# Meshfree Multiscale Methods for Solids

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Dynamische Simulation von Systemen mit großen Teilchenzahlen

# Solve Linear Elasticity with Discontinuous Basis Functions

## What are our goals?

- Fast simulation of material failure: crack nucleation and growth
- Linear Elasticity

$$\begin{aligned}\rho\ddot{u}(x,t) &= f\big(x, u(x,t), \nabla u(x,t), \nabla^2 u(x,t), \cdots, t\big) + b(x,t)\\ &\approx \mu\Delta u(x,t) + (\lambda+\mu)\nabla \operatorname{div} u(x,t) + b(x,t)\end{aligned}$$

- Explicit time integration
- Partition of Unity $0 \le \varphi_i \le 1$, suff. smooth, locally supported, $\sum_{i\in I}\varphi_i = 1$.
- Given some enrichment $\eta^n$ and $I^n \subseteq I$ find coefficients $c_i^n, d_i^n$ (higher order terms $v_i^n \in V_i$) such that

$$\boldsymbol{u}(\cdot, t_n) = \sum_{i\in I}\varphi_i\,(c_i^n 1 + v_i^n) + \sum_{i\in I^n} d_i^n\,(\boldsymbol{\varphi_i\eta^n})$$

- Resulting mass matrix and linear system from weak form (higher order terms $v_i^n$ left out for ease of notation)

$$\left(\begin{array}{c|c}\left(\int_\Omega \varphi_i\varphi_j\right)_{i,j\in N} & \left(\int_\Omega \varphi_i\,(\boldsymbol{\varphi_j\eta^n})\right)_{i\in I, j\in I^n}\\ \hline \left(\int_\Omega (\boldsymbol{\varphi_i\eta^n})\,\varphi_j\right)_{i\in I^n, j\in I} & \left(\int_\Omega (\boldsymbol{\varphi_i\eta^n})\,(\boldsymbol{\varphi_j\eta^n})\right)_{i\in I^n, j\in I^n}\end{array}\right)\begin{pmatrix}(c_i^n)_{i\in I}\\ (d_i^n)_{i\in I^n}\end{pmatrix} = \cdots$$

# Peridynamics

## Where do we get Discontinuous Basis Functions from?

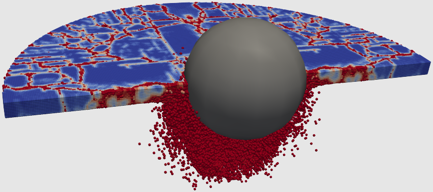
- Nonlocal equation of motion

$$\rho\ddot{u}(x,t) = \int_{\Omega(x)} f\Big( (u(\tilde{x},\cdot) - u(x,\cdot))|_{(-\infty,t]}, \tilde{x} - x, t \Big)\, d\tilde{x} + b(x,t)$$

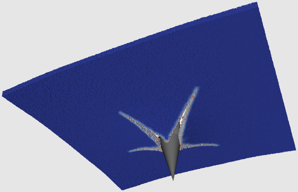
- No gradients, discontinuities occur naturally
- Discretization: fix $\boldsymbol{x_i}$, calculate $\boldsymbol{u_i^n} \approx u(\boldsymbol{x_i}, t_n)$

$$\rho\ddot{u}_i^n = \sum_{j\in N_i} f\Big( (u_j^m - u_i^m)\big|_{m\in(-\infty,n]}, x_j - x_i \Big) V_{i,j} + b(x_i, t_n)$$

### Damage everywhere



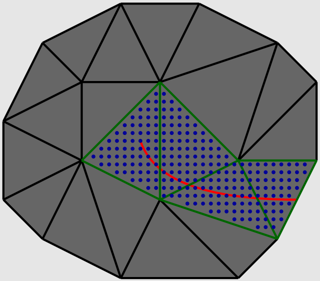
### Localized Damage



# Meshfree Multiscale Algorithm

## How do we put together our parts?

1. From **global solution** $u(\cdot, t_n)$ find **patches** where microscale information necessary.
2. On **these patches** use $u(\cdot, t_n)$ to seed $\boldsymbol{x_i}$ and run **local particle** simulation to get $\boldsymbol{x_i}, \boldsymbol{u_i^{n+1}}$.
3. From $\boldsymbol{x_i}, \boldsymbol{u_i^{n+1}}$ reconstruct vector field $\eta^{n+1}$ with gradients.
4. Use basis $\{\varphi_i\}_{i\in I} \cup \left\{\left(\varphi_i \eta^{n+1}\right)\right\}_{i\in I^{n+1}\subseteq I}$ to solve global problem yielding $u(\cdot, t_{n+1})$.



