Institut für Numerische Simulation
Universität Bonn

Meshfree Multiscale Methods for Solids

Sa Wu, Marc Alexander Schweitzer

wu@ins.uni-bonn.de, schweitzer@ins.uni-bonn.de

SFB 716
Sonderforschungsbereich 716
Universität Stuttgart

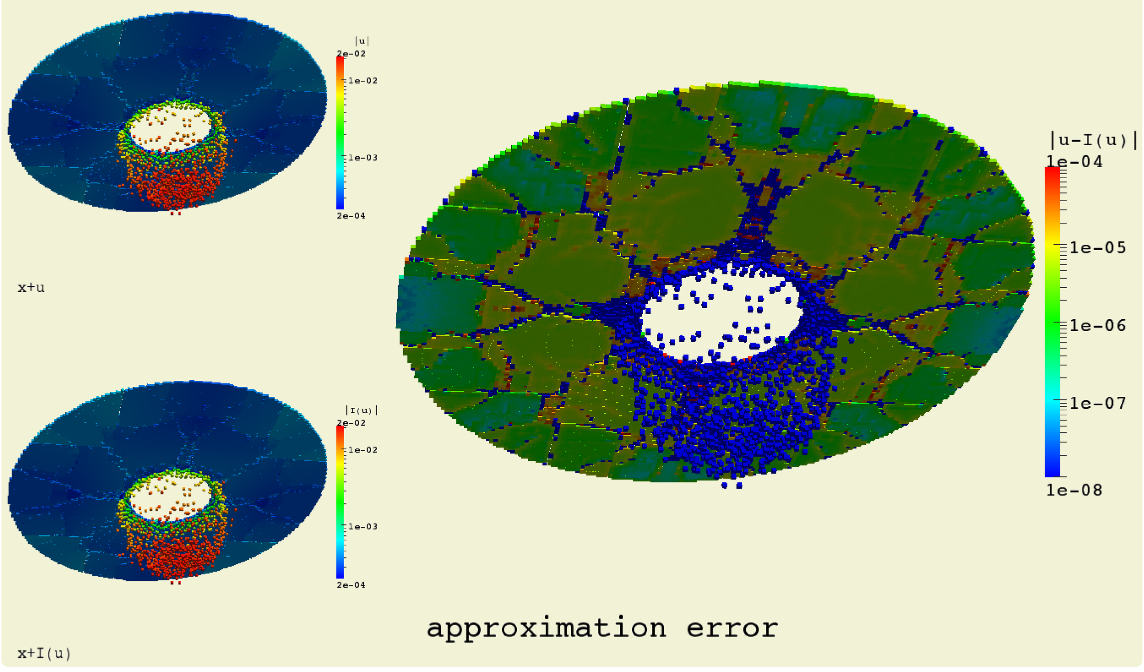
Objectives

Fast simulation of localized material failure for macroscale material samples through

- efficient (Generalized) Finite Elements for global linear elasticity problem and
- the use of solution from a local particle simulation
- as enrichment to construct discontinuous shape functions for the global problem.

Discontinuous Approximation of Peridynamics

- (Modified) Moving Least Squares approximation of displacements from impact simulation (not yet used as enrichment)
- upper left: coordinates $\mathbf{x_i} + \mathbf{u_i^n}$ from **Peridynamics**, color coding shows $\|\mathbf{u_i^n}\|$
- lower left: coordinates $\mathbf{x_i} + \eta^n(\mathbf{x_i})$ from MLS approximation $\mathbf{I}(\mathbf{u}) = \eta^n$ of Peridynamics data $\mathbf{x_i}, \mathbf{u_i^n}, \mathbf{A_{i,j}^n}$, color coding shows $\|\eta^n(\mathbf{x_i})\|$
- right: coordinates $\mathbf{x_i} + \mathbf{u_i^n}$, color coding shows $\|\mathbf{u_i^n} - \eta^n(\mathbf{x_i})\|$



