# TENSOR PRODUCTS OF SOBOLEV SPACES AND APPLICATIONS

REINHARD HOCHMUTH*, STEPHAN KNAPEK§, GERHARD ZUMBUSCH§

**Abstract.** In many cases the approximation of solutions to variational problems involving isotropic Sobolev spaces has a complexity which depends exponentially on the dimension. However, if the solutions possess dominating mixed derivatives one can find discretizations to the corresponding variational problems with a lower complexity – sometimes even independent of the dimension. In order to analyse these effects, we relate tensor products of Sobolev spaces with spaces with dominating mixed derivatives. Based on these considerations we construct families of finite dimensional anisotropic approximation spaces which generalize in particular sparse grids. The obtained estimates demonstrate, in which cases a complexity independent or nearly independent of the dimension can be expected. Finally numerical experiments demonstrate the usefulness of the suggested approximation spaces.

**Key words.** tensor product spaces, dominating mixed derivatives, best approximation, variational problems, sparse grids

**AMS subject classifications.** 41A65, 41A63, 65N15, 65N30, 65T10

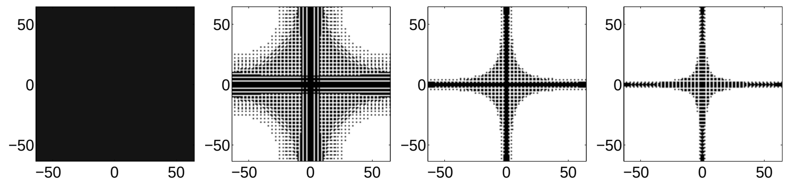
**1. Introduction.** A fundamental issue for the construction of a finite element method is the choice of a suitable approximation space. It is well known that for a large range of problems there exist trial and test functions such that a finite element discretization leads to stiffness matrices whose condition number is bounded independently of the number of unknowns in the finite element approximation space [5, 12, 26, 27, 40]. Hence, iterative methods can be expected to converge rapidly. However, on a regular full grid in $n$ dimensions the resulting linear system is of dimension $2^{Jn}$ ($J$ level in a multiscale discretization) and is therefore intractable for todays computers for large $J$ or large $n$. Here it is important to note that the complexity of many variational problems with solution in the standard isotropic Sobolev spaces is indeed exponentially depended on the dimension $n$ [34, 39].

However, the situation changes when we consider variational problems with solutions that posses a higher regularity such as functions with dominating mixed derivatives. Under this assumption and if one uses tensor-products of suitable univariate spaces as approximation spaces, subspaces with relatively large dimensions that contribute only "little" to the error reduction can be identified and omitted from the approximation spaces. This idea has appeared under various names (boolean blending schemes, hyperbolic crosses, sparse grids) in approximation and interpolation theory [2, 3, 4, 16, 33, 37] and has also been used for the solution of elliptic differential equations [7, 8, 10, 18, 41] as well as integral equations [15, 17, 20], see also [24].

A first impression of the usefulness of such sparse grid constructions and their range of applicability is obtained from the decay of the Fourier coefficients of the function $f = \chi_{(1/4,3/4)^2}$ on the torus $T^2$, where $\chi$ denotes the characteristic function. Notice, that the function $f$ is an element of $\mathcal{H}^{1/2-\epsilon}_{\text{mix}}$ for all $\epsilon > 0$ (see section 1 for a definition of these function spaces). The following figures show all those Fourier coefficients $c_{\mathbf{k}} = \sum_{j_1=0}^{2^7-1} \sum_{j_2=0}^{2^7-1} f(\frac{j_1}{2^7}, \frac{j_2}{2^7}) \exp(-2\pi i \langle \mathbf{j}, \mathbf{k}\rangle / 2^7)$ in black that have an absolute value larger than $0, 10^{-4}, 5 \cdot 10^{-4}, 10^{-3}$ (from left to right)

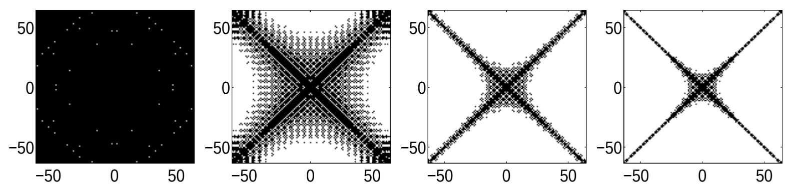
*Institut für Mathematik I, Freie Universität Berlin, Arnimallee 2-6, D-14195 Berlin, hochmuth@math.fu-berlin.de

§Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn, {knapek, zumbusch}@iam.uni-bonn.de




Stronger compression, i.e. a larger threshhold, leads to the typical cross structure for the coefficients of functions with dominating mixed smoothness. Sparse grids are specially adapted to this particular situation.

In contrast, for the $\pi/4$ around $(1/2, 1/2)$ rotated cube on $T^2$, which gives an element in $\mathcal{H}^{1/2-\epsilon}$ for all $\epsilon > 0$, an analogous compression of the discrete Fourier coefficients leads to the Figures




Hence, sparse grids will not outperform full grids when it comes to the approximation of this function. In other words, a simple rotation of the coordinate system could lead to very different approximation results if the mixed smoothness property changes.

The construction of suitable tensor product approximation spaces that exploit additional smoothness assumptions such as dominating mixed smoothness, can be based on norm equivalences with respect to tensor product bases functions, see [17]. For the application of this construction to e.g. elliptic boundary value problems it is therefore essential to identify circumstances under which their variational solutions are elements of such product spaces. To reach this goal we identify the arising tensor product Sobolev spaces as intersections of the energy space of the underlying variational problem with Sobolev spaces of functions with dominating mixed derivatives. In principle, this provides a feasible criterion whether the approximation results are applicable. A general regularity theory for elliptic boundary value problems with sufficient conditions on the data which enforces the desired mixed smoothness properties is postponed to a forthcoming paper. However, in a specific case, which is related to our numerical experiments, we trace back mixed regularity properties of the variational solution to related assumptions for the given data.

We proceed as follows: In section 2 we study tensor products of approximation spaces and Sobolev spaces. First we describe one-dimensional stable splittings and introduce related tensor products. By this way we may introduce approximation spaces for spaces of functions with different mixed smoothness properties. The approximation spaces give rise to a family of spaces starting from regular full grid spaces and including the sparse grid spaces. Then we investigate relations between tensor product Sobolev spaces and so called mixed smoothness spaces. Sobolev spaces of mixed type will disclose themselves as tensor product Sobolev spaces. The starting point are Sobolev spaces on $\mathbb{R}$. Then we consider intervals and finally intersections of Sobolev spaces with and without boundary conditions.

In section 3, we transfer results from [17] and obtain error estimates for specific approximation spaces with respect to tensor product Sobolev spaces. Next, complexity issues are discussed for different parameters of isotropic and mixed smoothness. Then we apply the approximation results to Sobolev spaces of mixed type. In that way we obtain error estimates for appropriate regular solutions to the Poisson problem. We conclude this section with some remarks on the optimal complexity solving the resulting discrete equations. Specifically we improve a result of [9] on the complexity of the Poisson problem.

In section 4 we sketch an approximation algorithm for this family of spaces that is based on an extrapolation technique. Of course, the final decision for the usefulness of one of these approximation spaces for a particular problem will rely on numerical tests that take everything into account, i.e. the number of degrees of freedom, the algorithmic complexity and also coding issues. Therefore we present numerical examples for the Poisson problem in section 5. Test functions belonging to different spaces of the scale are considered in order to demonstrate the complexity and approximation errors within suitable tensor product spaces.

**2. Tensor products of approximation spaces and of Sobolev spaces.** The complexity estimates in the next section are essentially based on (Hilbert space) tensor product arguments providing norm-equivalences with respect to appropriate tensor product bases. Therefore we start in this section with the standard definition of the Hilbert space tensor product $\otimes$, see [1, 13, 38]. Then we introduce our tensor-product approximation spaces. From e.g. [17] it is already known how those approximation spaces are related to tensor products of Sobolev spaces. In subsection 2.2 we proceed by presenting characterizations for those tensor products of Sobolev spaces, that are connected with the variational solutions of elliptic boundary value problems.

For real Hilbert spaces $(H_1, (\cdot,\cdot)_{H_1})$, $(H_2, (\cdot,\cdot)_{H_2})$ and their dual spaces $H_1'$ and $H_2'$ we denote by $L(H_1', H_2'; \mathbb{R})$ the space of all bilinear forms on $H_1' \times H_2'$. Then for $u \in H_1$ and $v \in H_2$ the tensor product $u \otimes v$ is defined by

$$(u \otimes v)(x', y') := x'(u)\, y'(v), \quad (x', y') \in H_1' \times H_2',$$

i.e. $u \otimes v \in L(H_1', H_2'; \mathbb{R})$, and $H_1 \hat{\otimes} H_2$ is simply the subspace of $L(H_1', H_2'; \mathbb{R})$ generated by the dyads $u \otimes v$. $H_1 \hat{\otimes} H_2$ becomes a Pre–Hilbert space by introducing a positive sesquilinear form $s : (H_1 \otimes H_2) \times (H_1 \otimes H_2) \longrightarrow \mathbb{R}$: For $f, g \in H_1 \otimes H_2$ with $f = \sum_{j=1}^m c_j u_j \otimes v_j$, $g = \sum_{k=1}^l d_k w_k \otimes z_k$, where $c_j, d_k \in \mathbb{R}$, $u_j, w_k \in H_1$, $v_j, z_k \in H_2$, we set

$$s(f, g) := \sum_{j=1}^m \sum_{k=1}^l c_j d_k (u_j, w_k)_{H_1} (v_j, z_k)_{H_2}\,.$$

The tensor product space $H_1 \otimes H_2$ is then defined as the completion of $H_1 \hat{\otimes} H_2$ with respect to $s(\cdot,\cdot)$ and we denote the corresponding norm by $\|\cdot\|_\otimes$.
Within our context it is worth to note that $\|\cdot\|_\otimes$ is up to equivalences the only tensor cross norm with the property that for an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ in $\ell_2$ the system $(e_k \otimes e_\ell)_{k,\ell\in\mathbb{N}}$ is at least an unconditional basis in $\ell_2 \otimes \ell_2$.