### Coupling

- vertical: shape functions for global problem from local solution
- horizontal: initial and boundary conditions for local problem from global solution

# Moving Least Squares

## How do we obtain Gradients from Peridynamics?

- Add Weights $W_i$ in least squares functional

$$\eta^n(x) := q_x(0) \qquad q_x := \operatorname*{argmin}_{p \in P} \boldsymbol{J_x^n}(p)$$

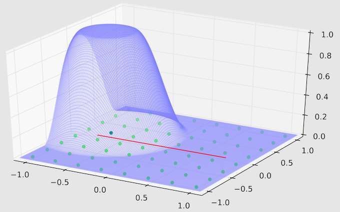
$$\boldsymbol{J_x^n}(p) := \sum_i W_i(x)\,\left(\boldsymbol{u_i^n} - p(\boldsymbol{x_i} - x)\right)^2$$

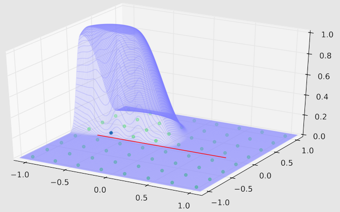
- *Adjacency* information from Peridynamics at time $t_n$:

$$\boldsymbol{A_{i,j}^n} := \begin{cases} 1 & \text{Bond between } x_i, x_j \\ 0 & \text{Broken bond between } x_i, x_j \end{cases}$$

- Take $\boldsymbol{w_{i,j}}(\boldsymbol{x_i}) = 1$, $\boldsymbol{w_{i,j}} \leq 1$ locally supported

$$\boldsymbol{\tilde{W}_i^n}(x) := W_i(x) \prod_{\left\{j : \boldsymbol{A_{i,j}^n} = 0\right\}} (1 - \boldsymbol{w_{j,i}}(x))$$

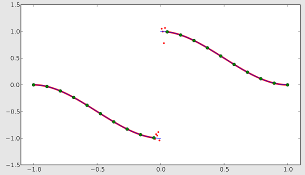


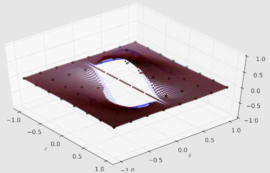


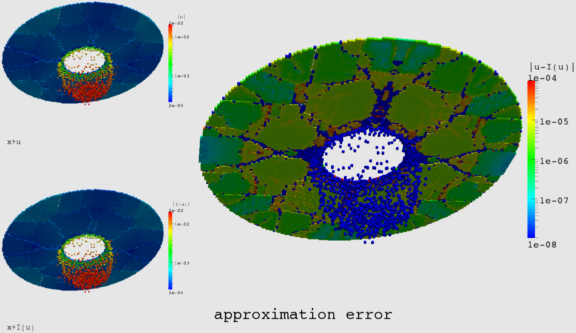
- in spirit similar to visibility criteria

# Approximation Examples

## What can we do with the modified weights?



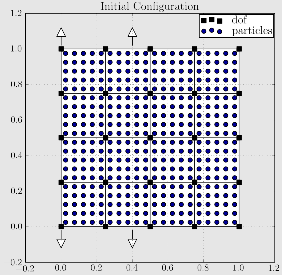


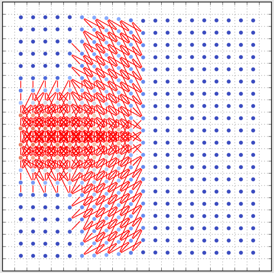


# Configurations

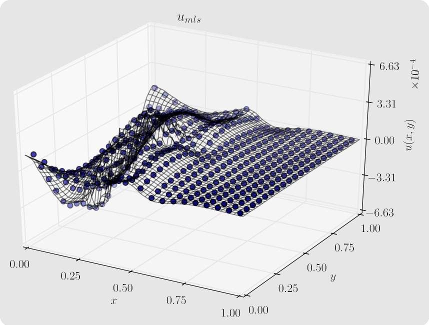
## Setup

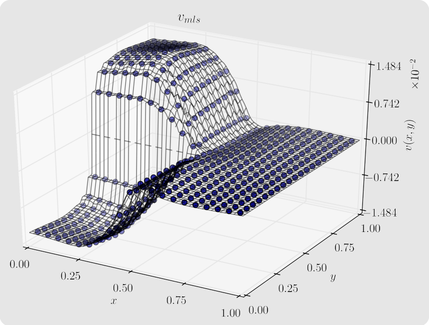
- Symmetric loads applied in left corners
- $4 \times 4$ bilinear Lagrange elements, 50 dof
- 400 Peridynamics particles throughout whole domain
- Automated choice of enriched dof
- Condition of mass matrix without enrichment 9
- 20 GFEM timesteps with $20 \times 5$ Peridynamics timesteps
- No global boundary conditions, new initial conditions for each Peridynamics run from previous GFEM solution