Acknowledgments

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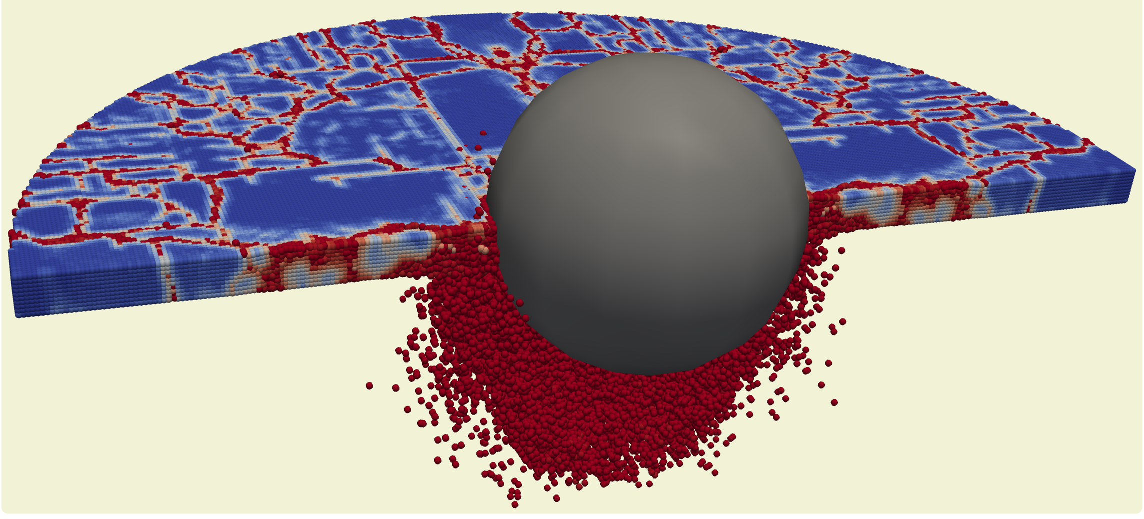
Damage everywhere

- Enrichments and particle simulation needed everywhere
- No speedup expected
- Example: Peridynamics simulation of brittle impact



Peridynamics Simulation with Particles

- Nonlocal equation of motion

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\Omega(\mathbf{x})} f\left(\left(\mathbf{u}(\tilde{\mathbf{x}},\cdot) - \mathbf{u}(\mathbf{x},\cdot)\right)|_{(-\infty,t]}, \tilde{\mathbf{x}} - \mathbf{x}\right) \mathrm{d}\tilde{\mathbf{x}} + b(\mathbf{x},t)$$

- No gradients, discontinuities occur naturally
- Discretization: fix $\mathbf{x_i}$, calculate $\mathbf{u_i^n} \approx \mathbf{u}(\mathbf{x_i}, t_n)$

$$\mathbf{u_i^{n+1}} = 2\mathbf{u_i^n} - \mathbf{u_i^{n-1}} + \frac{(\Delta t)^2}{\rho} \left(\sum_{j \in \mathbf{N_i}} f\left(\left(m \mapsto \mathbf{u_j^m} - \mathbf{u_i^m}\right)|_{(-\infty,n]}, \mathbf{x_j} - \mathbf{x_i}\right) \mathbf{V_{i,j}} + b(\mathbf{x_i}, t_n) \right)$$

- Explicitly track bonds and bond failure between particles

$$\mathbf{A_{i,j}^{n+1}} = \begin{cases} 1 & f\left(\left(m \mapsto \mathbf{u_j^m} - \mathbf{u_i^m}\right)|_{(-\infty,n]}, \mathbf{x_j} - \mathbf{x_i}\right) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Data $\mathbf{x_i}, \mathbf{u_i^{n+1}}, \mathbf{A_{i,j}^{n+1}}$ for (discontinuous) vector field approximation η^{n+1}

Moving Least Squares

- Scattered data $\mathbf{x_i}, \mathbf{u_i^{n+1}}$ approximation with

$$\eta^{n+1}(x) := q_x^{n+1}(0) \qquad q_x^{n+1} := \operatorname*{argmin}_{p \in P} \mathbf{J_x^{n+1}}(p)$$

