners B [3, 22] does not work for the level-dependent norms $\|\cdot\|_{j,\omega}$.

Let us define the preconditioner B via

$$\langle u, Bu \rangle_\omega = a(Q_0^\omega u, Q_0^\omega u) + \sum_{j=1}^{J} 2^{2j} \|u_j\|_\omega^2. \tag{4.1}$$

Using the orthogonality property $\langle u_j, u_l \rangle_\omega = \delta_{j,l}$ for $u_j = (Q_j^\omega - Q_{j-1}^\omega)u$ we obtain the operator formulation

$$B = A_0 Q_0^\omega + \sum_{j=1}^{J} 2^{2j} (Q_j^\omega - Q_{j-1}^\omega).$$

Similarly, we obtain the inverse operator

$$B^{-1} = A_0^{-1} Q_0^\omega + \sum_{j=1}^{J} 2^{-2j} (Q_j^\omega - Q_{j-1}^\omega).$$

Note that we can rewrite the sum of the differences of the projections also as a simple sum of the Q_j^ω, i.e.

$$\begin{aligned}\sum_{j=1}^{J} 2^{-2j} (Q_j^\omega - Q_{j-1}^\omega) &= 2^{-2J} Q_J^\omega - 2^{-2} Q_0^\omega + \sum_{j=1}^{J-1} 2^{-2j} Q_j^\omega - \sum_{j=1}^{J-1} 2^{-2j-2} Q_j^\omega \\ &= 2^{-2J} Q_J^\omega - 2^{-2} Q_0^\omega + \sum_{j=1}^{J-1} (1 - 2^{-2}) 2^{-2j} Q_j^\omega.\end{aligned}$$

With this we obtain the alternative representation of the operator B^{-1} as

$$B^{-1} = (A_0^{-1} - 2^{-2}) Q_0^\omega + \sum_{j=1}^{J-1} (1 - 2^{-2}) 2^{-2j} Q_j^\omega + 2^{-2J} Q_J^\omega.$$

Therefore, it is clear that the equivalent operator $\tilde{B}^{-1}$

$$\tilde{B}^{-1} = A_0^{-1} Q_0^\omega + \sum_{j=1}^{J} 2^{-2j} Q_j^\omega$$

can also be used as a preconditioner. However, the orthogonal projections Q_j^ω are rather expensive to compute. Instead we will employ the spectrally equivalent averaging operators (3.40)

$$M_j^\omega u = \sum_{i=1}^{n_j} \frac{\langle u, \psi_i^j \rangle_\omega}{\langle 1, \psi_i^j \rangle_\omega} \; \psi_i^j.$$

where $n_j = \#\mathcal{N}_j$ denotes the number of vertices of the triangulation $\mathcal{T}_j$. The operators M_j^ω satisfy

$$\gamma_0 \; \langle u, M_j^\omega u \rangle_\omega \leq \langle u, Q_j^\omega u \rangle_\omega \leq \gamma_1 \; \langle u, M_j^\omega u \rangle_\omega \;\; \text{uniformly in} \;\; j. \tag{4.2}$$

Finally, we obtain the substantially less expensive preconditioner $\hat{B}$ with

$$\hat{B}^{-1} = A_0^{-1} Q_0^\omega + \sum_{j=1}^{J} 2^{-2j} \sum_{i=1}^{n_j} \frac{\langle \cdot, \psi_i^j \rangle_\omega}{\langle 1, \psi_i^j \rangle_\omega} \; \psi_i^j.$$

It remains to show the spectral equivalence (4.2) which we obtain from the following lemma.

Lemma 4.1. *Let the weight ω but such that (2.13) is fulfilled. Then the piecewise linear finite element functions $\psi_l^{(j)}$ form a Riesz-basis for the spaces $\mathcal{V}_j$ in the weighted norm $\|\cdot\|_\omega$ arising from the weighted scalar product (2.11); i.e., there holds the estimate*

$$A \sum_{l=1}^{n_j} \langle 1, \psi_l^{(j)} \rangle_\omega |\alpha_l|^2 \;\leq\; \| \sum_{l=1}^{n_j} \alpha_l \psi_l^{(j)} \|_\omega^2 \leq\; B \sum_{l=1}^{n_j} \langle 1, \psi_l^{(j)} \rangle_\omega \; |\alpha_l|^2 \tag{4.3}$$

with constants A and B independent of the level j.

Proof. The inequality on the right-hand side follows from

$$\| \sum_{l=1}^{n_j} \alpha_l \psi_l^{(j)} \|_\omega^2 \le \left\| \Big(\sum_{l=1}^{n_j} |\alpha_l|^2 \psi_l^{(j)} \Big)^{1/2} \right\|_\omega^2 = \sum_{l \in \mathcal{N}_j} |\alpha_l|^2 \langle 1, \psi_l^{(j)} \rangle_\omega$$

With the help of Theorem 2.4 we obtain

$$\begin{aligned} \| \sum_{l=1}^{n_j} \alpha_l \psi_l^{(j)} \|_\omega^2 &\ge (1 + 2\, \gamma^{-1}\, C_0)\, \| \sum_{l=1}^{n_j} \alpha_l \psi_l^{(j)} \|_{j,\omega}^2 \\ &= (1 + 2\, \gamma^{-1}\, C_0) \sum_{T \in \mathcal{T}_j} \omega_T \int_T \Big| \sum_{\nu=1}^{3} \alpha_{\nu,T} \psi_{\nu,T} \Big|^2 \end{aligned}$$

for the inequality on the left-hand side, where the $\psi_{\nu,T}$ denote the three nodal functions of level j with $\operatorname{supp}(\psi_{\nu,T}) \supset T$. By the well-known stability properties of the $\psi_l^{(j)}$ there holds (cf. [23])

$$\int_T \Big| \sum_{\nu=1}^{3} \alpha_{\nu,T} \psi_{\nu,T}^{(j)} \Big|^2 \ \ge\ \sum_{\nu=1}^{3} |\alpha_{\nu,T}|^2 \langle 1, \psi_{\nu,T} \rangle_T .$$

