Orthogonal Projection onto a Subspace Spanned by a Nonorthogonal Basis

Let \( \mathbf{n}_1, \ldots, \mathbf{n}_m \in \mathbb{R}^d \) be a basis of dimension \( m \leq d \).

**Problem 1.**

Given \( \mathbf{v} \in \mathbb{R}^d \), find \( \mathbf{v}' \in \mathbb{R}^d \) such that

\[
\mathbf{v}' = \sum \lambda_i \mathbf{n}_i \quad \text{and} \quad (\mathbf{v}' - \mathbf{v}) \cdot \mathbf{n}_i = 0 \quad \text{for all} \quad i = 1, \ldots, m.
\]

With \( \mathbf{N} = (\mathbf{n}_1, \ldots, \mathbf{n}_m) \in \mathbb{R}^{d \times m} \), we reformulate, find \( \mathbf{N}' \in \mathbb{R}^{m \times m} \) such that

\[
\mathbf{v}' = \mathbf{N}' \Rightarrow \mathbf{N}'^T (\mathbf{v}' - \mathbf{v}) = 0. \quad (2)
\]

Combining the equation yields

\[
\mathbf{N}'^T (\mathbf{N}' - \mathbf{v}) = 0 \Rightarrow \lambda = (\mathbf{N}'^T \mathbf{N})^{-1} \mathbf{N}'^T \mathbf{v}. \quad (3)
\]

With the Gramian \( \mathbf{G} = \mathbf{N}'^T \mathbf{N} \in \mathbb{R}^{m \times m} \), we write the solution (and also verify uniqueness of \( \lambda \)) as
\[ v' = P_N v, \quad P_N = N G^{-1} N^T \] (4)

\( P_N \) is an orthogonal projector \((P_N^2 = P_N, P_N = P_N^T)\).

In the special case of orthonormal \( n_i \) we obtain the simplification \( P_N = NN^T \).

The complement operation can be written as

**Problem 2.**

Find the vector \( v'' \) with \( v'' \cdot n_i = 0 \) that minimizes \( \|v'' - v\|_2 \).

The solution is

\[ v'' = P_C v, \quad P_C = 1 - P_N = 1 - N G^{-1} N^T \] (5)

This result also follows by solving the constrained optimization problem through the Lagrangian functional

\[ L = \frac{1}{2} \|v'' - v\|_2^2 - \sum_i \lambda_i (n_i \cdot v'') \] (6)

In words, we compute the vector closest to \( v \) that is orthogonal to the given normal vectors \( n_i \), even if they are not orthogonal.