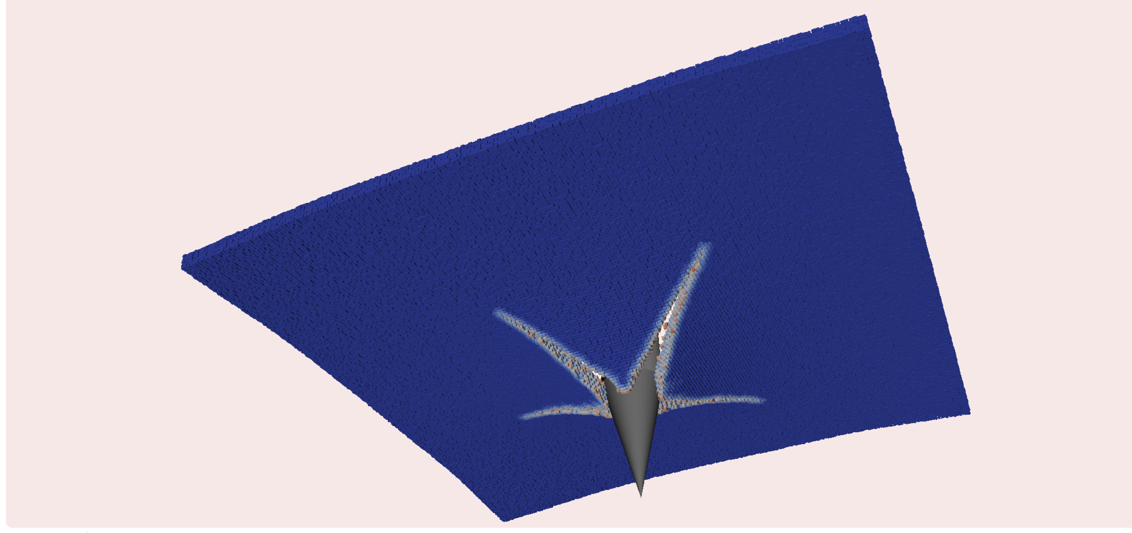
$$\mathbf{J_x^{n+1}}(p) := \sum_i \mathbf{W_i^{n+1}}(\mathbf{x}) \left(\mathbf{u_i^{n+1}} - p(\mathbf{x_i} - x)\right)^2$$

- $\mathbf{W_i^{n+1}}$ are normal Moving Least Squares weights W_i (some kernels) modified to take $\mathbf{A_{i,j}^{n+1}}$ into account
- Take $\mathbf{w_{i,j}}(\mathbf{x_i}) = 1, \mathbf{w_{i,j}} \leq 1$ locally supported

$$\mathbf{W_i^{n+1}}(\mathbf{x}) := W_i(\mathbf{x}) \prod_{\{j : \mathbf{A_{i,j}^{n+1}} = 0\}} (1 - \mathbf{w_{j,i}}(\mathbf{x}))$$

Focus: Localized Damage

- Enrichments and particle simulation needed only locally
- Speedup depending on localization of damage expected
- Example: Peridynamics simulation of petaling



Use global solution $\mathbf{u}(\cdot, t_n)$ to to choose domain $\bigcup_{i \in \mathbf{I^{n+1}}} \operatorname{supp}(\varphi_i) \ni \mathbf{x_i}$ and initial and boundary conditions (e.g. $\mathbf{u_i^n} = \mathbf{u}(\mathbf{x_i}, t_n), \ldots$) for local Peridynamics run(s)

Linear Elasticity with Generalized Finite Elements

- Local equation of motion

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = \mu\Delta\,\mathbf{u}(\mathbf{x},t) + (\lambda+\mu)\nabla \operatorname{div} \mathbf{u}(\mathbf{x},t) + b(\mathbf{x},t) = (L_t\,\mathbf{u}(\cdot,t))(\mathbf{x}) + b(\mathbf{x},t)$$