The following well-known fact is very useful: For $H_1 = H_2 = L_2(\mathbb{R})$, the Hilbert space $H_1 \otimes H_2 = L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$ is isomorphic to $L_2(\mathbb{R}^2)$ and the sesquilinear form $s(\cdot,\cdot)$ on $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$ can be identified with the standard scalar product in $L_2(\mathbb{R}^2)$ (see [38]).

We use the following standard notation: $x \sim y$ means that there exist constants $C_1, C_2$ independent of any parameters $x$ or $y$ may depend on, such that $C_1 \cdot y \le x \le C_2 \cdot y$. $x \lesssim y$ is defined analogously. Multi-indices (vectors) are written boldface, for example $\mathbf{j} = (j_1, \ldots, j_n)$.

**2.1. Tensor product approximation spaces.** The starting point for the construction of our approximation spaces are one-dimensional splittings $L_2 = \bigoplus_{j\geq 0} S_j$, where the $S_j \subset H_0^r(I)$, $r > 0$ (where $I := [0,1]$ and $H_0^r(I)$ is defined in subsection 2.2), are finite-dimensional and the complement spaces $W_j = S_j \ominus S_{j-1}$ are spanned by some basis $\{\psi_{jk}, k = 1, \dots, \dim(W_j)\}$, consisting of dilates and translates of a mother function. In the following we work exclusively with dyadic refinements, i.e. the dimension of the univariate subspaces $S_j$ behaves like $2^j$. As a rule, the considered basis functions should satisfy a refinement relation, to allow efficient computations via pyramidal algorithms. The particular choice of basis function is dictated by the problem at hand, specifically by the desired stability properties of the resulting splittings, the size of the local masks and, connected with this, the cost in implementing and even the taste of the user.

In the higher-dimensional case, $n > 1$, let $W_{\mathbf{j}}$ denote the corresponding tensor-product space, i.e.

$$W_{\mathbf{j}} := W_{j_1} \otimes \ldots \otimes W_{j_n}.$$

The approximation spaces we consider are given by

$$V_J^T := \bigoplus_{\mathbf{j}\in I_J^T} W_{\mathbf{j}} \tag{2.1}$$

with $T \in [-\infty, 1]$ and the index sets

$$I_J^T := \{\mathbf{j} \in \mathbb{N}^n : -|\mathbf{j}|_1 + T|\mathbf{j}|_\infty \geq -(n+J-1) + TJ\}. \tag{2.2}$$

They have been introduced in [17]. Here, $J$ is the maximal level in $V_J^T$, i.e. $j_i \leq J, i = 1, \dots, n$ for $\mathbf{j} \in I_J^T$. With $T$ varying between $-\infty$ and 1 (2.1) gives a scale of approximation spaces that are nested with respect to the parameters $J$ and $T$ and include regular full as well as sparse grid approximation spaces.

From [14, 17] we take the following estimates of the dimension of the spaces $V_J^T$,

$$dim\left(V_J^T\right) \lesssim \begin{cases} 2^J & \text{for } 1/J < T \leq 1, \\ 2^J J^{n-1} & \text{for } 0 \leq T \leq 1/J, \\ 2^{\frac{T-1}{T/n-1}J} & \text{for } T < 0, \\ 2^{nJ} & \text{for } T = -\infty. \end{cases} \tag{2.3}$$

**2.2. Tensor products of Sobolev spaces.** The main goal of this subsection is to identify tensor products of (isotropic) Sobolev spaces as Sobolev spaces of functions with dominating mixed smoothness properties. To be exemplarily more precise we relate

$$\mathcal{H}^{\mathbf{r}}_{\text{mix}}(\mathbb{R}^2) := H^{r_1}(\mathbb{R}) \otimes H^{r_2}(\mathbb{R}),$$

to

$$H^{\mathbf{r}}_{\text{mix}}(\mathbb{R}^2) := \{u \in L_2(\mathbb{R}^n) \mid \|u\|_{H^{\mathbf{r}}_{\text{mix}}} := (\sum_{\mathbf{k}\in\mathbb{Z}^2} \prod_{i=1}^{2} (1+|k_i|)^{2r_i} |\mathcal{F}u(\mathbf{k})|^2)^{1/2} < \infty\},$$

where $\mathbf{r} = (r_1, r_2) \geq 0$, and the Sobolev spaces $H^{r_i}(\mathbb{R})$ are defined via the Fourier transform in the usual way. In particular we are interested in Sobolev spaces with incorporated homogeneous boundary conditions, which appear as regularity spaces for solutions to elliptic boundary value problems.

We shall apply several times the following basic observation, see [38]: *If $H_1$ and $H_2$ are two Hilbert spaces and if $M_1 \subset H_1$ and $M_2 \subset H_2$ are total subsets, i.e. linear combinations of elements of $M_i$ lie dense in $H_i$, then the set $\{u \otimes v \mid u \in M_1, v \in M_2\}$ is total in $M_1 \otimes M_2$.*

We proceed as follows: First we treat tensor products of Sobolev spaces on $\mathbb{R}$, then tensor products of Sobolev spaces without boundary conditions on an interval and, finally, intersections of those tensor products with isotropic Sobolev spaces with boundary conditions.

Let us start with Sobolev spaces on $\mathbb{R}$. To show

$$\mathcal{H}^{\mathbf{r}}_{\text{mix}}(\mathbb{R}^2) = H^{\mathbf{r}}_{\text{mix}}(\mathbb{R}^2),$$

note that $C_0^\infty(\mathbb{R})$ is dense in $H^{r_i}(\mathbb{R})$. Hence it is sufficient, because of the above observation, to show that for $u_i \in C_0^\infty(\mathbb{R}),\ v_i \in C_0^\infty(\mathbb{R}), i = 1, 2$, holds

$$s(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1 \otimes' u_2, v_1 \otimes' v_2), \tag{2.4}$$

where $u_1 \otimes' u_2(x_1, x_2) := u_1(x_1) \cdot u_2(x_2)$. Then $\otimes'$ induces an isometric isomorphism from $\mathcal{H}^{\mathbf{r}}_{\text{mix}}(\mathbb{R}^2)$ onto $H^{\mathbf{r}}_{\text{mix}}(\mathbb{R}^2)$. But (2.4) follows immediately from the definitions and the identification of $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$ with $L_2(\mathbb{R}^2)$.

Next we consider Sobolev spaces on the cube $I^2$. Within our context a convenient way to introduce Sobolev spaces on $I$ is to apply differences, i.e. to take into account the notion of Besov spaces, cf. [36]: For $h \in \mathbb{R}$ and a function $u$ we define

$$\Delta_h(u)(x) := u(x+h) - u(x), \quad x \in \mathbb{R},$$

and for $d \geq 2$ we define inductively $\Delta_h^d(u) := \Delta_h^{d-1}(\Delta_h(u))$. Furthermore we introduce for $h \in \mathbb{R}$ and $d \in \mathbb{N}$

$$I_{h,d} := \{x \in I \mid x + dh \in I\},$$

and define for $r > 0$ and an arbitrary but fixed $m \in \mathbb{N}$ with $m > r$

$$(u, v)_{r;I} = \int_{\mathbb{R}} \Big( \int_{I_{h,m}} |h|^{-2r-1} \big(\Delta_h^m(u)(x)\big)\big(\Delta_h^m(v)(x)\big) dx \Big) dh,$$

$((u, v))_{r;I} := (u, v)_{0;I} + (u, v)_{r;I},\ |u|_{r;I} := (u, u)_{r;I}^{1/2},\ |||u|||_{r;I} := ((u, u))_{r;I}^{1/2}$
and

$$H^r(I) := \{u \in L^2(I) \mid |||u|||_{r;I} < \infty\}.$$

Note that another $m > r$ would give an equivalent norm $|||\cdot|||_{r;I}$. Moreover, for $r \in \mathbb{N}$ it is well-known, see [36], that

$$H^r(I) = \{u \in L^2(I) \mid ||u||_{r,I} := (\sum_{\alpha=0}^{r} ||u^{(\alpha)}||_{0,I}^2)^{1/2} < \infty\}$$

and $|||u|||_{r,I} \sim ||u||_{r,I},\ u \in H^r(I)$.