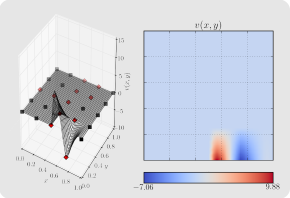


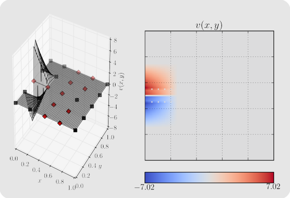



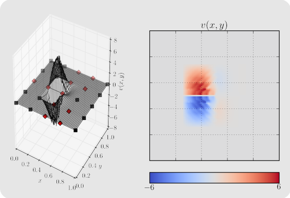
# Some Shape functions



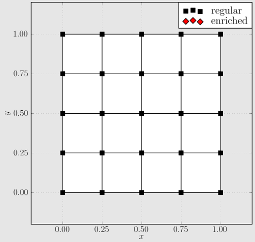



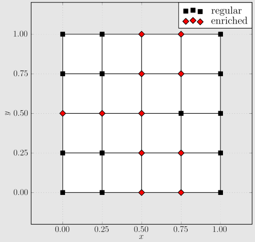



# Linear System in Last Timestep

## Enriched $x$ nodes



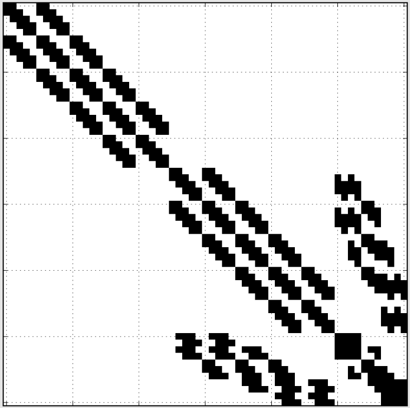

## Enriched $y$ nodes




## Numbers

- 11 additional enriched dof, total 61 dof
- Condition number of mass matrix $\sim 25$

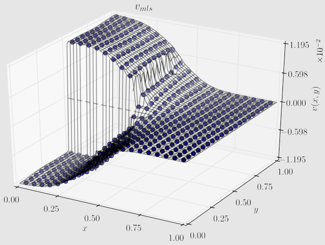
## Sparseness of Mass Matrix



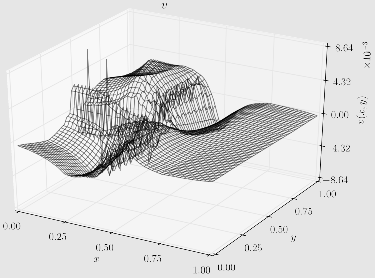
# Solutions

- Solution encompasses discontinuity
- Solution differs from pure Peridynamics solution
- Automatic choice of enriched nodes chooses only local enrichment

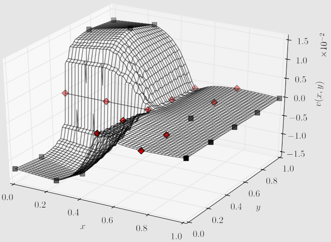
## Approximated Peridynamics




## Difference




## GFEM



# Summary

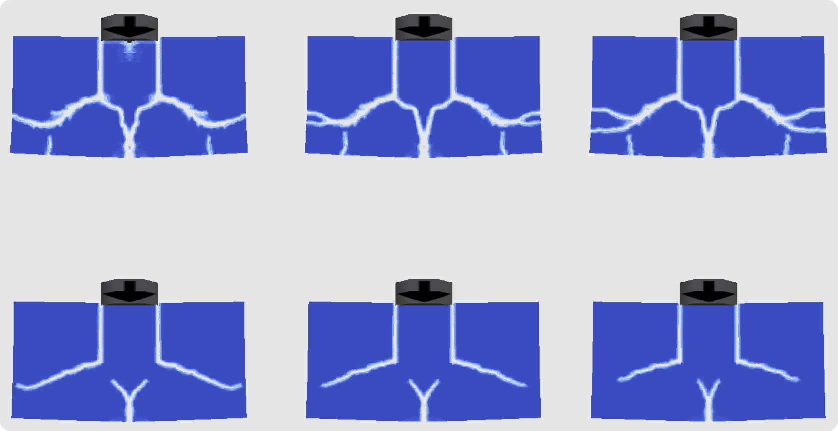
## Take Away

- Resolve fine-scaled features only where really necessary
- Use vector field reconstruction of solution from particle method as enrichment function

## Conclusions

- Particle methods on macroscale too expensive
- Enriching everywhere even more so
- Modified Moving Least Squares captures discontinuities (with adjacency information)
- Quadrature needs improvement
- Enriching everywhere leads to very badly conditioned system
- Scaling of new Shape Functions very important
- Partial monotonicity of energies

# Next?



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