- Partition of Unity $0 \leq \varphi_i \leq 1, i \in I \in \mathbb{N}$, suff. smooth, locally supported, $\sum_{i \in I} \varphi_i = 1$.
- Local approximation spaces P_i (polynomial(s)), $\mathbf{E_i^{n+1}} \overset{!}{\supseteq} \emptyset$ (**enrichments**) on $\operatorname{supp}(\varphi_i)$

$$E_i^{n+1} = \begin{cases} \operatorname{span}\{\eta^{\mathbf{n+1}}\} & i \in \mathbf{I^{n+1}} \\ \emptyset & i \notin \mathbf{I^{n+1}} \end{cases}$$

- Global shape functions for timestep t_{n+1}

$$\bigcup_{i \in I} \{\varphi_i p : p \in P_i\} \cup \bigcup_{i \in \mathbf{I^{n+1}}} \{(\varphi_i \eta^{\mathbf{n}})\}$$

- Discretization: find coefficients $\left(c_{i,\cdot}^{n+1}\right)_{i \in I}$ and $\left(\mathbf{d_i^{n+1}}\right)_{i \in \mathbf{I^{n+1}}}$ such that

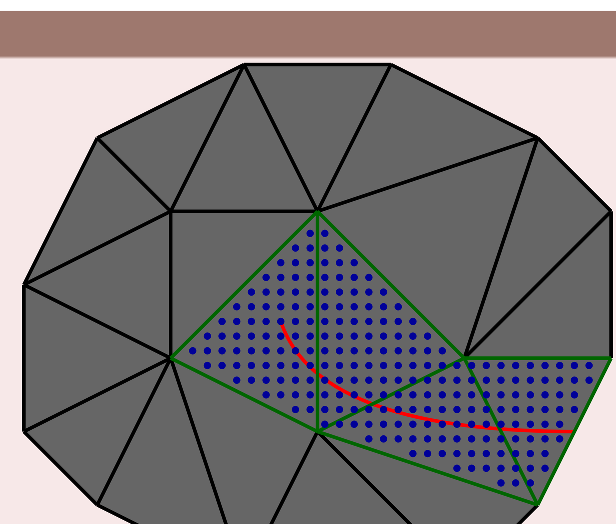
$$\mathbf{u}(\mathbf{x}, t_{n+1}) = \sum_{i \in I} \varphi_i(\mathbf{x}) \left(\sum_{p \in P_i} c_{i,p}^{n+1} p(\mathbf{x}) \right) + \sum_{i \in \mathbf{I^{n+1}}} \mathbf{d_i^{n+1}} \left(\varphi_i \eta^{\mathbf{n+1}}\right)(\mathbf{x})$$
$$\overset{!}{=} 2\mathbf{u}(\mathbf{x}, t_n) - \mathbf{u}(\mathbf{x}, t_{n-1}) + \frac{(\Delta t)^2}{\rho}\left((L_{t_n}\mathbf{u}(\cdot, t_n))(\mathbf{x}) + b(\mathbf{x}, t)\right)$$

(Discontinuous) enrichment η^{n+1} for construction of new shape functions $\varphi_i \eta^{\mathbf{n+1}}$ for global elasticity problem

Meshfree Multiscale (Coupled) Algorithm

Timestepping from $\mathbf{u}(\cdot, t_n)$ to $\mathbf{u}(\cdot, t_{n+1})$:

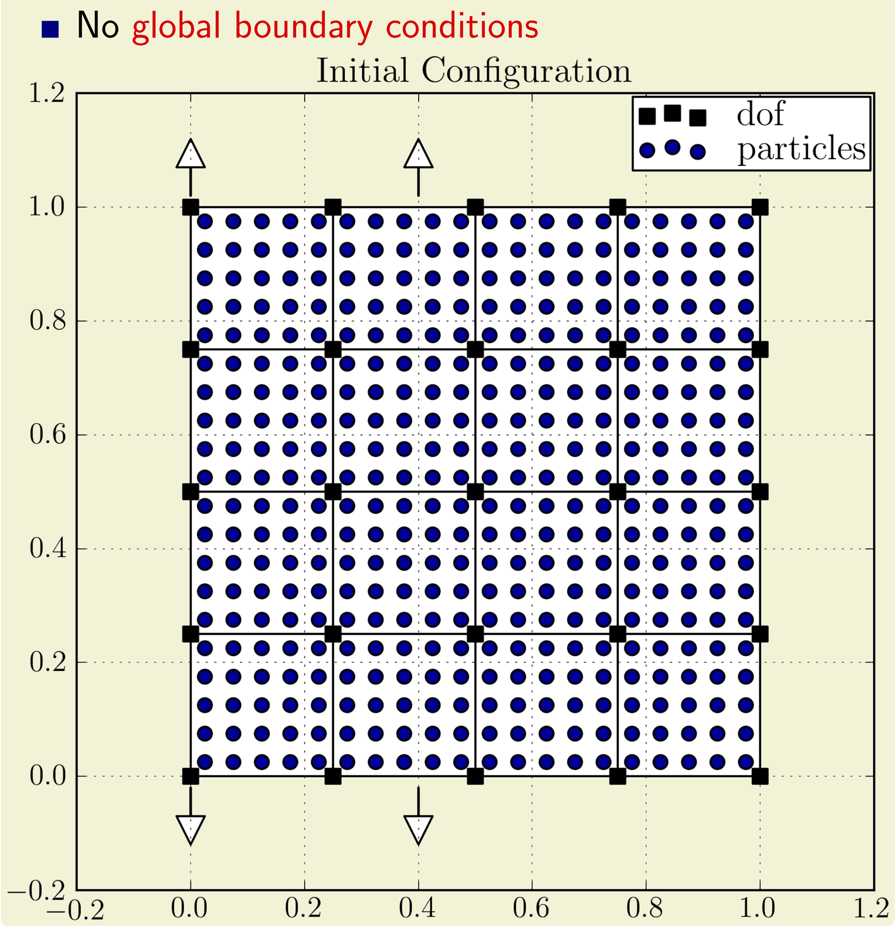
1. From **global solution** $\mathbf{u}(\cdot, t_n)$ find **patches** $\operatorname{supp}(\varphi_i), i \in \mathbf{I^{n+1}} \subseteq I$ where microscale information is necessary.
2. On $\bigcup_{i \in \mathbf{I^{n+1}}} \operatorname{supp}(\varphi_i)$ use $\mathbf{u}(\cdot, t_n)$ to seed $\mathbf{x_i}$ and find **initial and boundary conditions** to
3. Run **local particle** simulation to get $\mathbf{x_i}, \mathbf{u_i^{n+1}}$.
4. From $\mathbf{x_i}, \mathbf{u_i^{n+1}}$ reconstruct vector field η^{n+1} with gradients.
5. Use basis $\bigcup_{i \in I} \{\varphi_i p : p \in P_i\} \cup \left\{\left(\varphi_i \eta^{\mathbf{n+1}}\right)\right\}_{i \in \mathbf{I^{n+1}}}$ to solve **global problem** yielding $\mathbf{u}(\cdot, t_{n+1})$.