Next we introduce directional partial difference operators $\Delta_{h,i}^m,\ m \in \mathbb{N},\ h \in \mathbb{R},\ i = 1, 2$, by $(\Delta_{h,i}^1 f)(x) := f(x + he_i) - f(x)$ and $\Delta_{h,i}^m := \Delta_{h,i}^1 \Delta_{h,i}^{m-1},\ m \geq 2$, where

$e_i$ denote the canonical basis vectors in $\mathbb{R}^2$. Applying those directional differences we set for $\mathbf{r} = (r_1, r_2) > 0$ and arbitrary but fixed $m_1, m_2 \in \mathbb{N}$ with $m_i > r_i$

$$(u,v)_{\substack{r_1,0;I^2\\mix}} := \int_{\mathbb{R}} \Big( \int_{I_{h_1,m_1}\times I} |h_1|^{-2r_1-1} (\Delta^{m_1}_{h_1,1} u(\mathbf{x}))(\Delta^{m_1}_{h_1,1} v(x))\, d\mathbf{x} \Big)\, dh_1,$$

$$(u,v)_{\substack{0,r_2;I^2\\mix}} := \int_{\mathbb{R}} \Big( \int_{I\times I_{h_2,m_2}} |h_2|^{-2r_2-1} (\Delta^{m_2}_{h_2,2} u(\mathbf{x}))(\Delta^{m_2}_{h_2,2} v(x))\, d\mathbf{x} \Big)\, dh_2,$$

$$(u,v)_{\substack{\mathbf{r};I^2\\mix}} := \int_{\mathbb{R}^2} \Big( \int_{I_{h_1,m_1}\times I_{h_2,m_2}} |h_1|^{-2r_1-1}|h_2|^{-2r_2-1} \big(\Delta^{m_1}_{h_1,1} \otimes \Delta^{m_2}_{h_2,2} u(\mathbf{x})\big) \big(\Delta^{m_1}_{h_1,1} \otimes \Delta^{m_2}_{h_2,2} v(x)\big)\, d\mathbf{x} \Big)\, dh_1\, dh_2,$$

$((u,v))_{\substack{r_1,0;I^2\\mix}} := (u,v)_{0;I^2} + (u,v)_{\substack{r_1,0;I^2\\mix}}$, $((u,v))_{\substack{0,r_2;I^2\\mix}} := (u,v)_{0;I^2} + (u,v)_{\substack{0,r_2;I^2\\mix}}$,
$((u,v))_{\substack{\mathbf{r};I^2\\mix}} := (u,v)_{0;I^2} + (u,v)_{\substack{r_1,0;I^2\\mix}} + (u,v)_{\substack{0,r_2;I^2\\mix}} + (u,v)_{\substack{\mathbf{r};I^2\\mix}}$,
$\|u\|_{\substack{r_1,0;I^2\\mix}} := ((u,u))^{1/2}_{\substack{r_1,0;I^2\\mix}}$, $\|u\|_{\substack{0,r_2;I^2\\mix}} := ((u,u))^{1/2}_{\substack{0,r_2;I^2\\mix}}$, $|||u|||_{\substack{\mathbf{r};I^2\\mix}} := ((u,u))^{1/2}_{\substack{\mathbf{r};I^2\\mix}}$
and

$$H^{\mathbf{r}}_{\text{mix}}(I^2) := \{u \in L^2(I^2) \mid |||u|||_{\substack{\mathbf{r};I^2\\mix}} < \infty\}.$$

Again, different $m_i > r_i$ give equivalent norms and the same function spaces. As in the isotropic case one has for $\mathbf{r} = (r_1, r_2) \in \mathbb{N}^2$

$$H^{\mathbf{r}}_{\text{mix}}(I^2) = \{u \in L^2(I^2) \mid \|u\|_{\substack{\mathbf{r};I^2\\mix}} := \Big( \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ |\alpha_1|\le r_1, |\alpha_2|\le r_2}} \|\partial^\alpha u\|^2_{0,I^2} \Big)^{1/2} < \infty\}$$

and $|||u|||_{\substack{\mathbf{r};I^2\\mix}} \sim \|u\|_{\substack{\mathbf{r};I^2\\mix}}$ for $u \in H^{\mathbf{r}}_{\text{mix}}(I^2)$, cf. [30]. The reason for our choice of Besov type definitions is the identity

$$(u_1 \otimes v_1, u_2 \otimes v_2)_{H^{r_1}(I)\otimes H^{r_2}(I)} = ((u_1 \otimes' v_1, u_2 \otimes' v_2))_{H^{\mathbf{r}}_{\text{mix}}},$$

for $u_1, u_2 \in C^\infty(\overline{I})$ and $v_1, v_2 \in C^\infty(\overline{I})$, i.e. there is again an isometry on dyads $u \otimes v$. Thus, applying the fact that $C^\infty(\overline{I}) \otimes C^\infty(\overline{I})$ is total in $H^{\mathbf{r}}_{\text{mix}}(I^2)$, see [35], the same arguments as above, together with the identification $L_2(I) \otimes L_2(I) = L_2(I^2)$, give

$$H^{r_1}(I) \otimes H^{r_2}(I) = H^{\mathbf{r}}_{\text{mix}}(I^2)\,.$$

For another approach in the case $\mathbf{r} \in \mathbb{N}^2$ see [1].

In the case of products of intervals a simpler approach via Fourier series would also work. The advantage of our argument is that it can verbatim be transfered to Sobolev spaces in higher dimensions and products of domains satisfying the uniform cone condition, see [23].

Next we investigate Sobolev spaces with homogeneous boundary conditions. To present the typical situation that arises in the investigation of elliptic Dirichlet problems, we introduce for $r > 0$ the Sobolev space $H^r_0(I)$ as the completion of $C^\infty_0(I)$ in $H^r(I)$ and for $\mathbf{r} = (r_1, r_2) > 0$ the mixed space $H^{\mathbf{r}}_{0,mix}(I^2)$ as the completion of $C^\infty_0(I^2)$ in $H^{\mathbf{r}}_{\text{mix}}(I^2)$.

Since $C_0^\infty(I) \otimes C_0^\infty(I)$ is total in $C_0^\infty(I^2)$ we may apply again the above arguments to obtain

$$H_0^{r_1}(I) \otimes H_0^{r_2}(I) = H_{0,mix}^{\mathbf{r}}(I^2).$$

A little bit more involved is the following identity which considers intersections between Sobolev spaces with and without boundary conditions.

THEOREM 1. *For* $\mathbf{r} = (r_1, r_2) \geq \mathbf{l} = (l_1, l_2) > 0$ *holds*

$$(H_0^{l_1}(I) \cap H^{r_1}(I)) \otimes (H_0^{l_2}(I) \cap H^{r_2}(I)) = H_{0,mix}^{\mathbf{l}}(I^2) \cap H_{\mathrm{mix}}^{\mathbf{r}}(I^2) .$$

**Proof.** At first we notice that "$\subset$" follows directly by our above observations. For the other direction we apply an approximation argument using wavelets. From results in e.g. [11] follows the existence of biorthogonal wavelet bases $\Psi^{(i)} = (\psi_{\lambda_i}^{(i)})_{\lambda_i \in J_i}$, $\widetilde{\Psi}^{(i)} = (\widetilde{\psi}_{\lambda_i}^{(i)})_{\lambda_i \in J_i}$, $i = 1, 2$, $J_i$ suitable index sets, for $H_0^1(I) \cap H^{r_i}(I)$, which provide norm–equivalences

$$|||u|||_{r_i;I} \sim \Big( \sum_{\lambda_i \in J_i} 2^{2r_i|\lambda_i|} \, |(u, \widetilde{\psi}_{\lambda_i}^{(i)})_{0;I}|^2 \Big)^{1/2}, \quad \text{for } u \in H_0^{l_i}(I) \cap H^{r_i}(I) . \tag{2.5}$$

Applying arguments from [19], we then obtain

$$\|u\|_{\otimes_{\mathbf{r}}^{\mathbf{l}}} \sim \Big( \sum_{\substack{\lambda_1 \in J_1 \\ \lambda_2 \in J_2}} 2^{2(r_1|\lambda_1| + r_2|\lambda_2|)} |(u, \widetilde{\psi}_{\lambda_1}^{(1)} \otimes \widetilde{\psi}_{\lambda_2}^{(2)})_{0;I^2}|^2 \Big)^{1/2} \tag{2.6}$$

for $u \in (H_0^{l_1}(I) \cap H^{r_1}(I)) \otimes (H_0^{l_2}(I) \cap H^{r_2}(I))$. Here $\|\cdot\|_{\otimes_{\mathbf{r}}^{\mathbf{l}}}$ denotes the tensor product norm corresponding to $(H_0^{l_1}(I) \cap H^{r_1}(I)) \otimes (H_0^{l_2}(I) \cap H^{r_2}(I))$. Therefore it is sufficient to show that the right hand side in (2.6) is bounded for $u \in H_{0,mix}^{\mathbf{l}}(I^2) \cap H_{\mathrm{mix}}^{\mathbf{r}}(I^2)$.