Since the support of each $\psi_i^{(j)}$ consists only of a finite number of triangles it follows further that

$$\begin{aligned} \| \sum_{l=1}^{n_j} \alpha_l \psi_l^{(j)} \|_\omega^2 &\ge (1 + 2\, \gamma^{-1}\, C_0) \sum_{T \in \mathcal{T}_j} \omega_T \sum_{\nu=1}^{3} |\alpha_{\nu,T}|^2 \langle 1, \psi_{\nu,T} \rangle_T \\ &\ge C (1 + 2\, \gamma^{-1}\, C_0) \sum_{l=1}^{n_j} |\alpha_l|^2 \langle 1, \psi_l^{(j)} \rangle_{j,\omega}, \end{aligned} \tag{4.4}$$

with an absolute constant C. Now, with the L_1-counterpart of Theorem 2.4 we obtain the estimate

$$\| \sum_{l=1}^{n_j} \alpha_l \psi_l^{(j)} \|_\omega^2 \ge \bar{C} \sum_{l=1}^{n_j} |\alpha_l|^2 \langle 1, \psi_l^{(j)} \rangle_\omega .$$

□

Note that this equivalence involves the same constants as Theorem 2.4, i.e., involves some information on the coefficient function ω and holds for $\omega \in A_\infty$. With the help of this lemma we obtain (4.2) by the following consideration, see [23]. With $Q_j^\omega u = \alpha_l \psi_l^{(j)}$ we have the equivalence

$$\langle u, Q_j^\omega u \rangle_\omega = \langle Q_j^\omega u, Q_j^\omega u \rangle_\omega = \langle \vec{\alpha}, G_j \vec{\alpha} \rangle_{\mathbb{R}^{n_j}}$$

where $G_j := (\langle \psi_l^{(j)}, \psi_k^{(j)} \rangle_\omega)_{l,k}$ denotes the Gram matrix. Now observe that with $D_j := \operatorname{diag}(\langle 1, \psi_l^{(j)} \rangle_\omega)$ and $\hat{u} := Q_j^\omega u$, we also have

$$\langle G_j \vec{\alpha}, D_j^{-1} G_j \vec{\alpha} \rangle_{\mathbb{R}^{n_j}} = \langle \hat{u}, M_j^\omega \hat{u} \rangle_\omega = \langle Q_j^\omega u, M_j^\omega Q_j^\omega u \rangle_\omega = \langle Q_j^\omega u, M_j^\omega u \rangle_\omega = \langle u, M_j^\omega u \rangle_\omega .$$

since Q_j^ω is an orthogonal projection. Finally, with the notation $\vec{\beta} := G_j \vec{\alpha}$ and Lemma 4.1 we obtain

$$\langle \vec{\alpha}, G_j \vec{\alpha} \rangle_{\mathbb{R}^{n_j}} = \langle G_j^{-1} \vec{\beta}, \vec{\beta} \rangle_{\mathbb{R}^{n_j}} \asymp \langle \vec{\beta}, D_j^{-1} \vec{\beta} \rangle_{\mathbb{R}^{n_j}} = \langle G_j \vec{\alpha}, D_j^{-1} G_j \vec{\alpha} \rangle_{\mathbb{R}^{n_j}}$$

which is only the discrete form of (4.2). This completes the construction of our preconditioner based on the $\| \cdot \|_\omega$ norm.

Remark 4.2. Note, however, that this preconditioner still involves the weights ω_T with $T \in \mathcal{T}_J$ for all levels j. Hence, the evaluation of the $\langle 1, \psi_l^{(j)} \rangle_\omega$ terms is not trivial.

The development of a preconditioner based on the level-dependent norms $\| \cdot \|_{j,\omega}$ which should give a superior performance, i.e., should lead to a smaller condition number, is an open problem. Already the definition of a simple linear operator $\mathcal{B}$ similar to (4.1) is not trivial since the terms on the right-hand side now involve the level-dependent norms $\| \cdot \|_{j,\omega}$ and the orthogonality property holds for the norm $\| \cdot \|_\omega$ on the finest level only. The derivation of such a preconditioner is subject of current research and will be addressed in a forthcoming paper.

5. Concluding Remarks

We presented two optimal and (almost) robust norm equivalencies based on certain weighted norms for diffusion problems $-\nabla \omega \nabla u = f$ in two space dimensions with a scalar diffusion coefficient ω. We only require ω to be in the Muckenhoupt class $A_2 \subset A_\infty$ to obtain our optimal bounds which involve some information on the boundesness of the local variation of the coefficient function ω. This covers all piecewise constant functions independent of the location of jumps, their number or their frequency. This is in contrast to previous results, which require the resolution of the jumps on a particular level, i.e. the coarsest level.

Based on the weighted norm $\| \cdot \|_\omega$ which involves the evaluation of the piecewise constant approximation of the weights ω_T with respect to the finest triangulation $\mathcal{T}_J$ we have furthermore presented a simple and (nearly) robust BPX-type preconditioner. The design of a preconditioner based on the more sophisticated level-dependent norms $\| \cdot \|_{j,\omega}$ is more involved and subject of current research.

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