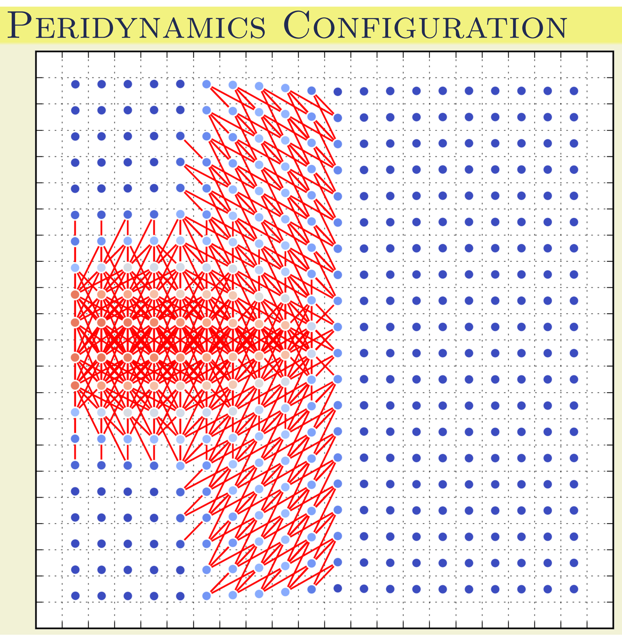
A first 2D Example

Setup

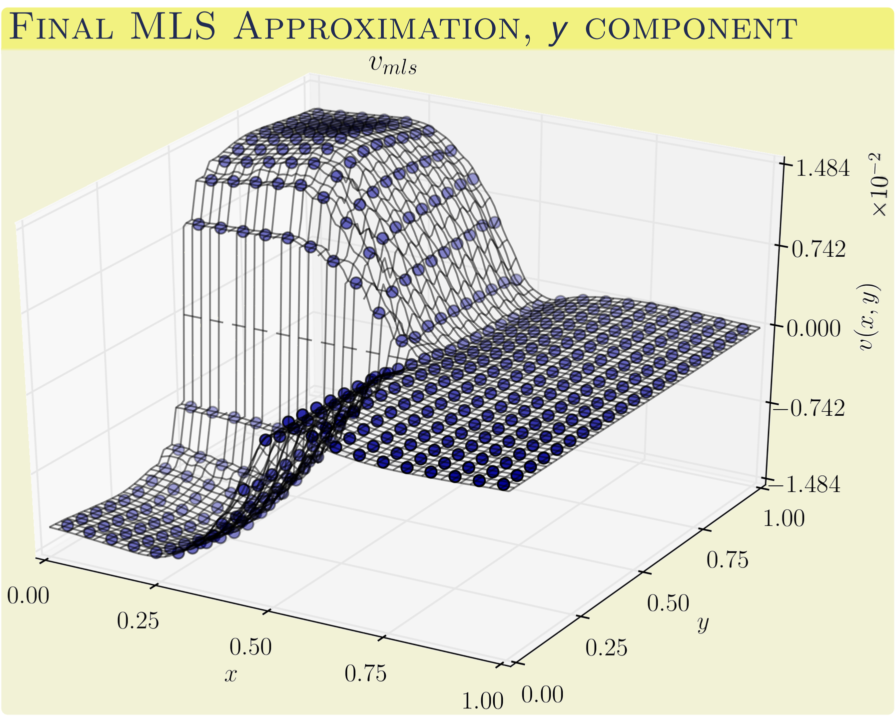
- Symmetric constant loads applied in left corners
- 4×4 **bilinear Lagrange elements**, 50 dof φ_i
- 400 **Peridynamics particles** throughout whole domain
- $P_i = \operatorname{span}\{1\}$, Automated choice of enriched dof
- 20 **GFEM timesteps** with 20×5 **Peridynamics timesteps**
- No global boundary conditions