To this end let $u \in H_{0,mix}^{\mathbf{l}}(I^2) \cap H_{\mathrm{mix}}^{\mathbf{r}}(I^2)$. For an arbitrary $\lambda_1 \in J_1$ the function $u_{\lambda_1}$ defined by

$$u_{\lambda_1}(x_2) := \int_I u(x_1, x_2) \, \tilde{\psi}_{\lambda_1}^{(1)}(x_1) \, dx_1 , \quad x_2 \in I ,$$

is an element of $H_0^{l_2}(I) \cap H^{r_2}(I)$. Then the Fubini theorem implies

$$(u_{\lambda_1}, \widetilde{\psi}_{\lambda_2}^{(2)})_{0;I} = (u, \widetilde{\psi}_{\lambda_1}^{(1)} \otimes \widetilde{\psi}_{\lambda_2}^{(2)})_{0;I^2}$$

for $\lambda_2 \in J_2$. From our assumptions on $\Psi^{(2)}$ and $\widetilde{\Psi}^{(2)}$ we then have

$$\begin{aligned} \sum_{\substack{\lambda_1 \in J_1 \\ \lambda_2 \in J_2}} 2^{2(r_1|\lambda_1| + r_2|\lambda_2|)} |(u, \widetilde{\psi}_{\lambda_1}^{(1)} \otimes \widetilde{\psi}_{\lambda_2}^{(2)})_{0;I^2}|^2 &= \sum_{\lambda_1 \in J_1} 2^{2r_1|\lambda_1|} \sum_{\lambda_2 \in J_2} 2^{2r_2|\lambda_2|} |(u_{\lambda_1}, \widetilde{\psi}_{\lambda_2}^{(2)})_{0;I}|^2 \\ &\lesssim \sum_{\lambda_1 \in J_1} 2^{2r_1|\lambda_1|} |||u_{\lambda_1}|||_{r_2;I}^2 . \end{aligned} \tag{2.7}$$

Then we observe that

$$\sum_{\lambda_1 \in J_1} 2^{2r_1|\lambda_1|} \|u_{\lambda_1}\|_{0;I}^2 = \sum_{\lambda_1 \in J_1} 2^{2r_1|\lambda_1|} \int_I |u_{\lambda_1}(x_2)|^2 \, dx_2$$

$$
\begin{aligned}
&\lesssim \int\limits_I \sum_{\lambda_1 \in J_1} 2^{2r_1|\lambda_1|} |(u(\cdot, x_2), \tilde{\psi}^{(1)}_{\lambda_1})_{0,I}|^2 \, dx_2 \\
&\lesssim \int\limits_I |||u(\cdot, x_2)|||^2_{r_1,I} \, dx_2 \qquad (2.8)\\
&\lesssim \|u\|^2_{r_1,0;I^2_{mix}} .
\end{aligned}
$$

Furthermore we have with respect to some arbitrary but fixed $m_2 > r_2$

$$
\begin{aligned}
&\int\limits_{\mathbb{R}} \int\limits_{I_{h_2,m_2}} |h_2|^{-2r_2-1} \|\Delta^{m_2}_{h_2,2}(u(\cdot, x_2))\|^2_{0;I} \, dx_2 dh_2 \\
&= \int\limits_I \int\limits_{\mathbb{R}} \int\limits_{I_{h_2,m_2}} |h_2|^{-2r_2-1} |\Delta^{m_2}_{h_2,2}(u(x_1, x_2))|^2 \, dx_1 dx_2 dh_2 \\
&\lesssim \|u\|^2_{0,r_2;I^2_{mix}} .
\end{aligned}
$$

Next we notice that for $h_2 \in \mathbb{R}$ and a.e. $x_2 \in I_{h_2,m_2}$ holds

$$
|\Delta^{m_2}_{h_2,2}(u_{\lambda_1}(x_2))|^2 = |(\Delta^{m_2}_{h_2,2} u(\cdot, x_2), \tilde{\psi}^{(1)}_{\lambda_1})_{0;I}|^2 ,
$$

which yields a.e.

$$
\sum_{\lambda_1 \in J_1} 2^{2r_1|\lambda_1|} |\Delta^{m_2}_{h_2,2}(u_{\lambda_1}(x_2))|^2 \lesssim |||\Delta^{m_2}_{h_2,2} u(\cdot, x_2)|||^2_{r_1;I} .
$$

Thus we obtain by monotone convergence and again the Fubini theorem with respect to some $m_1 > r_1$

$$
\begin{aligned}
\sum_{\lambda_1 \in J_1} 2^{2r_1|\lambda_1|} |u_{\lambda_1}|^2_{r_2;I} &= \int\limits_{\mathbb{R}} \int\limits_{I_{h_2,m_2}} \sum_{\lambda_1 \in J_1} 2^{2r_1|\lambda_1|} |h_2|^{-2r_2-1} |\Delta^{m_2}_{h_2,2}(u_{\lambda_1}(x_2))|^2 dx_2 dh_2 \\
&\lesssim \int\limits_{\mathbb{R}} \int\limits_{I_{h_2,m_2}} |h_2|^{-2r_2-1} |||\Delta^{m_2}_{h_2,2}(u(\cdot, x_2))|||^2_{r_1,I} dx_2 dh_2 \\
&\lesssim \|u\|^2_{0,r_2;I^2_{mix}} + \int\limits_{\mathbb{R}} \int\limits_{I_{h_2,m_2}} |h_2|^{-2r_2-1} \\
&\qquad \cdot \Big[ \int\limits_{\mathbb{R}} \int\limits_{I_{h_1,m_1}} |h_1|^{-2r_1-1} |\Delta^{m_1}_{h_1,1}(\Delta^{m_2}_{h_2,2} u(x_1, x_2))|^2 dx_1 dh_1 \Big] dx_2 dh_2 \qquad (2.9)\\
&\lesssim \|u\|^2_{0,r_2;I^2_{mix}} + \int\limits_{\mathbb{R}^2} \int\limits_{I_{h_1,m_1} \times I_{h_2,m_2}} |h_1|^{-2r_1-1} |h_2|^{-2r_2-1} \\
&\qquad \cdot |\Delta^{m_1}_{h_1,1} \otimes \Delta^{m_2}_{h_2,2} u(x_1, x_2)|^2 dx_1 dx_2 dh_1 dh_2 \\
&\lesssim |||u|||_{\mathbf{r};I^2_{mix}} .
\end{aligned}
$$

Now, (2.6)–(2.9) imply $\|u\|_{\otimes^1_{\mathbf{r}}} \lesssim |||u|||_{\mathbf{r};I^2_{mix}}$ for $u \in H^{\mathbf{1}}_{0,mix}(I^2) \cap H^{\mathbf{r}}_{\text{mix}}(I^2)$. ∎

Again a simple approach via Fourier series would also work here. However, the approach presented here is also appropriate for treating products of bounded domains with the uniform cone property.

For an application of Theorem 1 to homogeneous elliptic Dirichlet problems one needs a further argument with respect to Sobolev spaces with boundary conditions. To this end let $\gamma_l$, $l \in \mathbb{N}$, denote the usual trace operators defined on isotropic Sobolev spaces $H^l(I^2)$.

THEOREM 2. *For $l \in \mathbb{N}$ holds*

$$H^{l,l}_{0,mix}(I^2) = \{u \in H^{l,l}_{\text{mix}}(I^2) \mid \gamma_l u = 0\}\,. \tag{2.10}$$

**Proof.** Since the following is a transfered version of a more general approach presented in [22], we give only the sketch of a proof: "$\subset$" is clear, since $H^{l,l}_{0,mix}(I^2) \subset H^l_0(I^2)$. Let $P_i := \frac{\partial^{2l}}{\partial x_i^{2l}}$, $P = P_1 P_2$ and for $u \in C^\infty(\overline{I^2})$, $\phi \in C_0^\infty(I^2)$, let

$$b(u,\phi) := (Pu,\phi)_{0,I^2} = (\frac{\partial^{2l} u}{\partial x_1^l \partial x_2^l}, \frac{\partial^{2l}\phi}{\partial x_1^l \partial x_2^l})_{0,I^2}.$$

Note that $b(\cdot,\cdot)$ can be extended to a bounded bilinear-form

$$b : H^{l,l}_{\text{mix}}(I^2) \times H^{l,l}_{0,mix}(I^2) \to \mathbb{R}.$$

Now, let us consider $w \in H^{l,l}_{\text{mix}}(I^2)$ with $\gamma_l w = 0$ and $b(w,\phi) = 0$ for all $\phi \in C_0^\infty(I^2)$. Then for $\varphi_1 \in C_0^\infty(I)$ the definition

$$\tilde{w}(x_2) := (w(\cdot,x_2), P_1\varphi_1)_{0,I}, \quad x_2 \in I, \tag{2.11}$$

provides a function $\tilde{w} \in H^l_0(I)$ with

$$(\tilde{w}, P_2\varphi_2)_{0,I} = 0, \quad \varphi_2 \in C_0^\infty(I)$$

(use again the Fubini theorem). Thus, by standard elliptic regularity theory and uniqueness follows $\tilde{w} = 0$. Since $w(\cdot,x_1) \in H^l_0(I)$, equation (2.11) analogously implies $w = 0$.

Now let $u \in H^{l,l}_{\text{mix}}(I^2)$ with $\gamma_l u = 0$. Then, since $b(\cdot,\cdot)$ is obviously $H^{l,l}_{0,mix}(I^2)$-elliptic, there exists a unique $\tilde{u} \in H^{l,l}_{0,mix}(I^2)$ with

$$b(\tilde{u},\phi) = b(u,\phi), \quad \phi \in C_0^\infty(I^2).$$

Because $\gamma_l(u - \tilde{u}) = 0$ we must have $u = \tilde{u}$, i.e. $u \in H^{l,l}_{0,mix}(I^2)$. ∎

The characterization (2.10) allows to prove our final Sobolev space tensor product result.

