# Propagator

Lösung $x(t)$ existiert mit $x' = f(t, x(t))$, $x(t_0) = x_0$

Def. $\phi^{t,t_0} : \mathbb{R}^d \to \mathbb{R}^d$ durch $\phi^{t,t_0}(x_0) := x(t)$.

Def. $\frac{d}{dt}\phi^{t,t_0} : \mathbb{R}^d \to \mathbb{R}^d$ durch

(1) $$\left(\frac{d}{dt}\phi^{t,t_0}\right)(x_0) := \frac{d}{dt}\left(\phi^{t,t_0} x_0\right) = x' = f(t, x(t))$$

Def. $$W(t,t_0) = D_{x_0} x(t) = D_{x_0} \phi^{t,t_0} \in \mathbb{R}^{d \times d}. \quad (2)$$

Taylorreihe:

$$\phi^{t,t_0}(x_0 + \delta x_0) = \phi^{t,t_0} x_0 + \overbrace{D_{x_0}\phi^{t,t_0}}^{= W(t,t_0)} \delta x_0 + \mathcal{O}(\|\delta x_0\|^2) \quad (3)$$

Def. $$\delta x(t) = \phi^{t,t_0}(x_0 + \delta x_0) - \phi^{t,t_0} x_0 \approx W(t,t_0)\,\delta x_0. \quad (4)$$

$$\frac{d}{dt} W(t,t_0) = \frac{d}{dt}\left(D_{x_0}\phi^{t,t_0}\right) = D_{x_0}\left(\frac{d}{dt}\phi^{t,t_0}\right)$$

$$\overset{(1)}{=} D_{x_0} f(t, x(t)) \overset{\text{Kettenr.}}{=} D_{x(t)} f(t, \cdot) \circ \overbrace{D_{x_0} x(t)}^{= W(t,t_0)} \quad (5)$$

$$W(t_0,t_0) = D_{x_0} x(t_0) = D_{x_0} x_0 = \mathbb{1}_{\mathbb{R}^d} \quad (6)$$

(a) $$\delta x' \overset{(4)}{\approx} \left(\frac{d}{dt} W(t,t_0)\right)\delta x_0 \overset{(5)}{=} D_{x(t)} f(t, \cdot) \circ \underbrace{W(t,t_0)\,\delta x_0}_{\approx \delta x(t)} \quad (7)$$

mit $\delta x(0) = \delta x_0$.

(b) $$\delta x' \overset{(4)}{=} f(t, x(t) + \delta x(t)) - f(t, x(t)) \approx D_{x(t)} f(t, \cdot)\, \delta x(t) \quad (8)$$

Somit ist (7) $\Leftrightarrow$ (8) DGL für Störung $\delta x(t)$ zur 1. Ordnung und (5), (6) DGL für $W(t_0,t)$ (exakt).