Gauss theorem for tensor divergence

**Def. 1** Let $f,g : \mathbb{R} \rightarrow \mathbb{R}$ be functions, $u,v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ vector fields, $A : \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$ a tensor field.

**Def. 2** The tensor divergence is defined as

$$(\nabla \cdot A)_i = \sum_j A_{ij}.$$  (1)

**Thm. 3** (Gauss theorem)

$$\nabla \cdot \mathbf{F} = \oint_{\partial \Omega} \mathbf{F} \cdot d\mathbf{r} = \int_\Omega \nabla \cdot \mathbf{F} \, d\mathbf{r}$$  (2)

with outside normal vector $\mathbf{v}$ on $\partial \Omega$.

If we set $\mathbf{F} = u_i$ and sum over $i$ in (2), we obtain

$$\int_\Omega \nabla \cdot u = \oint_{\partial \Omega} u \cdot v. \quad (3)$$

Inserting $u = \nabla g$ into (3) yields

$$\int_\Omega \nabla g \cdot \nabla g = - \int_\Omega \nabla \cdot A g + \oint_{\partial \Omega} v \cdot \nabla g. \quad (4)$$
This equation is fundamental to the finite element discretization of Kirchhoff's equation. To discretize Stokes or linear elasticity we have to go one step further.

Note that

\[ \nabla \cdot (A \nabla v) = \sum_i \frac{\partial}{\partial x_i} \sum_{j} A_{ij} v_j = \]

\[ = \sum_{ij} (v_j \frac{\partial}{\partial x_i} A_{ij} + A_{ij} \frac{\partial v_j}{\partial x_i}) = \]

\[ = v \cdot \nabla A^T + A : (\nabla v)^T. \quad (5) \]

Thus using \( u = A^T v \) in (3) yields

\[ \int_\Omega A : \nabla v = - \int_\Omega v \cdot \nabla A + \int_\Omega v \cdot A \nabla v. \quad (6) \]

We have not assumed symmetry of \( A \). The symmetric form of Stokes etc. follows from setting

\[ A = \frac{1}{2} (\nabla u + (\nabla u)^T) \quad \Rightarrow \quad (7) \]

\[ \int_\Omega A : \nabla v = \int_\Omega \frac{1}{2} (\nabla u + (\nabla u)^T) : \frac{1}{2} (\nabla u + (\nabla u)^T). \]