THEOREM 3. *For $r_1, r_2 \geq l \in \mathbb{N}$ holds*

$$(H^l_0(I) \cap H^{r_1}(I)) \otimes (H^l_0(I) \cap H^{r_2}(I)) = H^l_0(I^2) \cap H^{r_1,r_2}_{\text{mix}}(I^2)\,. \tag{2.12}$$

**Proof.** Theorem 2 shows

$$H^{l,l}_{0,mix}(I^2) \cap H^{r_1,r_2}_{\text{mix}}(I^2) = \{u \in H^{r_1,r_2}_{\text{mix}}(I^2) \mid \gamma_l u = 0\}\,.$$

Since $H^l(I^2) \hookrightarrow H^{r_1,r_2}_{\text{mix}}(I^2)$ the well–known fact

$$H^l_0(I^2) = \{u \in H^l(I^2) \mid \gamma_l u = 0\}$$

implies (2.12). ∎

**3. Complexity estimates and Applications.** So far we have considered tensor products of a couple of Sobolev spaces. Corresponding results hold for tensor products of any finite number of Sobolev spaces. Thus we may apply our results to the more general class

$$H^{t,l}(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \mid \\ \|u\|_{H^{t,l}} = \Big(\sum_{\mathbf{k}\in\mathbb{Z}^n} \prod_{i=1}^{n} (1+|k_i|)^{2t}(1+|\mathbf{k}|_\infty)^{2l}|\mathcal{F}u(\mathbf{k})|^2\Big)^{\frac{1}{2}} < \infty\}, \quad t,l \geq 0$$

and to the corresponding spaces on domains. Note that the (standard) isotropic Sobolev spaces and the spaces of dominating mixed derivatives are special cases for $t = 0$ and $l = 0$, respectively.

Crucial for our approach are the observations

$$H^{t,l}(\mathbb{R}^n) = \mathcal{H}_{\text{mix}}^{t\mathbf{1}+l\mathbf{e}_1}(\mathbb{R}^n) \cap \ldots \cap \mathcal{H}_{\text{mix}}^{t\mathbf{1}+l\mathbf{e}_n}(\mathbb{R}^n),$$

where $\mathcal{H}_{\text{mix}}^{\mathbf{r}}(\mathbb{R}^n) := H^{r_1}(\mathbb{R}) \otimes \ldots \otimes H^{r_1}(\mathbb{R})$ for $\mathbf{r} = (r_1, \ldots, r_n) \geq 0$, $e_i$ the $i$-th unit-vector in $\mathbb{R}^n$, $\mathbf{1} = \{1, \ldots, 1\}$, as well as

$$H^{t,l}(I^n) \cap H_0^1(I^n) = (H^{t+l}(I) \cap H_0^1(I)) \otimes (H^t(I) \cap H_0^1(I)) \otimes \ldots \otimes (H^t(I) \cap H_0^1(I)) \\ \cap \ldots \cap (H^t(I) \cap H_0^1(I)) \otimes \ldots \otimes (H^t(I) \cap H_0^1(I)) \otimes (H^{t+l}(I) \cap H_0^1(I)).$$

**3.1. Approximation properties and complexity estimates.** Let us summarize the approximation properties of our approximation spaces $V_J^T$ from (2.1). Now we require that our basis functions have approximation order $m$ and that $S_j \subset H_0^r(I), 0 < r \leq m$, compare subsection 2.1.

As an example consider the case of piecewise linear splines. We assign the nodal basis function to each grid point in one dimension in the respective finite element space that takes the values 1 at that grid point and 0 at the others, i.e. $\psi_{jk}(x) := \phi\left(2^{j-1}x - k\right)$, where

$$\phi(x) := \begin{cases} 1-|x| & \text{for } x \in (-1,1), \\ 0 & \text{else.} \end{cases}$$

Then we have $m = 2$ and $r = 3/2 - \epsilon$ for $\epsilon > 0$.

Let $s \in \mathbb{R}$, $s < r$, $l, t \in \mathbb{R}_0^+$, $s < t + l \leq m$ and let $u = \sum_{\mathbf{j}} w_{\mathbf{j}} \in H^{t,l}(I^n) \cap H_0^s(I^n)$ with $w_{\mathbf{j}} \in W_{\mathbf{j}}$. Assume that the following estimate holds:

$$\|w_{\mathbf{j}}\|_{H^s} \lesssim 2^{(s-l)|\mathbf{j}|_\infty - t|\mathbf{j}|_1} \|u\|_{H^{t,l}} \;\forall \mathbf{j} \in \mathbb{N}^n.$$

Application of the triangle inequality then yields

$$\inf_{v\in V_J^T} \|u-v\|_{H^s} \leq \|\sum_{\mathbf{j}\notin I_J^T} w_{\mathbf{j}}\|_{H^s} \leq \sum_{\mathbf{j}\notin I_J^T} \|w_{\mathbf{j}}\|_{H^s} \lesssim \left(\sum_{\mathbf{j}\notin I_J^T} 2^{(s-l)|\mathbf{j}|_\infty - t|\mathbf{j}|_1}\right) \|u\|_{H^{t,l}}.$$

Now a straightforward estimate of the sum $\sum_{\mathbf{j}\notin I_J^T} 2^{(s-l)|\mathbf{j}|_\infty - t|\mathbf{j}|_1}$ shows

$$\text{(3.1)} \qquad \inf_{v\in V_J^T} \|u-v\|_{H^s} \lesssim \|u\|_{H^{t,l}} \cdot \begin{cases} 2^{(s-l-t+(Tt-s+l)\frac{n-1}{n-T})J} J^{n-1} & \text{for } T \geq \frac{s-l}{t}, \\ 2^{(s-l-t)J} & \text{for } T < \frac{s-l}{t}. \end{cases}$$

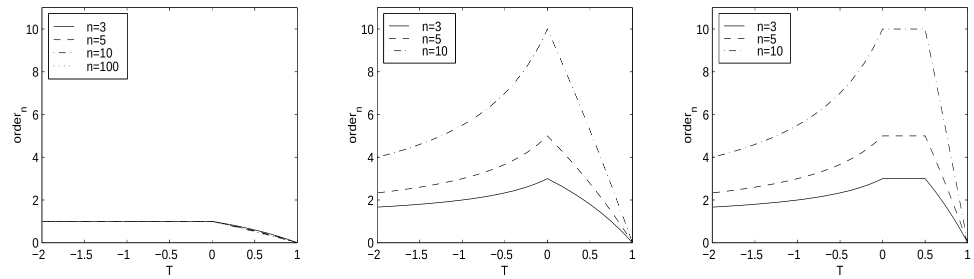



FIGURE 3.1. *Order of approximation for $s = 1$ and the spaces $H^{0,2}$, $H^{1,1}$ and $H^{2,0}$ (left to right)*

Inequality (3.1) states that the approximation power of the spaces $V_J^T$ with $T < (s-l)/t$ is the same as that of the full grid space $V_J^{-\infty}$. For approximation spaces with $T \geq (s-l)/t$ the order of approximation deteriorates. Note that in the case of stable splittings and at least for the parameter range $t + l < r$ the logarithmic term $J^{n-1}$ in the first inequality in (3.1) can be omitted, see [17].

Let $\mathcal{N}$ denote the number of degrees of freedom in the approximation space $V_J^T$. Now we combine (2.3) with (3.1) and obtain

$$\inf_{v \in V_J^T} \|u - v\|_{H^s} \lesssim \begin{cases} \mathcal{N}^{s-l-t} \cdot \|u\|_{H^{t,l}} & \text{for } 0 < T, T < \frac{s-l}{t}, \\ \left( \dfrac{\mathcal{N}}{\ln(\mathcal{N})^{n-1}} \right)^{s-l-t} \cdot \|u\|_{H^{t,l}} & \text{for } T = 0, T < \frac{s-l}{t}, \\ \mathcal{N}^{\frac{s-l-t}{n} \cdot \frac{T-n}{T-1}} \cdot \|u\|_{H^{t,l}} & \text{for } T < 0, T < \frac{s-l}{t}, \\ \mathcal{N}^{\frac{s-l-t}{n}} \cdot \|u\|_{H^{t,l}} & \text{for } T = -\infty. \end{cases} \tag{3.2}$$

From the requirement $\inf_{v \in V_J^T} \|u - v\|_{H^s} \leq \epsilon$ we then obtain the following: If $T < \frac{s-l}{t}$, then the spaces $V_J^T$ contain

$$\mathcal{N} \lesssim \begin{cases} \epsilon^{\frac{1}{s-l-t}} & \text{for } 0 < T, \\ \epsilon^{\frac{1}{s-l-t}} \cdot \ln(\epsilon^{-1})^{n-1} & \text{for } T = 0, \\ \epsilon^{\frac{1}{s-l-t} \frac{T-1}{T-n} n} & \text{for } -\infty < T < 0, \\ \epsilon^{\frac{n}{s-l-t}} & \text{for } T = -\infty. \end{cases} \tag{3.3}$$

grid points to obtain an accuracy $\epsilon$.

To visualize the implications of (3.2) and (3.3) we plot the negative exponents on the right hand side of (3.2) (where we disregard logarithmic terms) times the dimension $n$. Figure 3.1 shows these exponents against $T$ for various dimensions $n$ and various smoothness assumptions. These Figures show the dependence of the order of approximation from the dimension $n$ and from the parameter $T$. Specifically for the space $H^{0,2}$ the order of approximation deteriorates with larger dimension showing the intractability of higher dimensional problems. However, for the case $H^{2,0}$ of dominating mixed smoothness there is an interval $(0, 1/2]$ where the order of approximation is independent of the dimension $n$.

**3.2. Application to Poisson's equation.** As an application to elliptic boundary value problems let us consider the Poisson equation with homogeneous Dirichlet boundary conditions in its variational form on $H_0^1(I^n)$. Let $u \in H_0^1(I^n)$ be the unique solution of

$$(\nabla u, \nabla v) = (f, v), \quad v \in H_0^1(I^n) \tag{3.4}$$

and $u_J^T$ its Finite Element approximation in $V_J^T$.