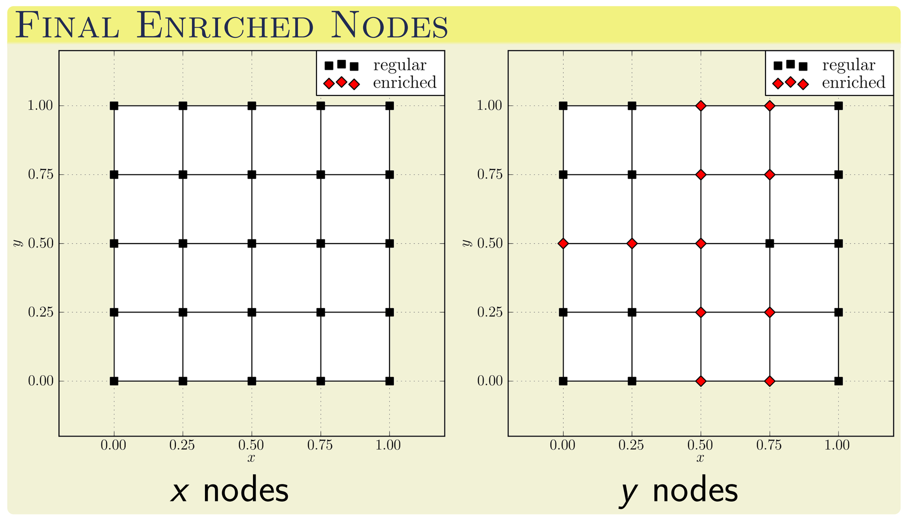
Final Peridynamics Configuration



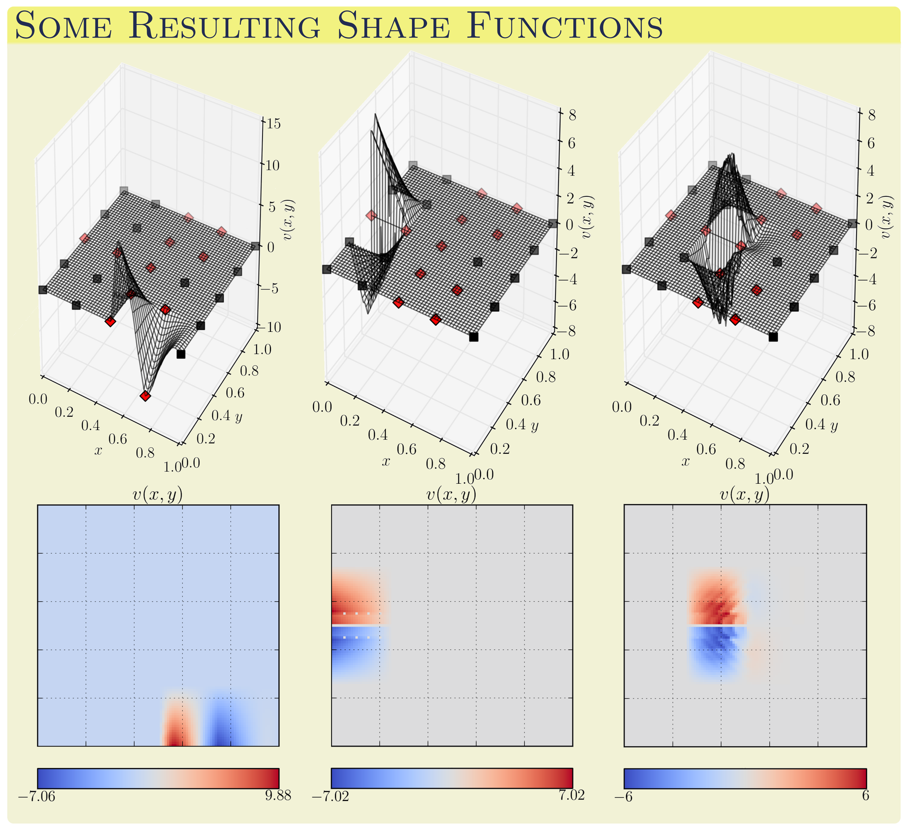
Final MLS Approximation, y component



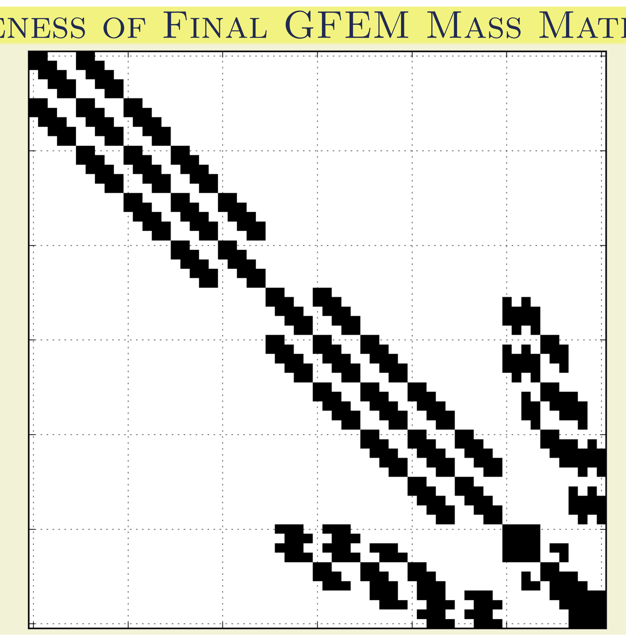
Final Enriched Nodes



Some Resulting Shape Functions



Sparseness of Final GFEM Mass Matrix



Final GFEM solution, y component