Sufficient conditions on the data $f$, which provide $u \in H^{t,l}(I^n)$ may easily be obtained by taking into account the spectral representation. It is known, that $f \in L_2(I^n)$ with $f(\mathbf{x}) = \sum_{\mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{N}^n} f_{\mathbf{k}} \Pi_{i=1}^n \sin(\pi k_i x_i)$ such that $(f_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^n} \in \ell^2$ gives the solution $u(\mathbf{x}) = \sum_{\mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{N}^n} u_{\mathbf{k}} \Pi_{i=1}^n \sin(\pi k_i x_i)$ with $u_{\mathbf{k}} = \frac{f_{\mathbf{k}}}{\pi^2 \sum_{i=1}^n k_i^2}$. Clearly, $\sum_{\mathbf{k}\in\mathbb{N}^n} \Pi_{i=1}^n (1+k_i)^{2t}(1+|\mathbf{k}|_\infty)^{2l}|u_{\mathbf{k}}|^2 < \infty$ implies $u \in H^{t,l}(I^n)$, which in turn provides a condition on the data $f$.

Estimates of the approximation error in the energy-norm using our approximation spaces (2.1) as test and trial spaces in the finite element method are easily obtained from (3.1) via the best approximation property of the finite element solution. If $u \in H^{t,l}(I^n)$, $t, l \in \mathbb{R}_0^+$, $1 \le t + l \le m$, $r \ge 1$, then we obtain

$$\|u - u_J^T\|_{H^1} \lesssim \begin{cases} 2^{(1-l-t+(Tt-1+l)\frac{n-1}{n-T})J} J^{n-1} & \text{for } T \ge \frac{1-l}{t}, \\ 2^{(1-l-t)J} & \text{for } T < \frac{1-l}{t}. \end{cases} \tag{3.5}$$

Unfortunately, best approximation results do not directly carry over to estimates in other norms than the energy-norm. However, from (3.5) together with the Lemma of Nitsche and the fact that $V_K^{-\infty} \subset V_J^T$ with $K = \frac{J(1-T)+(n-1)}{n-T}$ we obtain

$$\|u - u_J^T\|_{L_2} \lesssim \begin{cases} 2^{((1-l-t)(1+\frac{1-T}{n-T})+(Tt-1+l)\frac{n-1}{n-T})J} J^{n-1} & \text{for } T \ge \frac{1-l}{t}, \\ 2^{(1-l-t)(1+\frac{1-T}{n-T})J} & \text{for } T < \frac{1-l}{t}. \end{cases} \tag{3.6}$$

In several places in the literature [7, 9] it has been observed numerically that for smooth solutions, i.e. $u \in \mathcal{C}^\infty$, and piecewise linear splines as basis functions, the error w.r.t. the $L_2$-norm behaves like $2^{-2J} J^{n-1}$. This has to be compared with the weaker result $2^{-(1+1/n)}$ from (3.6).

We already mentioned in the introduction that for a large range of problems there exist trial and test functions, such that a finite element discretization leads to stiffness matrices whose condition number is bounded independently of the number of unknowns in the finite element approximation space. Therefore it is quite natural to ask for multilevel additive Schwarz preconditioners for the approximation spaces $V_J^T$ similar to the standard BPX schemes described in [6, 27, 40]. To this end let $R_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}_0^n$, be the tensor-product partition of the unit square with uniform stepsize $2^{-j_i}$ into the $i$-th coordinate direction and let $V_{\mathbf{j}}$ denote the space of multilinear finite element functions with respect to $R_{\mathbf{j}}$. Then the BPX-type preconditioned operator

$$P_{BPX}\, u = \sum_{\mathbf{j}\in I_J^T} \sum_{\mathbf{i}} \frac{a(u, \psi_{\mathbf{ji}})}{(\sum_{k=1}^n 2^{2|j_k|})(\psi_{\mathbf{ji}}, \psi_{\mathbf{ji}})_{L_2}}\ \psi_{\mathbf{ji}},\ \ u \in V_J^T,$$

for the Poisson problem on the unit square, does not lead to a spectral condition number that is independent of the number of levels, as in the non-tensor-product case [6]. The sum $\sum_{\mathbf{i}}$ is over all pairs $\mathbf{ji}$ in the index set of the (anisotropic) full grid space $V_{\mathbf{j}}$. For example for the index set $I_J^{-\infty}$ in 2D a condition number of $O(J^2)$ has been shown in [18]. However, this condition number can be slightly improved. The splitting

$$V_J^T = \sum_{l=0}^{\lfloor K \rfloor} V_{(l,\dots,l)} + \sum_{l=\lceil K \rceil}^{J} \sum_{\{\mathbf{j}\in I_J^T:\ \mathbf{j}+\mathbf{1}\notin I_J^T\ \&\ |\mathbf{j}|_\infty = l\}} V_{\mathbf{j}},$$

again with $K = \frac{J(1-T)+(n-1)}{n-T}$ (note that then $V_K^{-\infty} \subset V_J^T$) leads to the operator

$$\hat{P}_{BPX} u = \sum_{l=0}^{\lfloor K \rfloor} \sum_{\mathbf{i}} \frac{a(u, \psi_{(l,\dots,l),\mathbf{i}})}{2^{2l}(\psi_{(l,\dots,l),\mathbf{i}}, \psi_{(l,\dots,l),\mathbf{i}})_{L_2}} \psi_{(l,\dots,l),\mathbf{i}}$$
$$+ \sum_{l=\lceil K \rceil}^{J} \sum_{\{\mathbf{j}\in I_J^T:\ \mathbf{j}+\mathbf{1}\notin I_J^T\ \&\ |\mathbf{j}|_\infty = l\}} \sum_{\mathbf{i}} \frac{a(u, \psi_{\mathbf{ji}})}{(\sum_{k=1}^{n} 2^{2|j_k|})(\psi_{\mathbf{ji}}, \psi_{\mathbf{ji}})_{L_2}} \psi_{\mathbf{ji}},\ u \in V_J^T.$$

The fact that the number of subspaces $V_{\mathbf{j}}$ involved in the last sum is bounded by $J^{n-2}$ shows then that the spectral condition number of this operator is $O(J^{n-2})$ for $T \in (-\infty, 1)$. The proof is a simple generalization of the proof for the case of $V_J^0$ in 2D from [18] (compare equation (20) in [18]) to our more general approximation spaces and to arbitrary dimensions.

Optimal preconditioning can be obtained for example via wavelet preconditioning and multigrid methods. This has been described in detail in [19, 28] for the space $V_J^0$ and can be tuned for other values of $T$.

For differential operators it is important to note that one need not assemble the stiffness matrix, because all that is required in an iterative scheme is the application of the preconditioned stiffness matrix to a vector. Any efficient implementation must avoid the explicit construction of the stiffness matrix, which would be more densely populated than the usual finite element stiffness matrix using nodal (non-hierarchical) basis functions. Instead, because of the tensor-product structure of our basis functions, the action of the stiffness matrix on a function or the corresponding vector can be performed by the application of certain sparse operators, mapping a representation with respect to the multilevel nodal basis to the standard nodal basis representation. This has been explained in detail in [7, 19] for the spaces $V_J^0$ working with hyperbolic cross points and the full grid approximation space $V_J^{-\infty}$ and can be easily modified for other values of $T$. Then the cost of one matrix vector multiplication is of the order $O(dim(V_J^T))$. Hence, together with optimal preconditioning and nested iteration, the overall cost of computing an approximation to the solution of the problem (3.4) within discretization accuracy is $O(dim(I_J^T))$, compare [21]. This together with (3.5) and (3.2) tells us that an $\epsilon$-approximation of $u \in H^{t,l}(I^n) \cap H_0^1(I^n)$ with $l < 1, t > 0$, may be obtained with a cost of $O(\epsilon^{\frac{1}{1-l-t}})$. Specifically for $u \in H^{1+\delta,1-\delta}(I^n) \cap H_0^1(I^n)$ we have $s = 1 > l = 1 - \delta$. Hence we obtain $O(\epsilon^{\frac{1}{s-l-t}}) = O(\epsilon^{\frac{1}{1-(1-\delta)-(1+\delta)}}) = O(\epsilon^{-1})$. This implies the following Corollary.

COROLLARY 1. *Let the solution $u \in H_0^1(I^n)$ of (3.4) fulfil $u \in H^{1+\delta,1-\delta}(I^n) \cap H_0^1(I^n)$ with $0 < \delta \leq 1$. Then the solution can be obtained up to an error of $\epsilon$ in the energy norm with a cost of $cost(\epsilon) = O(\epsilon^{-1})$.*

This includes a result of [9] for the case $u \in H^{2,0}(I^n) \cap H_0^1(I^n)$. There the authors use index sets that are asymptotically equivalent to $I_J^{2/5}$.


FIGURE 4.1. *Index sets $I_8^0$ and $I_8^{0.5}$ (grey) and coefficients $r_{\mathbf{j}} \neq 0$ from the sum (4.2), 2D case.*

**4. Blending schemes, Combination technique.** Our approximation space $V_J^T$ is associated with the grid point sequence

$$S_J^T := \cup_{\mathbf{j} \in I_J^T} S_{\mathbf{j}}, \tag{4.1}$$

where the $S_{\mathbf{j}} := \{(a_1 2^{-j_1}, \ldots, a_n 2^{-j_n}), a_i = 0, \ldots, 2^{j_i} - 1, 1 \leq i \leq n\}$ are anisotropic full grids. Definition (4.1) indicates the possibility of constructing our approximation spaces as sums and differences of simpler spaces via blending schemes in analogy to Smolyak's blending scheme [31].

To see this, let $u_J^T = \sum_{\mathbf{j} \in I_J^T} w_{\mathbf{j}}$, $w_{\mathbf{j}} \in W_{\mathbf{j}}$ and $u_{\mathbf{j}} = \sum_{\mathbf{k} \leq \mathbf{j}} w_{\mathbf{k}}$ for $\mathbf{j} \in I_J^T$. Hence $u_J^T \in V_J^T$ and $u_{\mathbf{j}} \in V_{\mathbf{j}} := \bigoplus_{\mathbf{k} \leq \mathbf{j}} W_{\mathbf{k}}$. Then it holds

$$u_J^T = \sum_{\mathbf{j} \in I_J^T : r_{\mathbf{j}} \neq 0} r_{\mathbf{j}} u_{\mathbf{j}} \text{ with } r_{\mathbf{j}} = \sum_{k=0}^{n-1} (-1)^k \left( \sum_{\substack{\{\mathbf{l} : \mathbf{l} = \mathbf{j} + \sum_{i \in M} \mathbf{e}_i \in I_J^T, \\ M \subseteq \{1, \ldots, n\}, |M| = k\}}} 1 \right). \tag{4.2}$$

For a proof write $w_{\mathbf{j}}$ from $W_{\mathbf{j}}$ as sum and difference of elements $u_{\mathbf{k}} = \sum_{\mathbf{k} \leq \mathbf{j}} w_{\mathbf{j}}$ from the spaces $V_{\mathbf{k}}, \mathbf{k} \leq \mathbf{j}$, plug the result into $u_J^T = \sum_{\mathbf{j} \in I_J^T} w_{\mathbf{j}}$, collect equal terms in the sum and exploit the fact that $r_{\mathbf{j}} = 0$ if $\mathbf{j} + \mathbf{1} \in I_J^T$.

That is, every element $u_J^T$ from $V_J^T$ can be written as a weighted sum of approximations from anisotropic full grid approximation spaces and the coefficients of the multi-scale representation of $u_J^T$ follow directly from the coefficients of the multi-scale representation of the functions $u_{\mathbf{j}} \in V_{\mathbf{j}}$. Figure 4.1 shows some examples for the index sets $I_J^T$ and the corresponding coefficients $r_{\mathbf{j}}$ in the two-dimensional case. A short consideration shows that the number of grid points of the anisotropic grids $S_{\mathbf{j}}$ that are used in the blending scheme for $V_J^T$ is of the same order as $dim(V_J^T)$.

In the case that the exact $u_{\mathbf{j}}$ are not available, we can still compute an approximation to $u_J^T$ via some approximations

$$\overline{u}_{\mathbf{j}} = \sum_{\mathbf{m} \leq \mathbf{j}} \sum_{\mathbf{l} \in \tau_{\mathbf{m}}} c_{\mathbf{ml}}^{\mathbf{j}} \psi_{\mathbf{ml}} \tag{4.3}$$

of the $u_{\mathbf{j}}$. Here $\tau_{\mathbf{j}}$ is the index set defined from the subdivision rate of successive refinement levels. We can use

$$u_J^T \approx \sum_{\mathbf{j} \in I_J^T : \mathbf{j} + \mathbf{1} \notin I_J^T} r_{\mathbf{j}} \overline{u}_{\mathbf{j}}$$

$$= \sum_{\mathbf{j}\in I_J^T:\mathbf{j}+\mathbf{1}\notin I_J^T} r_{\mathbf{j}} \left( \sum_{\mathbf{m}\leq\mathbf{j}} \sum_{\mathbf{l}\in\tau_{\mathbf{m}}} c_{\mathbf{ml}}^{\mathbf{j}} \psi_{\mathbf{ml}} \right)$$

$$= \sum_{\mathbf{m}\in I_J^T} \sum_{\mathbf{l}\in\tau_{\mathbf{m}}} \left( \sum_{\mathbf{j}\in I_J^T:\mathbf{j}+\mathbf{1}\notin I_J^T,\mathbf{m}\leq\mathbf{j}} r_{\mathbf{j}}\, c_{\mathbf{ml}}^{\mathbf{j}} \right) \psi_{\mathbf{ml}}.$$

Hence we have for the coefficients $d_{\mathbf{ml}}$ of $u_J^T = \sum_{\mathbf{m}\in I_J^T} \sum_{\mathbf{l}\in\tau_{\mathbf{m}}} d_{\mathbf{ml}}\psi_{\mathbf{ml}}$

$$d_{\mathbf{ml}} \approx \sum_{\mathbf{j}\in I_J^T:\mathbf{j}+\mathbf{1}\notin I_J^T,\mathbf{m}\leq\mathbf{j}} r_{\mathbf{j}}\, c_{\mathbf{ml}}^{\mathbf{j}}. \tag{4.4}$$

Now the following algorithm may be used for the approximate solution of (3.4) in $V_J^T$.

ALGORITHM 1. *(Combination solution of Laplace equation)*

1. *Solve the variational problem (3.4) with respect to the Finite Element spaces* $V_{\mathbf{j}}, \mathbf{j} \in I_J^T, r_{\mathbf{j}} \neq 0$. *This results in the coefficients* $c_{\mathbf{ml}}^{\mathbf{j}}$ *of the functions* $\overline{u}_{\mathbf{j}} \in V_{\mathbf{j}}$ *in (4.3).*
2. *To obtain an approximation to* $u_J^T$ *combine these coefficients according to (4.4).*

In this algorithm the solution procedure is performed only with respect to regular anisotropic grids. Optimal solvers are then easy to obtain via Multilevel-preconditioners or multilevel methods using anisotropic refinement (semi-coarsening) [32]. The solution procedures on the anisotropic grids are completely independent of each other and can therefore be computed in parallel. Note that this algorithm can be implemented in a dimension recursive way.

Examples for the use of this algorithm for the approximate solution of the Laplace-equation in the spaces $V_J^0$ can be found in [25, 29, 42]. In [8] and [29] corresponding error estimates are given.

**5. Numerical example.** We describe a numerical example that highlights our theoretical results. In particular we consider the solution of the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = [0,1]^2, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

Based on piecewise bilinear splines, we use the spaces $V_J^T$ as trial- and test-spaces within the Finite Element method and we employ the blending schemes from section 4. A Galerkin discretization scheme is used on each regular grid that contributes to the blended global solution. The error is measured in two discrete norms that correspond to the $L_2$- and $H^1$-norm. For the evaluation of the norms we use the grid points of the full grid of the current level plus one. The discrete $L_2$-norm is computed from the sum of the squared values at the grid points and the $H^1$-norm is analogously computed via application of the discrete Laplacian on the full grid. All the Figures show the error with respect to the number of grid points that are used in the blending scheme (including the points on the boundary).

We consider three test cases with different right hand sides $f$ and different Dirichlet boundary values $g$, such that the solution $u$ is given by

$$
\begin{aligned}
1)\quad u(r,\phi) &= \begin{cases} (1-r)^2 & \text{for } r<1 \\ 0 & \text{else} \end{cases} \\
2)\ u(x_1,x_2) &= \sin(\pi x_1)\sinh(\pi x_2) \\
3)\quad u(r,\phi) &= e^{-r^2}
\end{aligned}
$$

The resulting convergence orders obviously depend on the approximation order of the bases and smoothness properties of the solutions $u$. Figure 5.1 shows graphs of the errors against the number of degrees of freedom. The Figures show the dependence of the convergence orders from the parameter $T$ and verify our theoretical results. Specifically the superiority of the full grid approximation space for test case 1 and the superiority of sparse grid constructions for the other two cases is verified. A proper choice of the parameter $T$ may lead to algorithms with lower cost than the choice $T=0$, compare example 3 Figure 5.1 bottom right.

Numerical results for the interpolation of the functions 1–3 gave qualitatively the same results.

REFERENCES


[1] J.-P. Aubin, *Applied functional analysis*, John Wiley & Sons, New York, 1979.

[2] K. I. Babenko, *Approximation by trigonometric polynomials in a certain class of periodic functions of several variables*, Dokl. Akad. Nauk SSSR 132 (1960), 672–675.

[3] G. Baszenski, F.-J. Delvos, *A discrete Fourier transform scheme for boolean sums of trigonometric operators*, Int. Series of Numerical Mathematics 90, Birkhäuser, Basel, 1989, 15-23.

[4] K. Bittner, *Fast algorithms for periodic spline wavelets on sparse grids*, SIAM J. Sci. Comput. 20(4) (1999), 1192-1213.

[5] F. A. Bornemann, H. Yserentant, *A basic norm equivalence for the theory of multilevel methods*, Numer. Math. 64 (1993), 455–476.

[6] J. Bramble, J. Pasciak, J. Xu, *Parallel multilevel preconditioners*, Math. Comp. 31 (1990), 333–390.

[7] H.-J. Bungartz, *Dünne Gitter und deren Anwendung bei der adaptiven Lösung der dreidimensionalen Poisson-Gleichung*, Dissertation, TU München, Institut für Informatik (1992).

[8] H.-J. Bungartz, M. Griebel, D. Röschke, C. Zenger, *Pointwise convergence of the combination technique for Laplace's equation*, East-West J. of Numerical Mathematics 1(2) (1994), 21–45.

[9] H.-J. Bungartz, M. Griebel, *A note on the complexity of solving Poisson's equation for spaces of bounded mixed derivatives*, J. Complexity 15 (1999), 167–199.

[10] J. C. Cavendish, W. J. Gordon, C. H. Hall, *Ritz-Galerkin approximations in blending function spaces*, Numer. Math. 26 (1976), 155-178.

[11] A. Cohen, *Wavelet methods in numerical analysis*, to appear in Handbook of Numerical Analysis, Editors P.G. Ciarlet and J.L. Lions, Elsevier Science Publishers, North Holland.

[12] W. Dahmen, A. Kunoth, *Multilevel preconditioning*, Numer. Math. 63 (1992), 315–344.

[13] A. Defant, K. Floret, *Tensor norms and operator ideals*, North-Holland, Amsterdam (1993).

[14] Din' Zung, *Number of integral points in a certain set and the approximation of functions of several variables*, Math. Notes (1984), 736-744.

[15] K. Frank, S. Heinrich, S. Pereverzev, *Information complexity of multivariate Fredholm equations in sobolev classes*, J. Complexity 12 (1996), 17-34.

[16] E. M. Galeev, *Approximation by Fourier sums of functions with several bounded derivatives*, Math. Notes 23 (1978), 109-117.

[17] M. Griebel, S. Knapek, *Optimized tensor-product approximation spaces*, to appear in Constructive Approximation.

[18] M. Griebel, P. Oswald, *On additive Schwarz preconditioners for sparse grid discretizations*, Numer. Math. 66(4) (1994), 449–464.

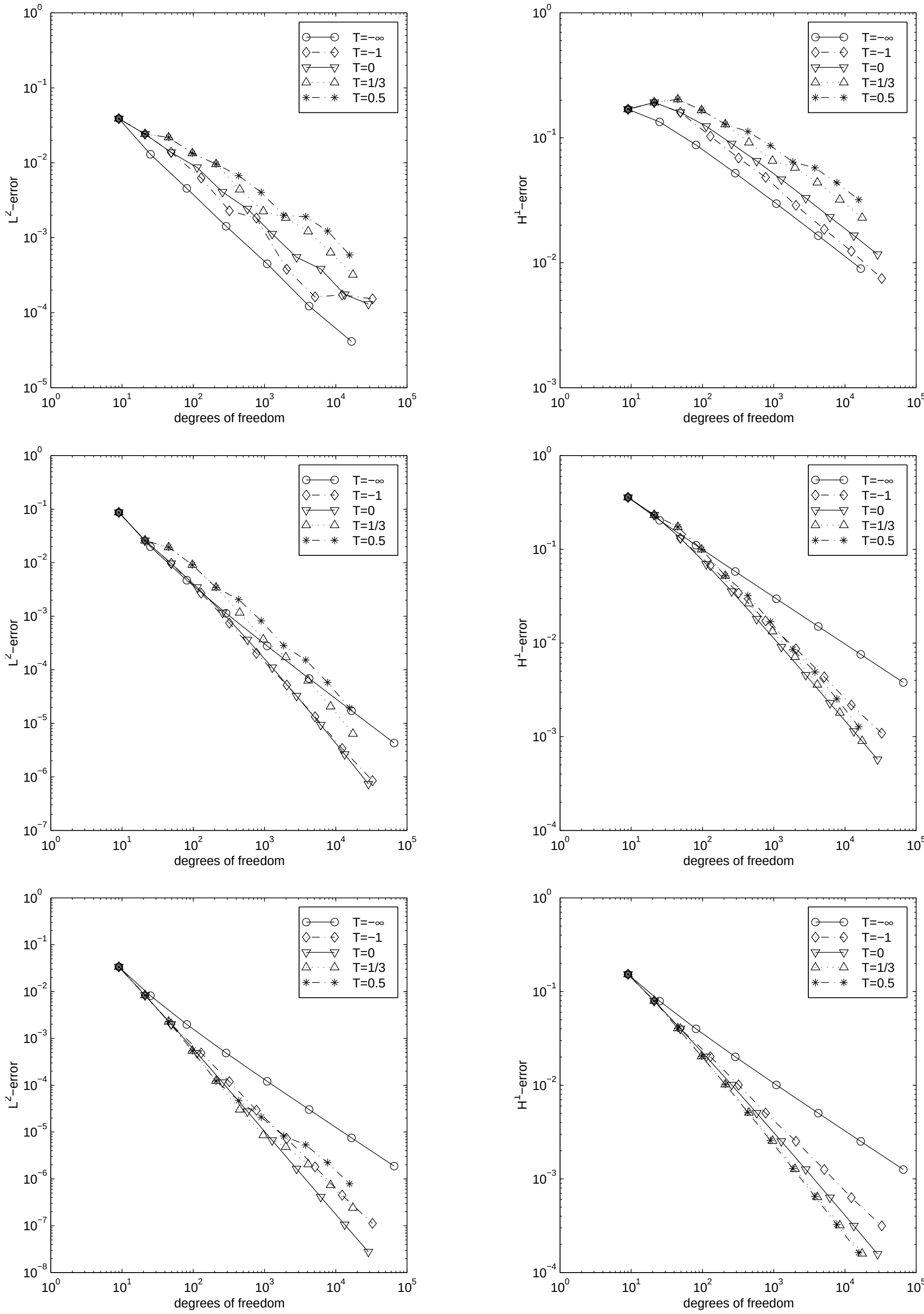



FIGURE 5.1. *Laplace Examples 1–3: $L_2$ (left) and $H^1$-error (right) against number of degrees of freedom.*

[19] M. Griebel, P. Oswald, *Tensor product type subspace splittings and multilevel iterative methods for anisotropic problems*, Advances in Computational Mathematics, 4 (1995), 171–206.

[20] M. Griebel, P. Oswald, T. Schiekofer, *Sparse grids for boundary integral equations*, Numer. Math. 83(2) (1999), 279-312.

[21] W. Hackbusch, *Iterative solution of large sparse systems of equations*, Springer, Berlin, 1994.

[22] R. Hochmuth, *An inhomogeneous Dirichlet problem for a non-hypoelliptic linear partial differential operator*, Ann. Acad. Sci. Fenn. Ser. A I Math. 21 (1996), 178–187.

[23] R. Hochmuth, *Wavelet bases in numerical analysis and restricted nonlinear approximation*, Habilitationsschrift, Freie Universität Berlin, 1999.

[24] S. Knapek, *Hyperbolic cross approximation of integral operators with smooth kernel*, SFB 256, report 665, Universität Bonn, 2000.

[25] J. Noordmans, P. W. Hemker, *Convergence results for 3D sparse grid approaches*, in Proc. Conf. on Preconditioned Iterative Solution Methods for Large Scale Problems in Scientific Computations, Nijmegen University, The Netherlands, (1997), 11–22.

[26] P. Oswald, *On discrete norm estimates related to multilevel preconditioners in the finite element method*, Constructive Theory of Functions, K.G. Ivanov, P. Petrushev, B. Sendov, (eds.), Proc. Int. Conf. Varna, 1991, Bulg. Acad. Sci., Sofia (1992), 203–214.

[27] P. Oswald, *Multilevel Finite Element Approximation, Theory and Applications*, Teubner Skripten zur Numerik, Teubner, Stuttgart, 1994.

[28] C. Pflaum, *A multilevel algorithm for the solution of second order differential equations on sparse grids*, Numer. Math. 79 (1998), 141–155.

[29] C. Pflaum, A. Zhou, *Error analysis of the combination technique*, Numer. Math. 84 (1999), 327-350.

[30] H. J. Schmeisser, H. Triebel, *Topics in Fourier analysis and function spaces*, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig (1987).

[31] S. A. Smolyak, *Quadrature and interpolation formulas for tensor products of certain classes of functions*, Dokl. Akad. Nauk SSSR 4 (1963), 240-243.

[32] K. Stüben, U. Trottenberg, *Multigrid Methods: Fundamental algorithms, model problem analysis and applications*, Lecture Notes in Mathematics 960, Springer, 1982.

[33] V. N. Temlyakov, *On approximate recovery of functions with bounded mixed derivative*, J. Complexity 9 (1993), 41–59.

[34] J. F. Traub, G. W. Wasilkowski, H. Wozniakowski, *Information-based complexity*, Academic Press, New York, 1988.

[35] F. Treves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, London, 1967.

[36] H. Triebel *Theory of function spaces*, Monographs in Mathematics 78, Birkäuser Verlag, Basel etc. (1983).

[37] G. W. Wasilkowski, H. Wozniakowski, *Explicit cost bounds of algorithms for multivariate tensor products*, J. Complexity 11 (1995), 1–56.

[38] J. Weidmann, *Linear operators in Hilbert spaces*, Springer, New York, 1980.

[39] A. G. Werschulz, *The computational complexity of differential and integral equations, An information based approach*, Oxford University Press, New York, 1991.

[40] J. Xu, *Iterative methods by space decompositions and subspace correction*, SIAM Review 34 (1992), 581-613.

[41] C. Zenger, *Sparse grids*, Parallel Algorithms for Partial Differential Equations, Notes on Num. Fluid Mech. 31, W. Hackbusch (ed.), Vieweg, Braunschweig, 1991.

[42] G. Zumbusch, *A sparse grid PDE solver*, chap. 4 in Advances in Software Tools for Scientific Computing, H. P. Langtangen, A. M. Bruaset, E. Quak, (eds.), Springer, Berlin (2000), 133-177.